Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2009, Article ID 195382, 11 pages
doi:10.1155/2009/195382

Research Article

Chaos in a Predator-Prey System with Impulsive Perturbations and Stage Structure for the Predator

Bin Zhang, Shuai Shi, and Woye Liu

Fifth Department, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China

Correspondence should be addressed to Shuai Shi, shishuai_lucky@sina.com

Received 23 May 2009; Accepted 18 September 2009

Recommended by Guang Zhang

We investigate a predator-prey model with stage structure for the predator and periodic constant impulsive perturbations. Conditions for extinction of prey and immature predator are given. By using the Floquet theory and small amplitude perturbation skills, we consider the local stability of prey, immature predator eradication periodic solution. Furthermore, by using the method of numerical simulation, the influence of the impulsive control strategy on the inherent oscillation is investigated, which shows rich complex dynamic (such as periodic doubling bifurcation, periodic halving bifurcation, nonunique attractors, chaos, and periodic windows).

Copyright © 2009 Bin Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Stage structure model has received much attention in the recent years [1–3]. In this paper, based on the model of [1], we study the dynamic complexities of a predator-prey system with stage structure for the predator and periodic constant impulsive perturbations. The model takes the form

$$
\begin{aligned}
&\dot{x}(t) = x(t)\left(r - ax(t) - \frac{by_2(t)}{1+mx(t)}\right),\\
&\dot{y}_1(t) = \frac{kbx(t)y_2(t)}{1+mx(t)} - (D+v_1)y_1(t), \quad t \neq (n+l-1)T,\ t \neq nT,\\
&\dot{y}_2(t) = Dy_1(t) - v_2y_2(t),\\
&\qquad \Delta x(t) = -\delta x(t),\\
&\qquad \Delta y_1(t) = -\delta_1 y_1(t), \quad t = (n+l-1)T,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta y_2(t) = -\delta_2 y_2(t),\\
&\Delta x(t) = 0,\\
&\Delta y_1(t) = 0, \quad t = nT,\\
&\Delta y_2(t) = q,
\end{aligned}
\tag{1.1}
$$

where $x(t)$ is the density of prey at time t, $y_1(t), y_2(t)$ are the densities of immature and mature predators at time t, respectively. All parameters are positive constants. $bx/(1+mx)$ is the response function of the mature predator. r is the intrinsic growth rate of the prey, $v_1(v_2)$ is the death rate of the immature (mature) predator, constant k denotes the coefficient in converting prey into a new immature predator, and the constant D denotes the rate of immature predator becoming mature predator. $0 < l \leq 1$, $\Delta x(t) = x(t^+) - x(t)$, $\Delta y_1(t) = y_1(t^+) - y_1(t)$, $\Delta y_2(t) = y_2(t^+) - y_2(t)$, and $0 < \delta \leq 1$ $(0 < \delta_1 \leq 1,\ 0 < \delta_2 \leq 1)$ represents the fraction of prey (immature predator, mature predator) which dies at $t = (n+l-1)T$, T is the period of the impulsive effect, and $q > 0$ is the release amount of mature predator at $t = nT$.

Recently, it is of great interests to investigate chaotic impulsive differential equations about biological control [4–6]. Gao and Chen [7] and Tang and Chen [8] investigated dynamic complexities in a single-species model with stage structure and birth pulses.

2. Preliminaries

In this section, we give some definitions and lemmas which will be useful to our results.

Let $R_+ = [0, \infty)$, $R_+^3 = \{x \in R^3 : x \geq 0\}$. Denote by $f = (f_1, f_2, f_3)$ the map defined by the right hand of the first, second, and third equations of system (1.1). Let $V : R_+ \times R_+^3 \to R_+$, and let V be said to belong to class V_0 if

(1) V is continuous at $((n-1)T, (n+l-1)T] \times R_+^3 \bigcup ((n+l-1)T, nT] \times R_+^3$ for each $x \in R_+^3$, $n \in N$, $\lim_{(t,y)\to((n+l-1)T^+,x)} V(t,y) = V((n+l-1)T^+, x)$, and $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+, x)$ exist;

(2) V is locally Lipschitzian in x.

Definition 2.1. Let $V \in V_0$, then for $(t,x) \in ((n-1)T, (n+l-1)T] \times R_+^3 \bigcup ((n+l-1)T, nT] \times R_+^3$, the upper right derivative of $V(t,x)$ with respect to the impulsive differential system (1.1) is defined as

$$
D^+V(t,x) = \lim_{h\to 0^+} \sup \frac{1}{h}\left[V\left(t+h, x+hf(t,x)\right) - V(t,x)\right]. \tag{2.1}
$$

Lemma 2.2. *Let $X(t)$ be a solution of system* (1.1) *with $X(0^+) \geq 0$, then $X(t) \geq 0$ for all $t \geq 0$. And further $X(t) > 0$, $t > 0$ if $X(0^+) > 0$.*

Lemma 2.3. *Suppose $V \in V_0$. Assume that*

$$
\begin{aligned}
&D^+V(t,X) \leq g(t, V(t,x)), \quad t \neq nT,\ t \neq (n+l-1)T,\\
&V\left(t, X(t^+)\right) \leq \psi_n(V(t,x))), \quad t = nT \text{ or } t = (n+l-1)T,
\end{aligned}
\tag{2.2}
$$

where $g : R_+ \times R_+ \longrightarrow R$ is continuous in $((n-1)T, (n+l-1)T] \times R_+ \bigcup ((n+l-1)T, nT] \times R_+$ and for $u \in R_+$, $n \in N$, then $\lim_{(t,v)\to(nT^+,u)} g(t,v) = g(nT^+, u)$ and $\lim_{(t,v)\to((n+l-1)T^+,u)} g(t,v) = g((n+l-1)T^+, u)$ exist, $\psi_n : R^+ \longrightarrow R^+$ is nondecreasing. Let $r(t) = r(t, t_0, u_0)$ be maximal solution of the scalar impulsive differential equation

$$
\begin{gathered}
\dot{u}(t) = g(t, u(t)), \quad t \neq nT,\ t \neq (n+l-1)T, \\
u(t^+) = \psi_n(u(t)), \quad t = nT \ \text{or}\ t = (n+l-1)T, \\
u(0^+) = u_0,
\end{gathered}
\tag{2.3}
$$

existing on $[0, \infty)$, then $V(0^+, x_0) \le u_0$, implies that $V(t, x(t)) \le r(t)$, $t \ge t_0$, where $x(t) = x(t, t_0, u_0)$ is any solution of system (1.1).

In the following, we give some basic properties about the following subsystem (2.4) and (2.7) of system (1.1):

$$
\begin{gathered}
\dot{y}_2(t) = -v_2 y_2(t), \quad t \neq nT,\ t \neq (n+l-1)T, \\
\Delta y_2(t) = -\delta_2 y_2(t), \quad t = (n+l-1)T, \\
\Delta y_2(t) = q, \quad t = nT, \\
y_2(0^+) = y_2^0.
\end{gathered}
\tag{2.4}
$$

Clearly

$$
y_2^*(t) = \begin{cases}
\dfrac{q \exp(-v_2(t-(n-1)T))}{1-(1-\delta_2)\exp(-v_2 T)}, & (n-1)T < t \le (n+l-1)T, \\[2ex]
\dfrac{q(1-\delta_2)\exp(-v_2(t-(n-1)T))}{1-(1-\delta_2)\exp(-v_2 T)}, & (n+l-1)T < t \le nT,
\end{cases}
\tag{2.5}
$$

$(y_2^*(0^+) = y_2^*(nT^+) = q/(1-(1-\delta_2)\exp(-v_2 T)),\ y_2^*(lT^+) = (q(1-\delta_2)\exp(-v_2 lT))/(1-(1-\delta_2)\exp(-v_2 T)))$ is a positive periodic solution of system (2.4). Since

$$
y_2(t) = \begin{cases}
(1-\delta_2)^{(n-1)}\left(y_2(0^+) - \dfrac{q}{1-(1-\delta_2)\exp(-v_2 T)}\right)\exp(-v_2 t) + y_2^*(t) \\
\qquad\qquad (n-1)T < t \le (n+l-1)T, \\[2ex]
(1-\delta_2)^{n}\left(y_2(0^+) - \dfrac{q}{1-(1-\delta_2)\exp(-v_2 T)}\right)\exp(-v_2 t) + y_2^*(t) \\
\qquad\qquad (n+l-1)T < t \le nT,
\end{cases}
\tag{2.6}
$$

is the solution of system (2.4) with initial value $y_2^0 \geq 0$, where $t \in ((n-1)T,\ (n+l-1)T) \bigcup ((n+l-1)T, nT),\ n \in N$, we get the following.

Lemma 2.4. *Let $y_2^*(t)$ be a positive periodic of system* (2.4)*, and for every solution $y_2(t)$ of system* (2.4) *with y_2^0, one has $|y_2(t) - y_2^*(t)| \to 0$, when $t \to \infty$,*

$$\begin{aligned} \dot{y}_1(t) &= -(D+v_1)y_1(t), \quad t \neq (n+l-1)T, \\ \Delta y_1(t) &= -\delta_1 y_1(t), \quad t = (n+l-1)T, \\ y_1(0^+) &= y_1^0. \end{aligned} \tag{2.7}$$

Lemma 2.5. *In subsystem* (2.7) *exists a equilibrium $y_1^*(t) = 0$.*

Therefore, we obtain a prey, immature predator eradication periodic solution $(0, 0, y_2^*(t))$ of system (1.1).

3. Extinction and Boundary

In this section, we study the stability of prey, immature predator eradication periodic solution of system (1.1), and we investigate the boundary of system (1.1).

Theorem 3.1. *Let $(x(t), y_1(t), y_2(t))$ be any solution of system* (1.1)*, then $(0, 0, y_2^*(t))$ is locally asymptotically stable if*

$$rT - \frac{bq(1-\delta_2 \exp(-v_2 lT) - (1-\delta_2)\exp(-v_2 T))}{v_2(1-(1-\delta_2)\exp(-v_2 T))} < \ln \frac{1}{1-\delta}. \tag{3.1}$$

Proof. The local stability of periodic solution $(0, 0, y_2^*(t))$ may be determined by considering the behavior of small amplitude perturbations of the solution. Define $x(t) = u(t)$, $y_1(t) = u(t)$, $y_2(t) = w(t) + y_2^*(t)$, then

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix} \quad 0 \leq t < T, \tag{3.2}$$

where $\Phi(t)$ satisfies

$$\dot{\Phi}(t) = \begin{pmatrix} r - by_2^* & 0 & 0 \\ kby_2^* & -(D+v_1) & 0 \\ 0 & D & -v_2 \end{pmatrix} \Phi(t), \tag{3.3}$$

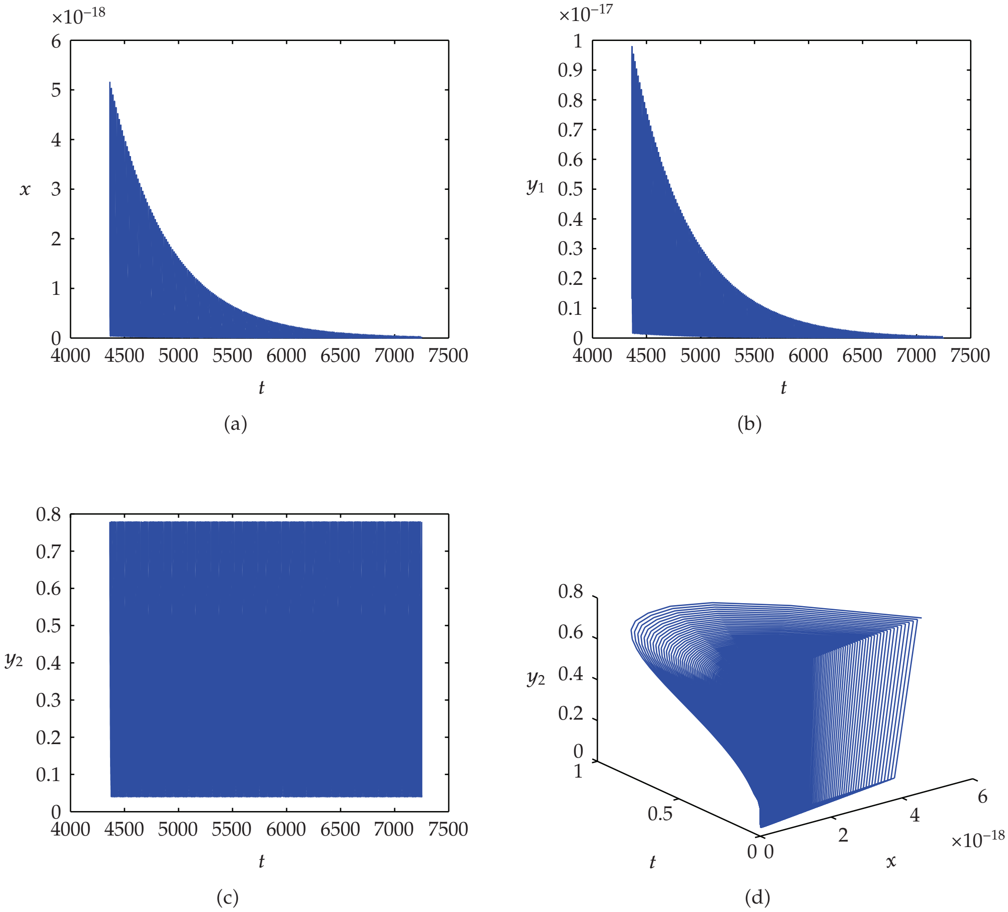


Figure 1: The dynamics of system (1.1) with $r = 1$, $a = 1$, $b = 4$, $k = 2$, $m = 0.8$, $D = 0.4$, $v_1 = v_2 = 0.2$, $l = 0.5$, $\delta = 0.1$, $\delta_1 = 0.5$, $\delta_2 = 0.1$, $q = 0.74$, $T = 14.5$ and initial values $x = 8.5$, $y_1 = 4.8$, $y_2 = 2.8$. (a) Time series of prey; (b) time series of immature predator; (c) time series of mature predator; (d) phase portrait of system (1.1).

and $\Phi(0) = 0$, the identity matrix. The impulsive perturbations of system (1.1) become

$$\begin{pmatrix} u((n+l-1)T^+) \\ v((n+l-1)T^+) \\ w((n+l-1)T^+) \end{pmatrix} = \begin{pmatrix} 1-\delta & 0 & 0 \\ 0 & 1-\delta_1 & 0 \\ 0 & 0 & 1-\delta_2 \end{pmatrix} \begin{pmatrix} u((n+l-1)T) \\ v((n+l-1)T) \\ w((n+l-1)T) \end{pmatrix},$$
$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \\ w(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \\ w(nT) \end{pmatrix}. \tag{3.4}$$

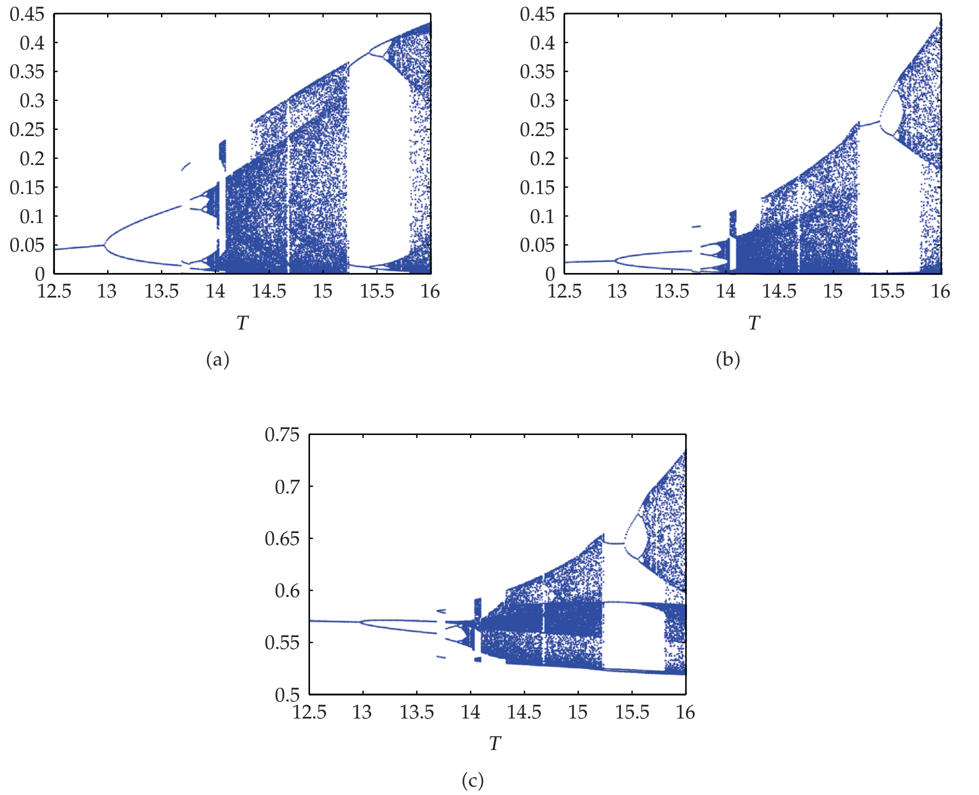


Figure 2: Bifurcation diagrams of system (1.1) showing the effect of T with $r = 1,\ a = 1,\ b = 4,\ k = 2,\ m = 0.8,\ D = 0.4,\ v_1 = v_2 = 0.2,\ l = 0.5,\ \delta = 0.1,\ \delta_1 = 0.5,\ \delta_2 = 0.1,\ q = 0.5,\ 12.5 \le T \le 16$ and initial values $x = 8.5,\ y_1 = 4.8,\ y_2 = 2.8$.

The stability of the periodic solution $(0, 0, y_2^*(t))$ is determined by the eigenvalues of

$$M = \begin{pmatrix} 1-\delta & 0 & 0 \\ 0 & 1-\delta_1 & 0 \\ 0 & 0 & 1-\delta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi(T), \tag{3.5}$$

where $\mu_1 = (1-\delta_1)\exp(-(D+v_1)T) < 1,\ \mu_2 = (1-\delta_2)\exp(-v_2 T) < 1,\ \mu_3 = (1-\delta)\exp(\int_0^T (r - by_2^*(t))dt)$ according to Floquet theory, the solution $(0, 0, y_2^*(t))$ is locally stable if $|\mu_3| < 1$, this is to say, $rT - (bq(1-\delta_2\exp(-v_2 lT) - (1-\delta_2)\exp(-v_2 T)))/v_2(1-(1-\delta_2)\exp(-v_2 T)) < \ln 1/(1-\delta)$. □

Theorem 3.2. *There exists a positive constant M such that $x(t) \le M,\ y_1(t) \le M,\ y_2(t) \le M$ for each solution $(x(t), y_1(t), y_2(t))$ of system* (1.1) *with all t large enough.*

Proof. Let $X(t) = (x(t), y_1(t), y_2(t))$ be any solution of system (1.1). Define function $V(t) = V(t, X(t)) = kx(t) + y_1(t) + y_2(t)$, when $t \ne (n+l-1)T,\ t \ne nT$, we select $0 < L < \min\{v_1, v_2\}$,

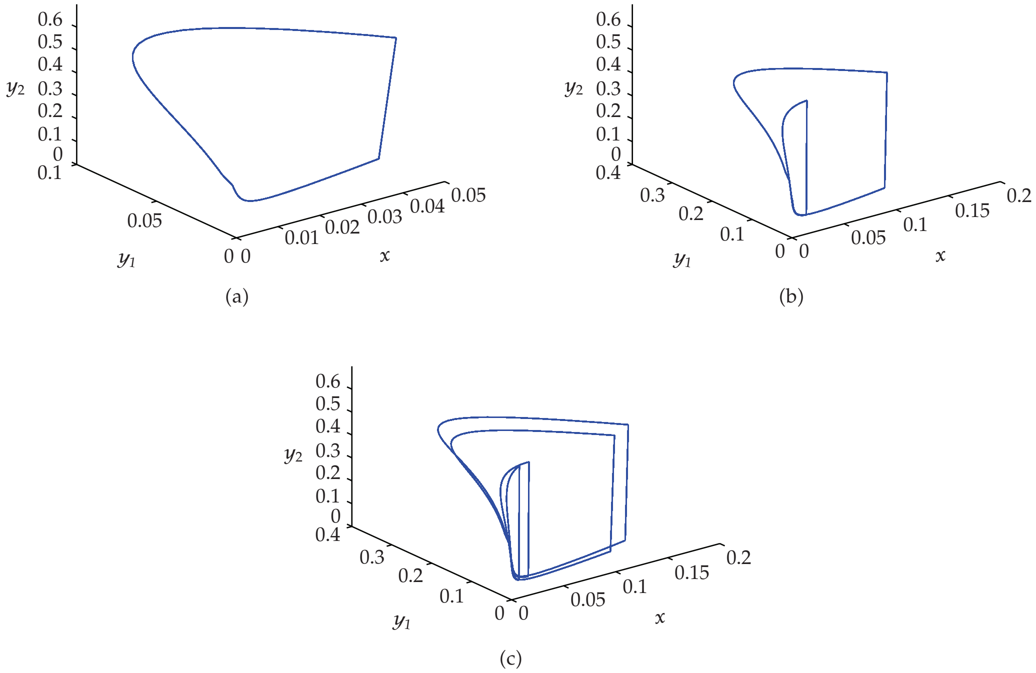


Figure 3: Period doubling cascade: (a) phase portrait of T-periodic solution for $T = 12.8$; (b) phase portrait of $2T$-periodic solution for $T = 13.5$; (c) phase portrait of $4T$ for $T = 13.8$.

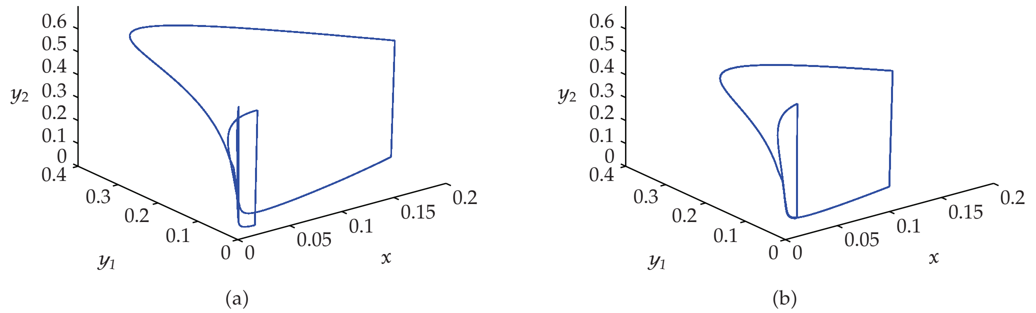
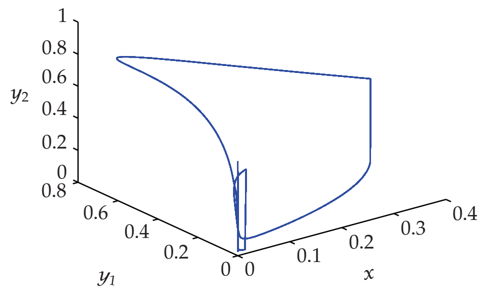


Figure 4: Nonunique attractor at $T = 13.7$: (a) Phase portrait of $3T$-periodic solution with initial values $x = 8.5$, $y_1 = 4.8$, $y_2 = 2.8$. (b) Phase portrait of $2T$-periodic solution with initial values $x = 18.5$, $y_1 = 14.8$, $y_2 = 12.8$.

then

$$D^+V(t) + LV(t) = k(r+L)x - kax^2 + (L - v_1)y_1 + (L - v_2)y_2 \leq k(r+L)x - kax^2 \leq M_0, \tag{3.6}$$

where $M_0 = k(r+L)^2/4a$. When $t = (n+l-1)T$, $V((n+l-1)T^+) \leq V((n+l-1)T)$, $t = nT$, $V(nT^+) \leq V(nT) + q$.



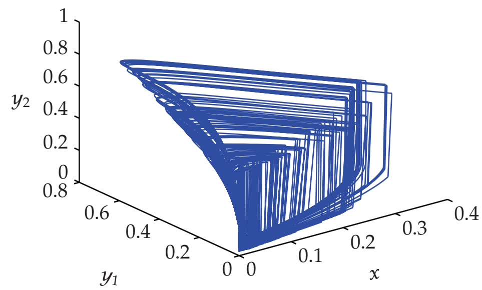


Figure 5: Non-unique attractor at $T = 15.22$: (a) Phase portrait of $3T$-periodic solution with initial values $x = 8.5$, $y_1 = 4.8$, $y_2 = 2.8$. (b) Phase portrait of chaotic periodic solution with initial values $x = 18.5$, $y_1 = 14.8$, $y_2 = 12.8$.

According to Lemma 2.3, for $t \geq 0$ we have

$$V(t) \leq V(0)e^{-Lt} + \frac{M_0}{l}\left(1 - e^{-Lt}\right) + q\frac{e^{-L(t-T)}}{1 - e^{LT}} + q\frac{e^{LT}}{e^{LT} - 1} \longrightarrow \frac{M_0}{l} + q\frac{e^{LT}}{e^{LT} - 1} \quad (t \longrightarrow \infty). \quad (3.7)$$

Therefore $V(t)$ is ultimately bounded, and we obtain that each positive solution of system (1.1) is uniformly ultimately bounded. □

4. Numerical Analysis

In this section, we study the influence of impulsive period T and impulsive perturbation q on complexities of the system (1.1).

Let $r = 1$, $a = 1$, $b = 4$, $k = 2$, $m = 0.8$, $D = 0.4$, $v_1 = v_2 = 0.2$, $l = 0.5$. Since the corresponding continuous system (1.1) cannot be solved explicitly and system (1.1) cannot be rewritten as equivalent difference equations, it is difficult to study them analytically. So we have to study system (1.1) numerically integrated by stroboscopically sampling one of the variables over a range of T, q values. The bifurcation diagram provides a summary of essential dynamical behavior of system.

From Theorem 3.1, we know that the prey, immature predator eradication periodic solution $(0, 0, y_2^*(t))$ is locally stable if $q > q_{\max} = 0.7397$, which is shown in Figure 1. when $q = 0.74$, we find that the variable $y_2(t)$ oscillates in a stable cycle. In contrast, the variables $x(t), y_1(t)$ rapidly decrease to zero.

The bifurcation diagram (Figure 2) shows that with T increasing from 12.5 to 16, system (1.1) has rich dynamics including periodic doubling bifurcation, chaos, periodic windows, and nonunique attractors. When $T < T_1 \approx 12.97$, system (1.1) has a stable T-periodic solution. When $T > T_1$, it is unstable and there is a cascade of periodic doubling bifurcations leading to chaos (Figure 3) (chaotic area with periodic windows). Figure 4 shows the phenomena of non-unique attractors [9]: different attractors can coexist, which one of the attractors is reached depends on the initial values.

The bifurcation diagram (Figure 6) shows that with q increasing from 0.01 to 0.8, system (1.1) has rich dynamics including periodic doubling bifurcation, chaos, periodic windows, and periodic halving bifurcation. When $q < q_1 \approx 0.08$, system (1.1) has a stable

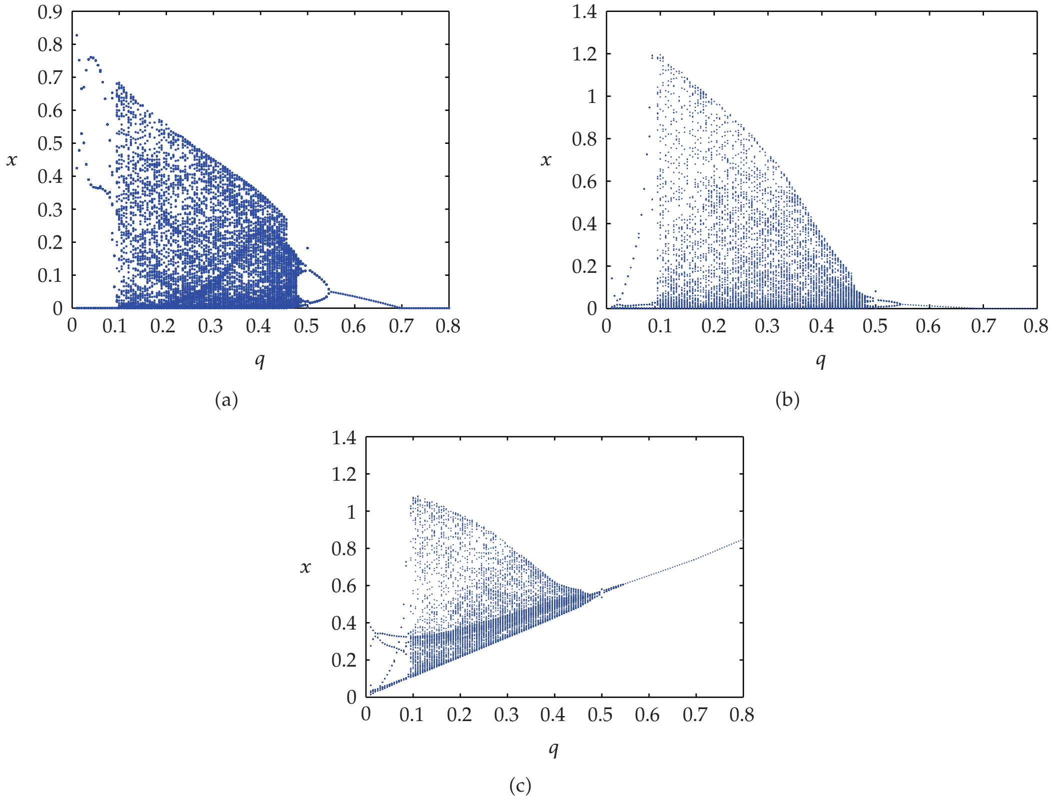


Figure 6: Bifurcation diagrams of system (1.1) showing the effect of q with $r = 1,\ a = 1,\ b = 4,\ k = 2,\ m = 0.8,\ D = 0.4,\ v_1 = v_2 = 0.2,\ l = 0.5,\ \delta = 0.1,\ \delta_1 = 0.5,\ \delta_2 = 0.1,\ T = 13.7,\ 0.01 \le q \le 0.8$ and initial values $x = 8.5,\ y_1 = 4.8,\ y_2 = 2.8$.

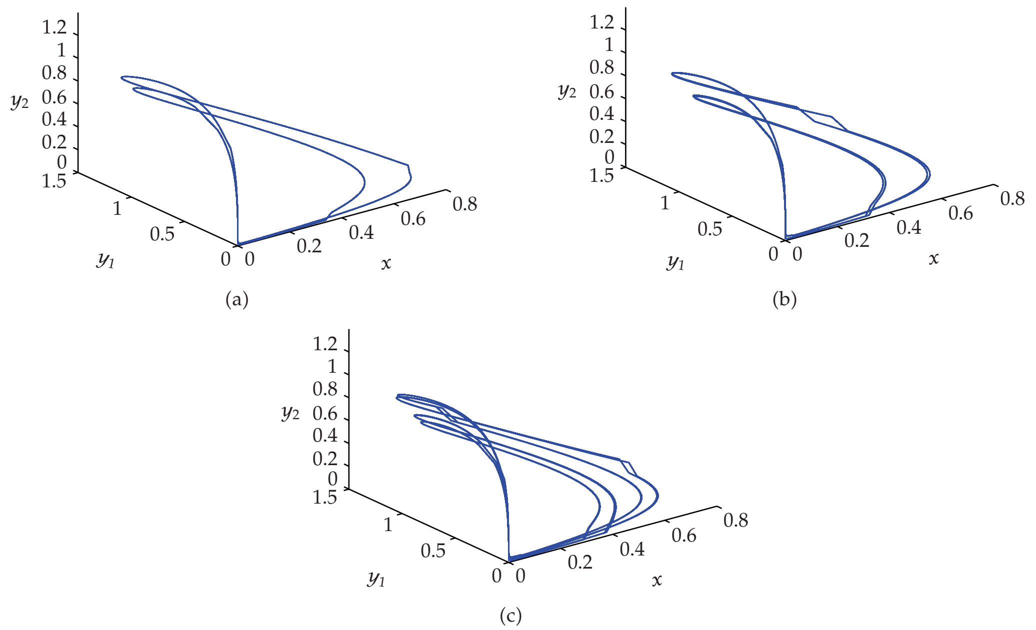


Figure 7: Period doubling cascade: (a) phase portrait of $2T$-periodic solution for $q = 0.05$; (b) phase portrait of $4T$-periodic solution for $q = 0.08$; (c) phase portrait of $4T$-periodic solution for $q = 0.085$.

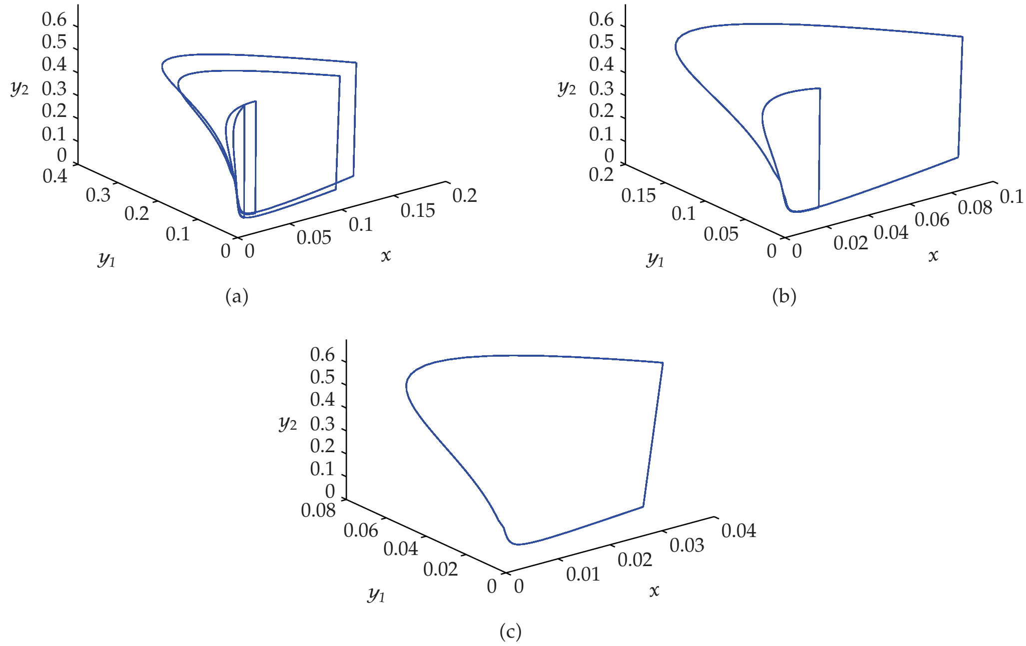


Figure 8: Period halving cascade: (a) phase portrait of T-periodic solution for $q = 0.49$; (b) phase portrait of $2T$-periodic solution for $q = 0.52$; (c) phase portrait of T-periodic solution for $q = 0.6$.

$2T$-periodic solution. When $q > q_1$, it is unstable and there is a cascade of periodic doubling bifurcations leading to chaos with periodic windows (Figure 8) which is followed by a cascade of periodic halving bifurcation from chaos to periodic solution.

5. Conclusion

In this paper, we investigated the dynamic complexities of a predator-prey system with impulsive perturbations and stage structure for the predator. Conditions for the local asymptotical stability of prey, immature predator eradication periodic solution $(0, 0, y_2^*(t))$ were given by using the Floquet theory and small amplitude perturbation skills. Using the method of numerical simulation, the influence of impulsive control strategy on inherent oscillation showed that there exists complexity for system (1.1) including periodic doubling bifurcation, periodic halving bifurcation, nonunique attractors, chaos, and periodic windows. All these results showed that dynamical behavior of system (1.1) becomes more complex under periodically impulsive control strategy.

References

[1] W. Wang and L. Chen, "A predator-prey system with stage-structure for predator," *Computers & Mathematics with Applications*, vol. 33, no. 8, pp. 83–91, 1997.

[2] Y. N. Xiao and L. Chen, "Global stability of a predator-prey system with stage structure for the predator," *Acta Mathematica Sinica*, vol. 20, no. 1, pp. 63–70, 2004.

[3] K. G. Magnusson, "Destabilizing effect of cannibalism on a structured predator-prey system," *Mathematical Biosciences*, vol. 155, no. 1, pp. 61–75, 1999.

[4] S. Zhang and L. Chen, “A Holling II functional response food chain model with impulsive perturbations,” *Chaos, Solitons & Fractals*, vol. 24, no. 5, pp. 1269–1278, 2005.
[5] S. Zhang, D. Tan, and L. Chen, “Dynamic complexities of a food chain model with impulsive perturbations and Beddington-DeAngelis functional response,” *Chaos, Solitons & Fractals*, vol. 27, no. 3, pp. 768–777, 2006.
[6] X. Liu and L. Chen, “Complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator,” *Chaos, Solitons & Fractals*, vol. 16, no. 2, pp. 311–320, 2003.
[7] S. Gao and L. Chen, “Dynamic complexities in a single-species discrete population model with stage structure and birth pulses,” *Chaos, Solitons & Fractals*, vol. 23, no. 2, pp. 519–527, 2005.
[8] S. Tang and L. Chen, “Multiple attractor stage-structure population model with birth pulses,” *Bulletin of Mathematical Biology*, vol. 65, no. 3, pp. 479–495, 2003.
[9] S. Tang and L. Chen, “Chaos in functional response host-parasitoid ecosystem models,” *Chaos, Solitons & Fractals*, vol. 13, no. 4, pp. 875–884, 2002.