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Research Article

Global Attractivity of Positive Periodic Solutions of Delay Differential Equations with Feedback Control

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By constructing suitable Liapunov functionals and estimating uniform upper and lower bounds of solutions, sufficient conditions are obtained for the global attractivity of positive periodic solutions of the delay differential system with feedback control $dy/dt = y(t)F(t,y(t-\tau_1(t)),\dots,y(t-\tau_n(t)),u(t-\delta(t)))$, $du/dt = -\eta(t)u(t)+a(t)y(t-\sigma(t))$. When these results are applied to the periodic logistic equation with several delays and feedback control, some new results are obtained.

1. Introduction

Recently, Huo and Li [1], Gopalsamy and Weng [2] introduced a feedback control variable into the delayed logistic model and discussed the asymptotic behavior of solutions in logistic models with feedback controls, in which the control variables satisfy certain differential equation. We also refer to Xiaoxing and Fengde [3], Li and Zhu [4] for further study on delay equations with feedback control.

In this paper, we consider the following general nonlinear nonautonomous delay differential system with feedback control:

$$\begin{aligned}\frac{dy}{dt} &= y(t)F(t,y(t-\tau_1(t)),\dots,y(t-\tau_n(t)),u(t-\delta(t))),\\ \frac{du}{dt} &= -\eta(t)u(t)+a(t)y(t-\sigma(t)),\end{aligned} \tag{1.1}$$

where $F(t,z_1,z_2,\dots,z_n,z_{n+1}) \in C(R^{n+2},R)$, $\tau_i(t)$ $(i=1,2,\dots,n)$, $\delta(t),\sigma(t) \in C(R,R)$, $\eta(t),a(t)\in C(R,(0,\infty))$, all functions are ω-periodic in t, and $\omega>0$ is a constant. By constructing some suitable Liapunov functionals, we obtain sufficient conditions of global attractivity of periodic solutions for system (1.1). When we apply the obtained results to the periodic logistic equation with several delays [5, 6] and feedback control, some new results are obtained. In view of [1], let

$$u(t)=\int_t^{t+\omega}G(t,s)a(s)y(s-\sigma(s))ds:=(\Phi y)(t), \tag{1.2}$$

where

$$G(t,s)=\frac{\exp\{\int_t^s\eta(r)dr\}}{\exp\{\int_0^\omega\eta(r)dr\}-1}. \tag{1.3}$$

We know that the existence of ω-periodic solution of (1.1) is equivalent to that of the equation

$$\frac{dy(t)}{dt}=y(t)F(t,y(t-\tau_1(t)),\dots,y(t-\tau_n(t)),(\Phi y)(t-\delta(t))). \tag{1.4}$$

2. Main results

From Huo and Li [1], we have the following theorem.

Theorem 2.1. *If the following conditions hold:*

(i) *there exists a constant $C>0$ such that if $x(t)$ and $u(t)$ are continuous ω-periodic functions and satisfy*

$$\int_0^\omega F(t,e^{x(t-\tau_1(t))},\dots,e^{x(t-\tau_n(t))},e^{u(t-\delta(t))})dt=0, \tag{2.1}$$

then

$$\int_0^\omega |F(t,e^{x(t-\tau_1(t))},\dots,e^{x(t-\tau_n(t))},e^{u(t-\delta(t))})|dt\le C; \tag{2.2}$$

(ii) *there exists a constant $H>0$ such that when $v_i\ge H$, $i=1,2,\dots,n+1$,*

$$F(t,e^{v_1},e^{v_2},\dots,e^{v_n},e^{v_{n+1}})>0,\qquad F(t,-e^{v_1},-e^{v_2},\dots,-e^{v_n},-e^{v_{n+1}})<0 \tag{2.3}$$

uniformly hold for $t\in[0,\infty)$, then system (1.1) has at least one positive ω-periodic solution.

Next we will derive sufficient conditions under which (1.1) has a unique positive ω-periodic solution that attracts all other positive solutions. From now on we always assume that $F \in C^1(R^{n+2}, R)$. Let $\{y^*(t), u^*(t)\}$ be a positive ω-periodic solution and set

$$y(t) = y^*(t)\exp\{x(t)\}, \qquad u(t) = u^*(t)\exp\{x(t)\}. \tag{2.4}$$

Then (1.1) can be reduced to

$$\frac{dx}{dt} = G(t, x(t-\tau_1(t)), \dots, x(t-\tau_n(t)), x(t-\delta(t))) - G(t, 0, \dots, 0, 0), \tag{2.5}$$

where

$$\begin{aligned} &G(t, u_1, \dots, u_n, u_{n+1}) \\ &\quad = F(t, y^*(t-\tau_1(t))\exp\{u_1\}, \dots, y^*(t-\tau_n(t))\exp\{u_n\}, u^*(t-\delta(t))\exp\{u_{n+1}\}). \end{aligned} \tag{2.6}$$

By the mean value theorem, we can rewrite (2.5) as

$$\frac{dx}{dt} = \sum_{i=1}^{n} J_i(t)x(t-\tau_i(t)) + J_{n+1}(t)x(t-\delta(t)), \tag{2.7}$$

where

$$\begin{aligned} J_i(t) &= \frac{\partial G(t, \eta_1(t), \dots, \eta_n(t), \eta_{n+1}(t))}{\partial \eta_i}, \quad i = 1, \dots, n, n+1, \\ \min\{y^*(t-\tau_i(t)), y(t-\tau_i(t))\} &\le y^*(t-\tau_i(t))\exp\{\eta_i(t)\} \\ &\le \max\{y^*(t-\tau_i(t)), y(t-\tau_i(t))\}, \quad i = 1, 2, \dots, n, \\ \min\{u^*(t-\delta(t)), u(t-\delta(t))\} &\le u^*(t-\delta(t))\exp\{\eta_{n+1}(t)\} \\ &\le \max\{u^*(t-\delta(t)), u(t-\delta(t))\}. \end{aligned} \tag{2.8}$$

Theorem 2.2. *In addition to the assumptions in Theorem 2.1, assume further that*

(i) $\tau_i(t) = \tau_i$ $(i = 1, 2, \dots, n)$ *and* $\delta(t) = \delta$ *are constants;*

(ii) *there exists* $T_1 > 0$ *such that*

$$J_i(t) < 0, \quad t > T_1, \ i = 1, 2, \dots, n+1; \tag{2.9}$$

(iii) *there exist* $T_2 > 0$ *and constants* $A_1, A_2 > 0$ *such that every solution* $\{y(t), u(t)\}$ *of (1.1) satisfies*

$$A_1 \le y(t) \le A_2, \quad A_1 \le u(t) \le A_2, \quad t > T_2; \tag{2.10}$$

(iv) $\max\{\beta,\gamma\}<2$, *where*

$$\begin{aligned}\beta&=\limsup_{t\to\infty}\Bigg[-\sum_{i=1}^{n}\int_{t-\tau_i}^{t}(J_i(s+\tau_i)+J_i(s+2\tau_i))ds-\int_{t-\delta}^{t}(J_{n+1}(s+\delta))ds\\&\qquad-\int_{t-\tau_i}^{t}(J_{n+1}(s+\delta+\tau_i))ds\Bigg],\\\gamma&=\limsup_{t\to\infty}\Bigg[-\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)ds-\int_{t-\delta}^{t}(J_{n+1}(s+\delta)+J_{n+1}(s+2\delta))ds\\&\qquad-\sum_{i=1}^{n}\int_{t-\delta}^{t}J_i(s+\delta+\tau_i)ds\Bigg],\end{aligned}\tag{2.11}$$

then system (1.1) has a unique positive ω-periodic solution $\{y^*(t),u^*(t)\}$ *such that every solution* $\{y(t),u(t)\}$ *of (1.1) satisfies*

$$\lim_{t\to\infty}[y(t)-y^*(t)]=0,\qquad \lim_{t\to\infty}[u(t)-u^*(t)]=0.\tag{2.12}$$

Proof. The existence of $\{y^*(t),u^*(t)\}$ follows from Theorem 2.1 and the uniqueness will follow from (2.12). Therefore, from the above discussion, it suffices to prove that every solution of (2.7) has the asymptotic behavior

$$\lim_{t\to\infty}x(t)=0.\tag{2.13}$$

To this end, we define a functional $V(t)=V(x(t))$ as

$$\begin{aligned}V(t)=&\left[x(t)+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)x(s)ds+\int_{t-\delta}^{t}J_{n+1}(s+\delta)x(s)ds\right]^2\\&+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+2\tau_i)\left[\sum_{i=1}^{n}\int_{s}^{t}J_i(h+\tau_i)x^2(h)dh\right]ds\\&+\int_{t-\delta}^{t}J_{n+1}(s+2\delta)\left[\int_{s}^{t}J_{n+1}(h+\delta)x^2(h)dh\right]ds\\&+\sum_{i=1}^{n}\int_{t-\delta}^{t}J_i(s+\delta+\tau_i)\left[\int_{s}^{t}J_{n+1}(h+\delta)x^2(h)dh\right]ds\\&+\int_{t-\tau_i}^{t}J_{n+1}(s+\delta+\tau_i)\left[\sum_{i=1}^{n}\int_{s}^{t}J_i(h+\tau_i)x^2(h)dh\right]ds.\end{aligned}\tag{2.14}$$

Then along the solutions of (2.7), we have

$$\begin{aligned}
\frac{dV(t)}{dt} &= 2\left[x(t)+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)x(s)ds+\int_{t-\delta}^{t}J_{n+1}(s+\delta)x(s)ds\right]\\
&\quad\times\left[x(t)\sum_{i=1}^{n}J_i(t+\tau_i)+x(t)J_{n+1}(t+\delta)\right]\\
&\quad+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+2\tau_i)ds\sum_{i=1}^{n}J_i(t+\tau_i)x^2(t)\\
&\quad-\sum_{i=1}^{n}J_i(t+\tau_i)\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)x^2(s)ds\\
&\quad+\int_{t-\delta}^{t}J_{n+1}(s+2\delta)dsJ_{n+1}(t+\delta)x^2(t)\\
&\quad-J_{n+1}(t+\delta)\int_{t-\delta}^{t}J_{n+1}(s+\delta)x^2(s)ds\\
&\quad+\sum_{i=1}^{n}\int_{t-\delta}^{t}J_i(s+\delta+\tau_i)dsJ_{n+1}(t+\delta)x^2(t)\\
&\quad-\sum_{i=1}^{n}J_i(t+\tau_i)\int_{t-\delta}^{t}J_{n+1}(s+\delta)x^2(s)ds\\
&\quad+\int_{t-\tau_i}^{t}J_{n+1}(s+\delta+\tau_i)ds\sum_{i=1}^{n}J_i(t+\tau_i)x^2(t)\\
&\quad-J_{n+1}(t+\delta)\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)x^2(s)ds.
\end{aligned}\tag{2.15}$$

When $t>T_1$, using inequality $2x(t)x(s)\le x^2(t)+x^2(s)$ to make estimation and simplification, we obtain

$$\begin{aligned}
\frac{dV(t)}{dt} &\le x^2(t)\left[\sum_{i=1}^{n}J_i(t+\tau_i)\right]\Bigg[2+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)ds+\int_{t-\delta}^{t}J_{n+1}(s+\delta)ds\\
&\qquad+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+2\tau_i)ds+\int_{t-\tau_i}^{t}J_{n+1}(s+\delta+\tau_i)ds\Bigg]\\
&\quad+x^2(t)J_{n+1}(t+\delta)\Bigg[2+\sum_{i=1}^{n}\int_{t-\tau_i}^{t}J_i(s+\tau_i)ds+\int_{t-\delta}^{t}J_{n+1}(s+\delta)ds\\
&\qquad+\int_{t-\delta}^{t}J_{n+1}(s+2\delta)ds+\sum_{i=1}^{n}\int_{t-\delta}^{t}J_i(s+\delta+\tau_i)ds\Bigg].
\end{aligned}\tag{2.16}$$

This and conditions (ii) and (iv) show that for $\epsilon_0 = (2-\beta)/4$, $\epsilon_1 = (2-\gamma)/4$, there exists a sufficiently large $T(\geq \max\{T_1, T_2\})$ such that

$$\frac{dV(t)}{dt} \leq x^2(t)\left[(2-\beta-\epsilon_0)\sum_{i=1}^{n} J_i(t+\tau_i) + (2-\gamma-\epsilon_1)J_{n+1}(t+\delta)\right], \quad t \geq T. \tag{2.17}$$

Thus, V is eventually nonincreasing. From (2.4), (2.5), and condition (iii), it follows that $x(t)$ is uniformly continuous on $[0,\infty)$. Moreover, by (2.6), (2.8), conditions (ii) and (iii), it follows that there exist a $T_3(\geq T)$ and constants $D_i, E_i > 0$, $i = 1,\dots,n,n+1$ such that

$$D_i \leq -J_i(t) \leq E_i, \quad t \geq T_3,\ i = 1,\dots,n,n+1. \tag{2.18}$$

Taking this into account and integrating (2.17) over $[T_3, t]$, we have

$$V(t) + (2-\beta-\epsilon_0)\left[\sum_{i=1}^{n} D_i\right]\int_{T_3}^{t} x^2(s)ds + (2-\gamma-\epsilon_1)D_{n+1}\int_{T_3}^{t} x^2(s)ds \leq V(T_3) < \infty. \tag{2.19}$$

Hence, $x^2 \in L^1(T_3, \infty)$. By Barbălat's lemma [7], we have

$$\lim_{t\to\infty} x^2(t) = 0. \tag{2.20}$$

This completes the proof. □

3. Applications

In this section, we apply the results obtained in the previous section to the following logistic model with several delays [5, 6] with feedback control:

$$\begin{aligned} \frac{dy}{dt} &= y(t)\left[r(t) - \sum_{i=1}^{n} a_i(t)y(t-\tau_i(t)) - c(t)u(t-\delta(t))\right], \\ \frac{du}{dt} &= -\eta(t)u(t) + a(t)y(t-\sigma(t)), \end{aligned} \tag{3.1}$$

where $\tau_i(t)$, $i = 1,2,\dots,n$, $\delta(t), \sigma(t) \in C(R,R)$, $r(t)$, $c(t)$, $a_i(t)$, $i = 1,2,\dots,n$, $\eta(t)$, $a(t) \in C(R,(0,\infty))$, all of the above functions are ω-periodic functions and $\omega > 0$ is a constant. By [1], we have the following theorem.

Theorem 3.1. *System (3.1) has at least one positive ω-periodic solution.*

In order to establish the uniqueness and global attractivity of the positive periodic solutions, we need to obtain certain upper and lower bounds for the positive solutions. For convenience, we introduce the following notations:

$$(f)_M = \max_{t\in[0,\omega]} f(t), \qquad (f)_m = \min_{t\in[0,\omega]} f(t), \qquad \tau^* = \max_{1\leq i\leq n}\left\{\max_{t\in[0,\infty]} \tau_i(t)\right\}, \tag{3.2}$$

where f is a continuous nonnegative ω-periodic solution.

Lemma 3.2. *Assume that* $\{y(t),u(t)\}$ *is a solution of (3.1) and*

$$(r)_m-(c)_M\overline{U}_1>0. \tag{3.3}$$

Then there exists a number T_3 *such that*

$$\underline{U}_1\le y(t)\le\overline{U}_1,\quad \underline{U}_1\le u(t)\le\overline{U}_1,\quad t\ge T_3, \tag{3.4}$$

where

$$\begin{aligned}\overline{U}_1&=\max\left\{\frac{(r)_M}{\sum_{i=1}^{n}(a_i)_m}\exp\left((r)_M\tau^*\right)\frac{(a)_M}{(\eta)_m},\frac{(r)_M}{\sum_{i=1}^{n}(a_i)_m}\exp\left((r)_M\tau^*\right)\right\},\\ \underline{U}_1&=\min\left\{\frac{(r)_m-(c)_M\overline{U}_1}{\sum_{i=1}^{n}(a_i)_M}\exp\left\{\left[(r)_m-\left(\sum_{i=1}^{n}(a_i)_M+(c)_M\right)\overline{U}_1\right]\tau^*\right\}\frac{(a)_m}{(\eta)_M},\right.\\ &\qquad\left.\frac{(r)_m-(c)_M\overline{U}_1}{\sum_{i=1}^{n}(a_i)_M}\exp\left\{\left[(r)_m-\left(\sum_{i=1}^{n}(a_i)_M+(c)_M\right)\overline{U}_1\right]\tau^*\right\}\right\}.\end{aligned} \tag{3.5}$$

Proof. Clearly, any solution $y(t)$ of (3.1) satisfies the delay differential inequality

$$\frac{dy}{dt}\le y(t)\left[(r)_M-\sum_{i=1}^{n}(a_i)_m y(t-\tau_i(t))\right]. \tag{3.6}$$

Now either $y(t)$ is oscillatory about

$$W=\frac{(r)_M}{\sum_{i=1}^{n}(a_i)_m} \tag{3.7}$$

or it is nonoscillatory. In the case when $y(t)$ is oscillatory about W, we let $\tau^*<t_1<t_2<\cdots<t_n<\cdots$ be a sequence of zeros of $y(t)-W$ with $\lim_{n\to\infty}t_n=\infty$ and $y(t_n)=W$. Our strategy is to establish the upper bound in each interval (t_n,t_{n+1}). For this, let t_n^* be a point where $y(t)$ attends its maximum in (t_n,t_{n+1}). Now, since $y(t_n^*)$ is the maximum, then we have $y(t_n^*)\ge y(t_n)=W$. Since $y'(t_n^*)=0$, it follows from (3.6) that we have

$$0=y'(t_n^*)<\frac{y(t_n^*)}{\sum_{i=1}^{n}(a_i)_m}\left[W-y(t_n^*-\tau^*)\right]. \tag{3.8}$$

Hence,

$$y(t_n^*-\tau^*)<W. \tag{3.9}$$

Since $y(t_n^*)\ge W$ and $y(t_n^*-\tau^*)<W$, we can let ξ be the first zero of $y(t)-W$ in $(t_n^*-\tau^*,t_n^*)$, that is, $y(\xi)=\dot{W}$. Integrating (3.6) from ξ to t_n^*, we have

$$\ln\left(\frac{y(t_n^*)}{y(\xi)}\right)\le\int_{\xi}^{t_n^*}(r)_M dt\le\int_{t_n^*-\tau_i(t_n^*)}^{t_n^*}(r)_M dt \tag{3.10}$$

or

$$y(t_n^*) \leq W \exp\left((r)_M \tau^*\right). \tag{3.11}$$

Since the right-hand side of (3.11) is independent of t, we conclude that

$$y(t) \leq W \exp\left((r)_M \tau^*\right), \quad t \geq t_1 + 2\tau^*. \tag{3.12}$$

Next, assume that $y(t)$ is nonoscillatory about W. Then we can see that for every $\epsilon > 0$, there exists a $T_1' = T_1'(\epsilon)$ such that

$$y(t) < W + \epsilon, \quad t > T_1'. \tag{3.13}$$

From this and (3.11) it follows that there exists a T_2' such that

$$y(t) \leq W \exp\left((r)_M \tau^*\right), \quad t > T_2'. \tag{3.14}$$

By (1.2) and (3.14), there exists a T_3', for $t > T_3'$, we have

$$\begin{aligned} u(t) &\leq W \exp\left((r)_M \tau^*\right) \int_t^{t+\omega} G(t,s) a(s) ds \\ &\leq W \exp\left((r)_M \tau^*\right) \frac{(a)_M}{(\eta)_m} \int_t^{t+\omega} G(t,s)\eta(s) ds = W \exp\left((r)_M \tau^*\right) \frac{(a)_M}{(\eta)_m}. \end{aligned} \tag{3.15}$$

From (3.12), (3.14), and (3.15), there exists a $T_4' > 0$ such that for $t > T_4'$,

$$y(t) \leq \max\left\{ W \exp\left((r)_M \tau^*\right) \frac{(a)_M}{(\eta)_m}, W \exp\left((r)_M \tau^*\right)\right\} = \overline{U}_1, \tag{3.16}$$

$$u(t) \leq \max\left\{ W \exp\left((r)_M \tau^*\right) \frac{(a)_M}{(\eta)_m}, W \exp\left((r)_M \tau^*\right)\right\} = \overline{U}_1. \tag{3.17}$$

In a similar way we can derive a lower bound for the solution of (3.1). In fact, by (3.1) and (3.17), we have

$$\frac{dy(t)}{dt} \geq y(t)\left[(r)_m - \sum_{i=1}^{n} (a_i)_M y(t - \tau_i(t)) - (c)_M \overline{U}_1 \right]. \tag{3.18}$$

Set

$$V = \frac{(r)_m - (c)_M \overline{U}_1}{\sum_{i=1}^{n} (a_i)_M}. \tag{3.19}$$

If $y(t)$ is oscillatory about V, let $\tau^* < s_1 < s_2 < \cdots < s_n < \cdots$ be a sequence of zeros of $y(t) - V$ with $\lim_{n\to\infty} s_n = \infty$ and $y(s_n) = V$. Our strategy is to establish the upper bound in each interval (s_n, s_{n+1}). For this, let s_n^* be a point where $y(t)$ attends its minimum in (s_n, s_{n+1}). Now, since $y(s_n^*)$ is the minimum, then we have $y(s_n^*) \leq y(s_n) = V$. Since $y'(s_n^*) = 0$, it follows from (3.18) that we have

$$0 = y'(s_n^*) > \frac{y(s_n^*)}{\sum_{i=1}^{n} (a_i)_M} \left[V - y(s_n^* - \tau^*) \right]. \tag{3.20}$$

Hence,

$$y(t_n^* - \tau^*) > V. \tag{3.21}$$

Since $y(t_n^*) \le W$ and $y(t_n^* - \tau^*) > V$, we can let ζ be the first zero of $y(t) - V$ in $(s_n^* - \tau^*, s_n^*)$, that is, $y(\zeta) = V$. Integrating (3.20) from ζ to s_n^*, we have

$$\begin{aligned} \ln\left(\frac{y(s_n^*)}{y(\zeta)}\right) &\ge \int_\zeta^{s_n^*}\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]dt \\ &\ge \left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right](s_n^* - \zeta) \end{aligned} \tag{3.22}$$

or

$$y(s_n^*) \ge V\exp\left\{\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]\tau^*\right\}. \tag{3.23}$$

Thus,

$$y(t) \ge V\exp\left\{\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]\tau^*\right\}, \quad t \ge s_1 + 2\tau^*. \tag{3.24}$$

Now assume that $y(t)$ is nonoscillatory about V. Then for every $\epsilon > 0$, there exists a $T_1'' = T_1''(\epsilon) > 0$ such that

$$y(t) > V - \epsilon, \quad t > T_1''. \tag{3.25}$$

From this and (3.24) it follows that there exists a $T_2'' > 0$ such that

$$y(t) \ge V\exp\left\{\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]\tau^*\right\}, \quad t > T_2''. \tag{3.26}$$

By (1.2), (3.24), and (3.26), we have $T_3'' > 0$ such that when $t > T_3''$,

$$\begin{aligned} u(t) &\ge V\exp\left\{\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]\tau^*\right\}\int_t^{t+\omega} G(t,s)a(s)ds \\ &\ge V\exp\left\{\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]\tau^*\right\}\frac{(a)_m}{(\eta)_M}\int_t^{t+\omega} G(t,s)\eta(s)ds \\ &= V\exp\left\{\left[(r)_m - \left(\sum_{i=1}^n (a_i)_M + (c)_M\right)\overline{U}_1\right]\tau^*\right\}\frac{(a)_m}{(\eta)_M}. \end{aligned} \tag{3.27}$$

From (3.24), (3.26), and (3.27), there exists a $T_4'' > 0$ such that for $t > T_4''$,

$$
\begin{aligned}
y(t) \geq \min\Bigg\{ & V\exp\Bigg\{\Bigg[(r)_m - \Bigg(\sum_{i=1}^{n}(a_i)_M + (c)_M\Bigg)\overline{U}_1\Bigg]\tau^*\Bigg\}\frac{(a)_m}{(\eta)_M}, \\
& V\exp\Bigg\{\Bigg[(r)_m - \Bigg(\sum_{i=1}^{n}(a_i)_M + (c)_M\Bigg)\overline{U}_1\Bigg]\tau^*\Bigg\}\Bigg\} = \underline{U}_1, \\
u(t) \geq \min\Bigg\{ & V\exp\Bigg\{\Bigg[(r)_m - \Bigg(\sum_{i=1}^{n}(a_i)_M + (c)_M\Bigg)\overline{U}_1\Bigg]\tau^*\Bigg\}\frac{(a)_m}{(\eta)_M}, \\
& V\exp\Bigg\{\Bigg[(r)_m - \Bigg(\sum_{i=1}^{n}(a_i)_M + (c)_M\Bigg)\overline{U}_1\Bigg]\tau^*\Bigg\}\Bigg\} = \underline{U}_1.
\end{aligned}
\tag{3.28}
$$

The proof is complete. □

By Theorem 3.1, Lemma 3.2, and Theorem 2.2, we have the following results.

THEOREM 3.3. *Assume that* $\tau_i(t) = \tau_i$ $(i = 1,2,\ldots,n)$, $\delta(t) = \delta$ *are constants. If*
(i) $(r)_m - (c)_M\overline{U}_1 > 0$;
(ii) $\max\{\beta,\gamma\} < 2$, *where*

$$
\begin{aligned}
\beta = \limsup_{t\to\infty}\Bigg\{\overline{U}_1\Bigg[& \sum_{i=1}^{n}\int_{t-\tau_i}^{t}\big(b_i(s+\tau_i) + b_i(s+2\tau_i)\big)ds \\
& + \int_{t-\delta}^{t} c(s+\delta)ds + \int_{t-\tau_i}^{t} c(s+\delta+\tau_i)ds\Bigg]\Bigg\}, \\
\gamma = \limsup_{t\to\infty}\Bigg\{\overline{U}_1\Bigg[& \sum_{i=1}^{n}\int_{t-\tau_i}^{t} b_i(s+\tau_i)ds + \int_{t-\delta}^{t}\big(c(s+\delta) + c(s+2\delta)\big)ds \\
& + \sum_{i=1}^{n}\int_{t-\delta}^{t} b(s+2\delta+\tau_i)ds\Bigg]\Bigg\},
\end{aligned}
\tag{3.29}
$$

then system (3.1) has a unique positive ω-periodic solution $\{y^*(t),u^*(t)\}$ *such that every solution* $\{y(t),u(t)\}$ *of (3.1) satisfies (2.12).*

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