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Research Article

On the Difference Equation $x_{n+1} = \sum_{j=0}^{k} a_j f_j(x_{n-j})$

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This paper studies the boundedness character and the global attractivity of positive solutions of the difference equation $x_{n+1} = \sum_{j=0}^{k} a_j f_j(x_{n-j})$, $n \in \mathbb{N}_0$, where a_j are positive numbers and f_j are continuous decreasing self-maps of the interval $(0,\infty)$ for $j = 0,1,\dots,k$.

1. Introduction

Recently, there has been a great interest in studying the behavior of rational and nonlinear difference equations; see, for example, [1–20]. One of the most intriguing properties of solutions of difference equations is their boundedness character. There are numerous papers devoted, among others, to this research area, see; for example, [1–6, 9–19], and related references therein.

It is said that a function f is decreasing on an interval J if for all $x, y \in J$ such that $x < y$, $f(x) > f(y)$.

Consider the nonlinear higher-order difference equation of the form

$$x_{n+1} = \sum_{j=0}^{k} a_j f_j(x_{n-j}), \quad n \in \mathbb{N}_0, \tag{1.1}$$

where for each $j = 0,1,\dots,k$, the functions $f_j : (0,\infty) \to (0,\infty)$, $j = 0,1,\dots,k$, are continuous and a_j are positive numbers.

In all the sequel, we assume the following.

(H_1) All f_j are continuous decreasing bijections of the interval $(0,+\infty)$.

(H_2) For each $j = 0,1,\dots,k$, the function $x \to x f_j(x)$ is nondecreasing on $(0,+\infty)$.

From condition (H_1) and since a_j, $j = 0,1,\dots,k$ are positive numbers, it follows that $\sum_{j=0}^{k} a_j f_j(x)$ is a continuous decreasing self-map of $(0,\infty)$, hence there exists $\gamma > 0$ such that

$$0 < \gamma < \sum_{j=0}^{k} a_j f_j(\gamma). \tag{1.2}$$

Further, (H_1) implies that the algebraic equation

$$x = \sum_{j=0}^{k} a_j f_j(x) \tag{1.3}$$

has a unique positive equilibrium $x = \overline{x}$ such that

$$\left[x - \sum_{j=0}^{k} a_j f_j(x)\right](x - \overline{x}) > 0, \quad \forall x \neq \overline{x}. \tag{1.4}$$

It is obvious that

$$\gamma < \overline{x} < \sum_{j=0}^{k} a_j f_j(\gamma) =: \Gamma, \tag{1.5}$$

where γ satisfies (1.2).

Also, note that condition (H_1) implies that

$$\lim_{x\to+0} f_j(x) = +\infty, \qquad \lim_{x\to+\infty} f_j(x) = 0 \tag{1.6}$$

for $j = 0,1,\dots,k$.

Our aim here is to investigate the boundedness character and global attractivity of positive solutions of (1.1), where $a_j > 0$, $j = 0,1,\dots,k$, and $f_j : (0,\infty) \to (0,\infty)$, $j = 0,1,\dots,k$, satisfy conditions (H_1) and (H_2). Some special cases of (1.1) has been investigated, for example, in [4, 5, 9, 10, 12] from which our motivation stems.

Our main result of this paper is the following.

Theorem 1.1. *Consider (1.1), where for each $j = 0,1,\dots,k$, the functions f_j satisfy conditions (H_1) and (H_2). Then every positive solution of (1.1) converges to the equilibrium $\overline{x}$.*

The paper is organized as follows. In Section 2.1, we prove several auxiliary results which will be used in the proof of Theorem 1.1. The theorem will be proved in Section 2.2.

2. Boundedness and global attractivity of (1.1)

In view of assumption (H_1), any solution (x_n) of (1.1), starting from positive values x_{-k}, $x_{-k+1},\dots,x_0$, is positive.

For each $j = 0,1,\dots,k$, we define the following auxiliary functions:

$$F_j(x) := a_j f_j(x),$$
$$G_j(x) := a_j f_j(x) + \sum_{i \neq j} a_i f_i(\overline{x}). \tag{2.1}$$

Notice that the functions F_j and G_j, $j = 0,1,\dots,k$ are positive and decreasing on the interval $(0,+\infty)$.

Before we formulate our results, we recall definitions of semicycles.

Let $(x_n)_{n=-k}^{\infty}$ be a solution of (1.1). A positive semicycle of the solution $(x_n)_{n=-k}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to the equilibrium point $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\begin{aligned} &\text{either } l = -k \quad \text{or} \quad l > -k, \quad x_{l-1} < \overline{x}, \\ &\text{either } m = \infty \quad \text{or} \quad m < \infty, \quad x_{m+1} < \overline{x}. \end{aligned} \tag{2.2}$$

Let $(x_n)_{n=-k}^{\infty}$ be a solution of (1.1). A negative semicycle of the solution $(x_n)_{n=-k}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than the equilibrium point $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\begin{aligned} &\text{either } l = -k \quad \text{or} \quad l > -k, \quad x_{l-1} \geq \overline{x}, \\ &\text{either } m = \infty \quad \text{or} \quad m < \infty, \quad x_{m+1} \geq \overline{x}. \end{aligned} \tag{2.3}$$

2.1. Some auxiliary facts. Here are some important properties of the semicycles of (1.1).

Lemma 2.1. *Assume that, for all $j = 0,1,\dots,k$, the functions f_j satisfy condition (H_1). Then every semicycle of an eventually nonequilibrium solution contains at most $k+1$ terms.*

Proof. Let (x_n) be an eventually nonequilibrium solution of (1.1) with terms $x_{n-j} \geq \overline{x}$ for $j = 0,1,\dots,k$, and at least one of them is greater than $\overline{x}$. Then from (1.1) and (H_1), we obtain

$$x_{n+1} = \sum_{j=0}^{k} a_j f_j(x_{n-j}) < \sum_{j=0}^{k} a_j f_j(\overline{x}) = \overline{x}. \tag{2.4}$$

The case $x_n, x_{n-1}, \dots, x_{n-k} < \overline{x}$ is similar. □

Lemma 2.2. *Assume that all functions f_j, $j = 0,1,\dots,k$ satisfy condition (H_1). Then the following statements are true.*

(a) *If for some n it holds that*

$$x_n \leq \min_i f_i^{-1}\left(\frac{\Gamma}{\min_j a_j}\right), \tag{2.5}$$

then $x_{n+l} > \overline{x}$ for all $l = 1,2,\dots,k+1$.

(b) *If for some $n \geq k+2$ it holds that*

$$x_n \leq \min_j F_j(\Gamma), \tag{2.6}$$

then $x_{n-l} > \overline{x}$ for $l = 1,2,\dots,k+1$.

(c) *If for some $n \geq k+2$ it holds that*

$$x_n \leq m, \tag{2.7}$$

where

$$m := \min\left\{\min_i f_i^{-1}\left(\frac{\Gamma}{\min_j a_j}\right), \min_i F_i(\Gamma)\right\}, \tag{2.8}$$

then

$$x_{n\pm l} > \overline{x} \tag{2.9}$$

for $l = 1,2,\dots,k+1$.

Proof. (a) Assume that for some n condition (2.5) holds and consider any $l \in \{1,2,\dots,k+1\}$. Since f_{l-1} is a decreasing map, from (2.5), we have

$$f_{l-1}(x_n) \geq \frac{\Gamma}{\min_j a_j}, \tag{2.10}$$

and therefore

$$x_{n+l} = \sum_{j=1}^{k+1} a_{j-1} f_{j-1}(x_{n+l-j}) > a_{l-1} f_{l-1}(x_n) \geq \min_j a_j \frac{\Gamma}{\min_j a_j} > \overline{x}, \tag{2.11}$$

where we have used (1.5).

(b) Suppose, on the contrary, that for some n satisfying (2.6) and $l \in \{1,2,\dots,k+1\}$ it holds $x_{n-l} \leq \overline{x}$. Then we observe that

$$x_n = \sum_{j=1}^{k+1} a_{j-1} f_{j-1}(x_{n-j}) > a_{l-1} f_{l-1}(x_{n-l}) = F_{l-1}(x_{n-l}) \geq F_{l-1}(\overline{x}) > F_{l-1}(\Gamma), \tag{2.12}$$

and so

$$x_n > \min_j F_j(\Gamma), \tag{2.13}$$

which is a contradiction.

(c) The proof is a direct consequence of statements (a) and (b). □

Now consider the function

$$\phi(x) := \frac{1}{x}\sum_{j=0}^{k} a_j f_j\big(G_{k-j}(x)\big), \quad x > 0. \tag{2.14}$$

Notice that $\phi(\overline{x}) = 1$. Moreover, the following statement holds true.

Lemma 2.3. *If conditions (H_1), (H_2) are satisfied, then ϕ is decreasing, and so $\overline{x}$ is the unique solution of the equation*

$$\phi(u) = 1. \tag{2.15}$$

Proof. The function ϕ can be written in the following form;

$$\phi(x) := \sum_{j=0}^{k} a_j \frac{G_{k-j}(x) f_j\big(G_{k-j}(x)\big)}{xG_{k-j}(x)}. \tag{2.16}$$

Here we observe that for each j, the numerator is nonincreasing, while in view of condition (H_2) and the definition of G_j, the denominator increases. This proves the monotonicity. The rest of the proof is obvious. □

Lemma 2.4. *Assume that f satisfies condition (H_1) and set*

$$M := \min\{\overline{x}, m\}, \tag{2.17}$$

where m is defined in Lemma 2.2. Suppose that (x_n) is a solution of (1.1) such that $x_n \le M$ for some $n \ge k+2$. Then

$$\begin{gathered} x_{n\pm l} > \overline{x}, \quad l = 1,2,\dots,k+1, \\ x_n < x_{n+k+2} < \overline{x}. \end{gathered} \tag{2.18}$$

Proof. By Lemma 2.2(c), we have $x_{n\pm l} > \overline{x}$ for $l = 1,2,\dots,k+1$. Hence for all $l \in \{1,2,\dots, k+1\}$, we have

$$\begin{aligned} x_{n+l} &= \sum_{j=1}^{k+1} a_{j-1} f_{j-1}(x_{n+l-j}) = a_{l-1} f_{l-1}(x_n) + \sum_{j\ne l} a_{j-1} f_{j-1}(x_{n-j}) \\ &< a_{l-1} f_{l-1}(x_n) + \sum_{j\ne l} a_{j-1} f_{j-1}(\overline{x}), \end{aligned} \tag{2.19}$$

namely,

$$x_{n+l} < G_{l-1}(x_n). \tag{2.20}$$

Consequently, we obtain

$$x_{n+k+2} = \sum_{j=1}^{k+1} a_{j-1} f_{j-1}(x_{n+k+2-j}) > \sum_{j=0}^{k} a_j f_j\big(G_{k-j}(x_n)\big) = x_n\phi(x_n) \ge x_n, \tag{2.21}$$

where the last inequality holds because of the fact that $x_n \le \overline{x}$, ϕ is decreasing, and $\phi(\overline{x}) = 1$. By Lemma 2.1, it follows that $x_{n+k+2} < \overline{x}$, as desired. □

2.2. Proof of the main result. In this subsection, we prove the main result of this paper. In the proof, we need the following result by Karakostas (see [7, 8]).

Theorem 2.5. *Let J be some interval of real numbers, $f \in C[J^{k+1}, J]$, and let $(x_n)_{n=-k}^{\infty}$ be a bounded solution of the difference equation*

$$x_{n+1} = f(x_n, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{2.22}$$

with $I = \liminf_{n\to\infty} I_n$, $S = \limsup_{n\to\infty} x_n$, and with $I, S \in J$. Then there exist two solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of the difference equation

$$x_{n+1} = f(x_n, \ldots, x_{n-k}) \tag{2.23}$$

which satisfy the equation for all $n \in \mathbb{Z}$, with $I_0 = I$, $S_0 = S$, $I_n, S_n \in [I, S]$ for all $n \in \mathbb{Z}$ and such that for every $N \in \mathbb{Z}$, I_N and S_N are limit points of $(x_n)_{n=-k}^{\infty}$. Furthermore, for every $m \le -k$, there exist two subsequences (x_{r_n}) and (x_{l_n}) of the solution $(x_n)_{n=-k}^{\infty}$ such that the following are true:

$$\lim_{n\to\infty} x_{r_n+N} = I_N, \quad \lim_{n\to\infty} x_{l_n+N} = S_N \quad \text{for every } N \ge m. \tag{2.24}$$

The solutions $(I_n)_{n=-\infty}^{\infty}$ and $(S_n)_{n=-\infty}^{\infty}$ of (2.23) are called full limiting solutions of (2.23) associated with the solution $(x_n)_{n=-k}^{\infty}$ of (2.22).

Proof of Theorem 1.1. If a solution (x_n) of (1.1) is eventually equal to the equilibrium $\overline{x}$, the result is obvious. Hence, we may assume that (x_n) is not eventually equal to $\overline{x}$. First, we show that any such solution is bounded and it stays away from zero. Notice that by Lemma 2.3, the function ϕ defined in (2.14) is decreasing.

Let M be defined as in (2.17) and set

$$M_1 := \sum_{j=0}^{k} a_j f_j(M), \tag{2.25}$$

$$m_0 := \min\left\{M, \sum_{j=0}^{k} a_j f_j(M_1)\right\}. \tag{2.26}$$

Let (x_n) be a solution of (1.1). If b is a lower bound of (x_n), then $B := \sum_{j=0}^{k} a_j f_j(b)$ is an upper bound and vice-versa. Hence, it suffices to show that (x_n) is bounded from below by a positive constant.

Now, the number m_0 is a lower bound of the solution (x_n), or not. If the first case occurs, then we finished. In the second case, assume that there is an $n_0 \ge k+2$ such that $x_{n_0} < m_0$. We will show that

$$x_{n_0} \le x_n \tag{2.27}$$

for all $n \geq n_0$. On the contrary, assume that there is an $N > n_0$ such that $x_N < x_{n_0}$. We can assume that N is the smallest index with this property.

Since

$$x_N < x_{n_0} < m_0 \leq M, \tag{2.28}$$

by Lemma 2.4, we have

$$x_{N \pm j} > \overline{x} \geq m, \quad j = 1, 2, \dots, k+1. \tag{2.29}$$

If $N = n_0 + j$ for some $j \in \{1, 2, \dots, k+1\}$, then from (2.17), (2.26), and (2.29), we have

$$x_{n_0} = x_{N-j} > \overline{x} \geq m_0, \tag{2.30}$$

a contradiction. Thus, it holds that $N \geq n_0 + k + 2$, and therefore

$$x_{n_0} \leq x_{N-(k+2)}, \tag{2.31}$$

in view of the choice of N.

We claim that

$$x_{N-j} < M_1 \quad \text{for } j = 1, 2, \dots, k+1. \tag{2.32}$$

Indeed, to this end, suppose that

$$x_{N-(k+2)} \leq M. \tag{2.33}$$

Since $N - (k+2) \geq k+2$, by Lemma 2.4, we obtain

$$x_{N-(k+2)} < x_N, \tag{2.34}$$

and so $x_{n_0} < x_N$ because of (2.31). But this contradicts the choice of N. Thus we have $x_{N-(k+2)} > M$.

Also, if $x_{N-j-(k+2)} \leq M$, for some $j \in \{1, 2, \dots, k+1\}$, then by Lemma 2.4, it holds that $x_{N-j-(k+2)} < x_{N-j} \leq \overline{x}$. On the other hand, from (2.29), we have $x_{N-j} > \overline{x}$, thus we arrive to a contradiction. Therefore, we have

$$x_{N-j-(k+2)} > M \quad \text{for } j = 1, 2, \dots, k+1. \tag{2.35}$$

Hence for $j = 1, 2, \dots, k+1$, it follows that

$$x_{N-j} = \sum_{i=1}^{k+1} a_{i-1} f_{i-1}(x_{N-j-i}) < \sum_{i=1}^{k+1} a_{i-1} f_{i-1}(M) = M_1. \tag{2.36}$$

This proves the claim.

Now, from (1.1) and (2.36), it follows that

$$x_{n_0} > x_N = \sum_{i=1}^{k+1} a_{i-1} f_{i-1}(x_{N-i}) \geq \sum_{i=1}^{k+1} a_{i-1} f_{i-1}(M_1) \geq m_0 > x_{n_0}, \tag{2.37}$$

which is a contradiction. From this, the boundedness of (x_n) follows.

Next, we use Theorem 2.5 to show the convergence to the equilibrium point $\bar{x}$. As we proved above, every positive solution (x_n) is bounded and it stays away from zero. This means that (x_n) is a compact solution in the sense of limiting sequences. Consider two full limiting sequences z_n, y_n, $n \in \mathbb{Z}$ such that

$$0 < \liminf x_n = z_0 \leq z_n, \qquad y_n \leq y_0 = \limsup x_n. \tag{2.38}$$

By taking subsequences, we have that

$$\begin{aligned} y_0 &= \sum_{i=0}^{k} a_i f_i(y_{-1-i}) \leq \sum_{i=0}^{k} a_i f_i(z_0), \\ z_0 &= \sum_{i=0}^{k} a_i f_i(z_{-1-i}) \geq \sum_{i=0}^{k} a_i f_i(y_0). \end{aligned} \tag{2.39}$$

Then, from condition (H_2) and (2.39), it follows that

$$z_0 y_0 \leq \sum_{i=0}^{k} a_i z_0 f_i(z_0) \leq \sum_{i=0}^{k} a_i y_0 f_i(y_0) \leq z_0 y_0. \tag{2.40}$$

This means that (2.39) hold as equalities, and consequently,

$$\sum_{i=0}^{k} a_i f_i(z_0) = y_0, \qquad \sum_{i=0}^{k} a_i f_i(y_0) = z_0. \tag{2.41}$$

Thus it follows that $y_0 = z_{-i}$ for $i = 1, 2, \ldots, k+1$. From, (1.1) and (2.41), we obtain

$$y_0 = z_{-k} = \sum_{i=0}^{k} a_i f_i(z_{-k-1-i}) \leq \sum_{i=0}^{k} a_i f_i(z_0) = y_0, \tag{2.42}$$

and so $z_{-k-i} = z_0$ for all $i = 1, 2, \ldots, k+1$. In particular, we obtain $z_0 = z_{-k-1} = y_0$, and since $\bar{x}$ in a unique equilibrium of (1.1), it follows that $z_0 = y_0 = \bar{x}$, which proves our theorem. □

Example 2.6. Theorem 1.1 can be applied, for example, to the following difference equation

$$x_{n+1} = \frac{a}{x_n^p + x_n^{p+\alpha}} + \frac{b}{x_{n-1}^q + x_{n-1}^{q+\beta}} + \frac{c}{x_{n-2}^r + x_{n-2}^{r+\delta}}, \tag{2.43}$$

when $a, b, c, p, q, r > 0$, $\alpha, \beta, \delta \geq 0$, $a + b + c > 2$, and $\max\{p + \alpha, q + \beta, r + \delta\} \leq 1$.

Indeed, it is easy to see that conditions (H_1) and (H_2) are satisfied. Note that condition (1.2) is satisfied for $\gamma = 1$ (here we use the condition $a + b + c > 2$).

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References

[1] K. S. Berenhaut, J. D. Foley, and S. Stević, "Quantitative bounds for the recursive sequence $y_{n+1} = A + y_n/y_{n-k}$," *Applied Mathematics Letters*, vol. 19, no. 9, pp. 983–989, 2006.

[2] K. S. Berenhaut and S. Stević, "The behaviour of the positive solutions of the difference equation $x_n = A + (x_{n-2}/x_{n-1})^p$," *Journal of Difference Equations and Applications*, vol. 12, no. 9, pp. 909–918, 2006.

[3] B. Dai and X. Zou, "Permanence for a class of nonlinear difference systems," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 78607, 10 pages, 2006.

[4] R. DeVault, G. Ladas, and S. W. Schultz, "Necessary and sufficient conditions for the boundedness of $x_{n+1} = A/x_n^p + B/x_{n-1}^q$," *Journal of Difference Equations and Applications*, vol. 3, no. 3-4, pp. 259–266, 1998.

[5] H. M. El-Owaidy, A. A. Ragab, and M. M. El-Afifi, "On the recursive sequence $x_{n+1} = A/x_n^p + B/x_{n-1}^q + C/x_{n-2}^s$," *Applied Mathematics and Computation*, vol. 112, no. 2-3, pp. 277–290, 2000.

[6] H. M. El-Owaidy, A. M. Ahmed, and M. S. Mousa, "On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + (x_{n-1}^p/x_n^p)$," *Journal of Applied Mathematics & Computing*, vol. 12, no. 1-2, pp. 31–37, 2003.

[7] G. L. Karakostas, "A discrete semi-flow in the space of sequences and study of convergence of sequenses defined by difference equations," *M. E. Greek Math. Soc.*, vol. 30, pp. 66–74, 1989.

[8] G. L. Karakostas, "Convergence of a difference equation via the full limiting sequences method," *Differential Equations and Dynamical Systems*, vol. 1, no. 4, pp. 289–294, 1993.

[9] G. L. Karakostas and S. Stević, "On the difference equation $x_{n+1} = Af(x_n) + f(x_{n-1})$," *Applicable Analysis*, vol. 83, no. 3, pp. 309–323, 2004.

[10] Ch. G. Philos, I. K. Purnaras, and Y. G. Sficas, "Global attractivity in a nonlinear difference equation," *Applied Mathematics and Computation*, vol. 62, no. 2-3, pp. 249–258, 1994.

[11] S. Stević, "On the recursive sequence $x_{n+1} = -1/x_n + A/x_{n-1}$," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 1, pp. 1–6, 2001.

[12] S. Stević, "A note on the difference equation $x_{n+1} = \sum_{j=0}^{k} \alpha_i/x_{n-i}^{p_i}$," *Journal of Difference Equations and Applications*, vol. 8, no. 7, pp. 641–647, 2002.

[13] S. Stević, "A global convergence results with applications to periodic solutions," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 1, pp. 45–53, 2002.

[14] S. Stević, "On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 249–259, 2003.

[15] S. Stević, "On the recursive sequence $x_{n+1} = \alpha + x_{n-1}^p/x_n^p$," *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 229–234, 2005.

[16] S. Stević, "On the recursive sequence $x_{n+1} = \alpha + \beta x_{n-k}/f(x_n, \dots, x_{n-k+1})$," *Taiwanese Journal of Mathematics*, vol. 9, no. 4, pp. 583–593, 2005.

[17] S. Stević, "On the recursive sequence $x_n = 1 + \sum_{i=1}^{k} \alpha_i x_{n-p_i} / \sum_{j=1}^{m} \beta_j x_{n-q_j}$," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 39404, 7 pages, 2007.

[18] S. Stević, "On the recursive sequence $x_{n+1} = A + x_n^p/x_{n-1}^p$," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 34517, 9 pages, 2007.

[19] T. Sun, H. Xi, and H. Wu, "On boundedness of the solutions of the difference equation $x_{n+1} = x_{n-1}/(p + x_n)$," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 20652, 7 pages, 2006.

[20] S.-E. Takahasi, Y. Miura, and T. Miura, "On convergence of a recursive sequence $x_{n+1} = f(x_{n-1}, x_n)$," *Taiwanese Journal of Mathematics*, vol. 10, no. 3, pp. 631–638, 2006.

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