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Research Article

A Global Convergence Result for a Higher Order Difference Equation

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Let $f(z_1,\dots,z_k) \in C(I^k,I)$ be a given function, where I is (bounded or unbounded) subinterval of $\mathbb{R}$, and $k \in \mathbb{N}$. Assume that $f(y_1,y_2,\dots,y_k) \ge f(y_2,\dots,y_k,y_1)$ if $y_1 \ge \max\{y_2,\dots,y_k\}$, $f(y_1,y_2,\dots,y_k) \le f(y_2,\dots,y_k,y_1)$ if $y_1 \le \min\{y_2,\dots,y_k\}$, and f is nondecreasing in the last variable z_k. We then prove that every bounded solution of an autonomous difference equation of order k, namely, $x_n = f(x_{n-1},\dots,x_{n-k})$, $n = 0,1,2,\dots$, with initial values $x_{-k},\dots,x_{-1} \in I$, is convergent, and every unbounded solution tends either to $+\infty$ or to $-\infty$.

1. Introduction and the main results

Let us consider an autonomous difference equation of order $k \in \mathbb{N}$:

$$x_n = f(x_{n-1},\dots,x_{n-k}), \quad n \in \mathbb{N}_0 := \{0,1,2,\dots\}, \tag{1.1}$$

where $f : I^k \to I$, interval I is bounded or unbounded interval of the real line $\mathbb{R}$, and f is a function that satisfies the condition $f(x,\dots,x) \le x$ for all $x \in I$. The difference equation (1.1) was investigated by many authors (see, e.g., [1–11] and the references cited therein). In most of these results the monotonicity of the function $f(z_1,\dots,z_k)$ in its variables $z_1,\dots,z_k$ plays the main role.

For example, in [7], Stević proved the following theorem.

Theorem 1.1. *Assume that f is a continuous real-valued function defined on $\mathbb{R}^k$ satisfying the following conditions:*

(a) *f is nondecreasing in each of its arguments;*

(b) *$f(z_1,\dots,z_k)$ is strictly increasing in z_1;*

(c) *for every* $x \in \mathbb{R}$ *it holds*

$$f(x,x,\dots,x) \le x. \tag{1.2}$$

Then every bounded solution of the difference inequality

$$x_n \le f(x_{n-1},\dots,x_{n-k}), \quad n \in \mathbb{N}_0, \tag{1.3}$$

converges.

This result generalizes the main results in [1, 3]. For closely related results see also [2, 6, 9, 10, 12–18] and the references cited therein. The asymptotic behavior of the solutions of (1.1) was investigated in [11].

The following result was proved (also by Stević) in [8] (for a particular case of the result in which condition (c) below is replaced with the fact that every real number $\overline{x}$ is an equilibrium point of $x_n = f(x_{n-1},\dots,x_{n-k})$, see [4, Theorem 2.2]).

Theorem 1.2. *Assume that f is a continuous real-valued function on $\mathbb{R}^k$, where f satisfies conditions (a) and (c) from Theorem 1.1, and where $f(z_1,\dots,z_k)$ is strictly increasing in at least two of its arguments z_i and z_j, where i and j are relatively prime integers. Then any bounded solution of difference inequality (1.3) converges.*

Remark 1.3. The fact that the last condition in Theorem 1.2 is sharp, in the sense that primeness cannot be omitted, was shown in [9].

Here we will improve these results in some sense. We present a global convergence result regarding the solutions of (1.1) in which the function f need not be monotonous in each variable, and the condition $f(x,\dots,x) \le x$ can be omitted as an irrelevant one.

Theorem 1.4. *Let $f(z_1,\dots,z_k) \in C(I^k, I)$ be a given function, where I is bounded or unbounded interval of $\mathbb{R}$, which satisfies the following conditions:*

(a)

$$f(y_1,y_2,\dots,y_k) \ge f(y_2,\dots,y_k,y_1) \tag{1.4}$$

if $y_1 \ge \max\{y_2,\dots,y_k\}$;

(b)

$$f(y_1,y_2,\dots,y_k) \le f(y_2,\dots,y_k,y_1) \tag{1.5}$$

if $y_1 \le \min\{y_2,\dots,y_k\}$;

(c) *f is nondecreasing in the last variable z_k.*

Then every bounded solution of (1.1) with initial values $x_{-k},\dots,x_{-1} \in I$ converges, and every unbounded solution of (1.1) tends either to $+\infty$ or to $-\infty$.

2. Proof of the main result

In this section, we prove the main result of this paper. Before this, we formulate and prove an auxiliary result.

Lemma 2.1. *Assume that (x_n) is a solution of (1.1) for which there is an $n_0 \in \mathbb{N}$ such that*

$$x_{n_0-1} \le \min\{x_{n_0-2},\dots,x_{n_0-k-1}\} \tag{2.1}$$

or

$$x_{n_0-1} \ge \max\{x_{n_0-2},\dots,x_{n_0-k-1}\}. \tag{2.2}$$

Then under the conditions of Theorem 1.4, the solution (x_n) is eventually monotonous.

Proof. Assume that condition (2.1) holds. From (2.1) and (1.5), and by employing conditions (b) and (c) from Theorem 1.4, it follows that

$$\begin{aligned} x_{n_0} &= f(x_{n_0-1},x_{n_0-2},\dots,x_{n_0-k}) \\ &\le f(x_{n_0-2},\dots,x_{n_0-k},x_{n_0-1}) \\ &\le f(x_{n_0-2},\dots,x_{n_0-k},x_{n_0-k-1}) = x_{n_0-1} \end{aligned} \tag{2.3}$$

and, consequently,

$$x_{n_0} \le \min\{x_{n_0-1},\dots,x_{n_0-k}\}. \tag{2.4}$$

From this and by induction, it follows that $x_{n+1} \le x_n$ for $n \ge n_0 - 1$, which means that the solution (x_n) of (1.1) is eventually nonincreasing, as desired. The proof of the lemma under condition (2.2) is very similar and, therefore, will be omitted. This finishes the proof. □

Proof of Theorem 1.4. By Lemma 2.1, we may suppose that for every $n \in \mathbb{N}$,

$$\min\{x_{n-1},\dots,x_{n-k}\} < x_n < \max\{x_{n-1},\dots,x_{n-k}\}. \tag{2.5}$$

Otherwise, the sequence is eventually monotonous and, as such, it converges to a finite limit or to $+\infty$ or $-\infty$.

Since $\min\{x_{n-1},\dots,x_{n-k}\} \le \min\{x_{n-1},\dots,x_{n-k+1}\}$ and from (2.5), we have $\min\{x_{n-1},\dots,x_{n-k}\} < x_n$. It is easily seen that the sequence

$$m_n = \min\{x_n,\dots,x_{n-k+1}\}, \quad n \in \mathbb{N}, \tag{2.6}$$

is nondecreasing; similarly, the sequence

$$M_n = \max\{x_n,\dots,x_{n-k+1}\}, \quad n \in \mathbb{N}, \tag{2.7}$$

is nonincreasing.

Our aim is to prove that

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n. \tag{2.8}$$

Since sequences (m_n) and (M_n) are bounded, this is equivalent with

$$\lim_{n\to\infty}\left(\max\{x_n,\dots,x_{n-k+1}\} - \min\{x_n,\dots,x_{n-k+1}\}\right) = 0, \tag{2.9}$$

that motivates us to consider the function $h : I^k \to \mathbb{R}^+ \cup \{0\}$ given by

$$h(z_1,\dots,z_k) = \max\{z_1,\dots,z_k\} - \min\{z_1,\dots,z_k\}. \tag{2.10}$$

Let us introduce the sequence

$$X_n = (x_n, x_{n-1},\dots,x_{n-k+1}), \quad n \in \mathbb{N}, \tag{2.11}$$

and $\mathcal{A} = \overline{\{X_n : n \in \mathbb{N}\}}$, that is, the closure of the set $\{X_n : n \in \mathbb{N}\}$. Note that set $\mathcal{A}$ is compact, as a bounded and a closed subset of $\mathbb{R}^k$. Since X_n can be written in the form

$$X_n = (f(x_{n-1},\dots,x_{n-k}), x_{n-1},\dots,x_{n-k+1}), \quad n \in \mathbb{N}, \tag{2.12}$$

we also have that set $\mathcal{A}$ is invariant under the vector function

$$F(z_1,\dots,z_k) = (f(z_1,\dots,z_k), z_1,\dots,z_{k-1}). \tag{2.13}$$

Function h, as a continuous one, attains its minimum on $\mathcal{A}$ at some $\mathbf{c}_0 \in \mathcal{A}$. Let us remember that $h(\mathbf{c}_0) \geq 0$. Let (y_n) be the solution of (1.1) with initial values equal to $\mathbf{c}_0 = (y_{-1},\dots,y_{-k})$. We claim that, for this solution, there is a number $n_0 \in \mathbb{N}$ such that

$$y_{n_0-1} \leq \min\{y_{n_0-2},\dots,y_{n_0-k-1}\} \quad \text{or} \quad y_{n_0-1} \geq \max\{y_{n_0-2},\dots,y_{n_0-k-1}\}. \tag{2.14}$$

Since the set $\mathcal{A}$ is invariant under F,

$$h(y_{k-1},\dots,y_0) \geq h(y_{-1},\dots,y_{-k}), \tag{2.15}$$

where the right-hand side of the inequality is the minimum of function h, we have that there is an index $i_0 \in \{0,1,\dots,k-1\}$ such that

$$y_{i_0} = \max\{y_{k-1},\dots,y_0\} \geq \max\{y_{-1},\dots,y_{-k}\} \tag{2.16}$$

or

$$y_{i_0} = \min\{y_{k-1},\dots,y_0\} \leq \min\{y_{-1},\dots,y_{-k}\}. \tag{2.17}$$

Indeed, if $\max\{y_{k-1},\dots,y_0\} \geq \max\{y_{-1},\dots,y_{-k}\}$, we finish. Otherwise, from $\max\{y_{k-1},\dots,y_0\} - \min\{y_{k-1},\dots,y_0\} \geq \max\{y_{-1},\dots,y_{-k}\} - \min\{y_{-1},\dots,y_{-k}\} = h(\mathbf{c}_0) \geq 0$, we deduce $\min\{y_{-1},\dots,y_{-k}\} - \min\{y_{k-1},\dots,y_0\} \geq \max\{y_{-1},\dots,y_{-k}\} - \max\{y_{k-1},\dots,y_0\} > 0$, which leads to $\min\{y_{-1},\dots,y_{-k}\} \geq \min\{y_{k-1},\dots,y_0\}$. Note that the indices $-k, -k+1,\dots,-1,0,1,\dots,i_0$ are consecutive, with $i_0 \leq k-1$. Hence,

$$y_{i_0} \geq \max\{y_{i_0-1},\dots,y_{i_0-k}\} \tag{2.18}$$

or

$$y_{i_0} \leq \min\{y_{i_0-1},\dots,y_{i_0-k}\}, \tag{2.19}$$

as claimed.

By Lemma 2.1, we have that (y_n) is eventually nondecreasing or nonincreasing. Now, note that (y_n) cannot be unbounded, since $\mathbf{c}_0 \in \mathcal{A}$ and $\mathcal{A}$ is invariant under F. Hence, the sequence

$$Y_n = (y_n, y_{n-1}, \dots, y_{n-k+1}), \quad n \in \mathbb{N}, \tag{2.20}$$

converges to a point, which must be of the form $\mathbf{c}_0 = (x^*, x^*, \dots, x^*) \in I^k$, and for which $h(\mathbf{c}_0) = 0$. In particular, we have that

$$\lim_{n\to\infty} \sum_{j=0}^{k-1} |y_{n-j} - x^*| = 0. \tag{2.21}$$

On the other hand, since $Y_n \in \mathcal{A}$, and set $\mathcal{A}$ is closed, we have that for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a vector X_m such that

$$\sum_{j=0}^{k-1} |y_{n-j} - x_{m-j}| < \varepsilon. \tag{2.22}$$

From this and (2.21), it follows that for every $\varepsilon > 0$, there is an index $m \in \mathbb{N}$ such one that

$$\sum_{j=0}^{k-1} |x_{m-j} - x^*| < 2\varepsilon. \tag{2.23}$$

Using (2.5) and (2.23), we obtain

$$x^* - 2\varepsilon < \min\{x_m, \dots, x_{m-k+1}\} < x_{m+1} < \max\{x_m, \dots, x_{m-k+1}\} < x^* + 2\varepsilon. \tag{2.24}$$

From this, (2.5), and the fact that (m_n) and (M_n) are nondecreasing and nonincreasing, respectively, we obtain that $|x_n - x^*| < 2\varepsilon$, for $n \geq m+1$. Since ε is an arbitrary positive real number, we have $\lim_{n\to\infty} x_n = x^*$, finishing the proof of the main result. □

Example 2.2. There are many natural examples for a difference equation (1.1) in which the function f satisfies conditions (1.4) and (1.5) from Theorem 1.4, such as particular maps; for instance, $f(z_1, z_2) = \alpha z_1 + \beta z_2$ with constants $\alpha \geq \beta \geq 0$.

Now, let us consider the function f and suppose that it is a linear combination of any monotonous nondecreasing function g with respect to variables z_i, $i = 1, 2, \dots, k$, that is,

$$f(z_1, \dots, z_k) = a_1 g(z_1) + \cdots + a_k g(z_k), \quad a_1 \geq a_2 \geq \cdots \geq a_k \geq 0. \tag{2.25}$$

The case when all constants a_i, $i = 1, 2, \dots, k$, are the same, obviously satisfies all conditions. So, suppose that $a_1 > a_k$. Now, assume that $z_1 \geq z_i$ for all $i = 1, 2, \dots, k$. It is well known that for any monotonous nondecreasing function g (which need not necessarily be a differentiable one), inequality $g(z_1) \geq g(z_i)$ holds for all $i = 1, 2, \dots, k$. After the multiplication of all these inequalities by $(a_{i-1} - a_i)/(a_1 - a_k)$, respectively, and the summation from $i = 2$ to $i = k$, for some $\eta \leq z_1$, we get (notice that $\sum_{i=2}^{k}((a_{i-1} - a_i)/(a_1 - a_k)) = 1$)

$$g(z_1) \geq g(\eta) = \frac{1}{a_1 - a_k}\left(\sum_{i=2}^{k} a_{i-1}g(z_i) - \sum_{i=2}^{k} a_i g(z_i)\right) \tag{2.26}$$

that is

$$\sum_{i=1}^{k} a_i g(z_i) \geq \sum_{i=2}^{k} a_{i-1}g(z_i) + a_k g(z_1), \tag{2.27}$$

which constitutes the condition (1.4) from Theorem 1.4. A similar algebraic manipulation can connect $y_1 \leq \min\{y_2,\dots,y_k\}$ (i.e., the condition (b)) from Theorem 1.4 and inequality (1.5) for such functions g. This implies that any function f from the previously described class of functions (2.25) satisfies the conditions from Theorem 1.4. Hence, this theorem is applicable to the respective difference equation (1.1).

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