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Research Article

Inverse Eigenvalue Problem of Unitary Hessenberg Matrices

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Let $H \in \mathbb{C}^{n\times n}$ be an $n \times n$ unitary upper Hessenberg matrix whose subdiagonal elements are all positive, let H_k be the kth leading principal submatrix of H, and let $\widetilde{H}_k$ be a modified submatrix of H_k. It is shown that when the minimal and maximal eigenvalues of $\widetilde{H}_k$ $(k = 1, 2, \ldots, n)$ are known, H can be constructed uniquely and efficiently. Theoretic analysis, numerical algorithm, and a small example are given.

1. Introduction

Direct matrix eigenvalue problems are concerned with deriving and analyzing the spectral information and, hence, predicting the dynamical behavior of a system from a priori known physical parameters such as mass, length, elasticity, inductance, and capacitance. Inverse eigenvalue problems (IEPs), in contrast, are concerned with the determination, identification, or construction of the parameters of a system according to its observed or expected behavior.

The inverse eigenvalue problems arise in a remarkable variety of applications, such as mathematics physics, control theory, vibration project, structure design, system parameter identification, and the revise of mathematics models [1–8]. Recent years, inverse eigenvalue problem of matrices has become an active topic of computational mathematics for needs of project and technology, and it has resolved a great deal of concrete problem. Especially, the inverse eigenvalue problems have many applications in engineering design, for example, they arise in aviation, civil structure, nucleus engineering, bridge design, shipping construction, and so on. Pole assignment problem have been of major interest in system identification and control theory, we can use optimization techniques to get a solution which is least sensitive to perturbation of problem data. Byrnes [9], Kautsky et al. [10], and

Chu and Li [11] gave an excellent recount of activities in this area. Joseph [7] presented a method for the design of a structure with specified low-order natural frequencies, and the method can further be used to generate initial feasible designs for optimum design problems with frequency constraints. By measuring the changes in the natural frequencies, the IEP idea can be employed to detect the size and location of a blockage in a duct or a crack in a beam, see [12–15] for additional references. Starek and Inman [16] discussed the applications of IEPs to model updating problems and fault detection problems for machine and structure diagnostics. Applications to other types engineering problems can be found in the books [4, 17] and articles [18–23].

Throughout this paper we use I_j to denote the $j \times j$ identity matrix, e_j to denote the jth column of the identity matrix, $\Lambda(H)$ to denote the spectrums of a square matrix H, $\overline{x}$ to denote the complex conjugate of x, and $\mathscr{H}_n$ to denote the set of unitary upper Hessenberg matrices of order n with positive subdiagonal elements.

It is known [24] that any $H \in \mathscr{H}_n$ can be written uniquely as the products

$$H = G_1(\gamma_1) \cdots G_{n-1}(\gamma_{n-1}) \widetilde{G}_n(\gamma_n), \tag{1.1}$$

where

$$G_k(\gamma_k) = \begin{pmatrix} I_{k-1} & & & \\ & -\gamma_k & \sigma_k & \\ & \sigma_k & \overline{\gamma}_k & \\ & & & I_{n-k-1} \end{pmatrix}, \quad k = 1, 2, \dots, n-1, \tag{1.2}$$

$$\widetilde{G}_n(\gamma_n) = \operatorname{diag}(I_{n-1}, -\gamma_n). \tag{1.3}$$

In (1.1) and (1.2), the parameters $\gamma_k \in \mathbb{C}$ $(k = 1, 2, \dots, n)$ are called reflection coefficients or Schur parameters in signal processing, $\sigma_k \in \mathbb{R}$ $(k = 1, 2, \dots, n-1)$ are said to be complementary parameters and satisfy $|\gamma_k|^2 + \sigma_k^2 = 1$, $\sigma_k > 0$, $k = 1, \dots, n-1$, and $|\gamma_n| = 1$. We refer to (1.1) as Schur parametric form of H [25], it plays a fundamental role in the development of efficient algorithms for solving eigenproblems for unitary Hessenberg matrices. However, (1.2) is called the complex Givens matrices. H in (1.1) is of the explicit form

$$H = \begin{pmatrix} -\gamma_1 & -\sigma_1\gamma_2 & -\sigma_1\sigma_2\gamma_3 & \cdots & -\sigma_1 \cdots \sigma_{n-1}\gamma_n \\ \sigma_1 & -\overline{\gamma}_1\gamma_2 & -\overline{\gamma}_1\sigma_2\gamma_3 & \cdots & -\overline{\gamma}_1\sigma_2 \cdots \sigma_{n-1}\gamma_n \\ & \sigma_2 & -\overline{\gamma}_2\gamma_3 & \cdots & -\overline{\gamma}_2\sigma_3 \cdots \sigma_{n-1}\gamma_n \\ & & \ddots & \ddots & \vdots \\ & & & \sigma_{n-1} & -\overline{\gamma}_{n-1}\gamma_n \end{pmatrix}, \tag{1.4}$$

and is uniquely determined by $\gamma_1, \gamma_2, \dots, \gamma_n$. We denote this $n \times n$ unitary Hessenberg matrix by $H(\gamma_1, \gamma_2, \dots, \gamma_n)$, each $H \in \mathscr{H}_n$ is therefore determined by the $2n-1$ real parameters. Let H_k be the kth leading principal submatrix of H. The matrix H_k is not unitary for $k < n$ and its eigenvalues are inside the unit circle. However, H_k will become unitary if γ_k is replaced by ρ_k which is any number on the unit circle [24]. We introduce the following sequence of modified unitary submatrices:

$$\widetilde{H}_k = \begin{pmatrix} -\gamma_1 & -\sigma_1\gamma_2 & \cdots & -\sigma_1\cdots\sigma_{k-1}\rho_k \\ \sigma_1 & -\overline{\gamma}_1\gamma_2 & \cdots & -\overline{\gamma}_1\sigma_2\cdots\sigma_{k-1}\rho_k \\ & \ddots & \ddots & \vdots \\ & & \sigma_{k-1} & -\overline{\gamma}_{k-1}\rho_k \end{pmatrix}, \quad k = 1, 2, \ldots, n. \tag{1.5}$$

Because all ρ_k are of modulus one, the modified submatrices $\widetilde{H}_k$ are unitary and its eigenvalues lie on the unit circle, $\widetilde{H}_k = H(\gamma_1, \ldots, \gamma_{k-1}, \rho_k)$. Assume that -1 is not an eigenvalue of H, then $\lambda_j^{(k)} \in \Lambda(\widetilde{H}_k)$ can be described as

$$\lambda_j^{(k)} = \exp\left(i\theta_j^{(k)}\right), \quad j = 1, 2, \ldots, k. \tag{1.6}$$

If we number the roots of $\widetilde{H}_k$ starting from $-\pi$ moving counterclockwise along the unit circle, that is,

$$-\pi < \theta_1^{(k)} \leq \theta_2^{(k)} \leq \cdots \leq \theta_k^{(k)} \leq \pi, \tag{1.7}$$

then we also call $\lambda_1^{(k)} = \exp(i\theta_1^{(k)})$, $\lambda_k^{(k)} = \exp(i\theta_k^{(k)})$ are, respectively, the minimal and maximal eigenvalues of $\widetilde{H}_k$.

Hessenberg matrices arise naturally in several signal processing applications including the frequency estimation procedure and harmonic retrieval problem for radar or sonar navigation [26, 27]. Two kinds of inverse eigenvalue problems for unitary Hessenberg matrices have been considered up to now. Ammar et al. [28] discussed $H = H(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathscr{H}_n$ is uniquely determined by its eigenvalues and the eigenvalues of $\widehat{H}$, where $\widehat{H} = H(\alpha\gamma_1, \alpha\gamma_2, \ldots, \alpha\gamma_n) = (I - (1-\alpha)e_1e_1^T)H(\gamma_1, \gamma_2, \ldots, \gamma_n)$, that is, $\widehat{H}$ a multiplicative rank-one perturbation of H, and the methods are described in [28, 29]. Ammar and He in [24] considered that $H \in \mathscr{H}_n$ can also be determined by its eigenvalues and the eigenvalues of a modified $(n-1) \times (n-1)$ leading principal submatrix of H.

In this paper, we consider the following inverse eigenvalue problem.

Problem 1. For $2n-1$ given real numbers $\theta_1^{(k)}, \theta_k^{(k)} \in (-\pi, \pi]$ $(k = 1, 2, \ldots, n)$, find unitary Hessenberg matrices $H \in \mathscr{H}_n$, such that $\lambda_1^{(k)} = \exp(i\theta_1^{(k)})$, $\lambda_k^{(k)} = \exp(i\theta_k^{(k)})$ are, respectively, the minimal and the maximal eigenvalues of $\widetilde{H}_k$ for all $k = 1, 2, \ldots, n$.

This paper is organized as follows. In Section 2, we discussed the properties of unitary Hessenberg matrix. Then the necessary and sufficient conditions for solvability of Problem 1 are derived in Section 3. Section 4 gives the algorithm and numerical example for the problem.

2. The Properties of Unitary Hessenberg Matrix

We denote the characteristic polynomials of $\widetilde{H}_k$ by φ_k, that is, $\varphi_k(\lambda) = \det(\lambda I_k - \widetilde{H}_k)$. We can appropriately choose ρ_k such that $\varphi_k(\lambda)$ satisfy the three-term recurrence relations [30, 31], the following lemma give a special method to define ρ_k.

Lemma 2.1 (see [32]). *Let $H = H(\gamma_1, \ldots, \gamma_n) \in \mathscr{H}_n$, assume -1 is not an eigenvalue of H, define*

$$\begin{aligned} \rho_n &= \gamma_n, \\ \rho_k &= \frac{\gamma_k - \rho_{k+1}}{1 - \overline{\gamma}_k \rho_{k+1}}, \quad k = n-1, n-2, \ldots, 1. \end{aligned} \tag{2.1}$$

Let $\widetilde{H}_k \in \mathscr{H}_k$ $(k = 1, \ldots, n)$ be the modified unitary submatrices defined by (1.5)*. If one number the eigenvalues of $\widetilde{H}_k$ starting from -1 moving counterclockwise along the unit circle, then the eigenvalues of $\widetilde{H}_k$ interlace those of $\widetilde{H}_{k+1}$ in the following sense: the jth eigenvalue of $\widetilde{H}_k$ lies on the arc between the jth and the $j+1$st eigenvalue of $\widetilde{H}_{k+1}$.*

If ρ_k are defined by (2.1), we get the following lemma.

Lemma 2.2 (see [32]). *The characteristic polynomials $\varphi_k(\lambda) = \det(\lambda I_k - \widetilde{H}_k)$ of $\widetilde{H}_k$ defined by* (1.5) *satisfy the following three-term recurrence relations:*

$$\begin{aligned} \varphi_0(\lambda) &= 1, \\ \varphi_1(\lambda) &= \lambda + \rho_1, \\ \varphi_k(\lambda) &= \left(\lambda + \rho_k \overline{\rho}_{k-1}\right)\varphi_{k-1}(\lambda) - \alpha_{k-1}\lambda\varphi_{k-2}(\lambda), \quad k = 2, 3, \ldots, n, \end{aligned} \tag{2.2}$$

where

$$\alpha_k = \overline{\gamma}_{k-1}\left(\rho_k - \gamma_k\right) + \rho_{k+1}\left(\overline{\rho}_k - \overline{\gamma}_k\right), \quad \gamma_0 = 1. \tag{2.3}$$

Lemma 2.3. *If ρ_k defined by* (2.1)*, α_k defined by* (2.3)*, then*

$$\begin{aligned} \gamma_0 &= 1, \\ \gamma_k &= \frac{\alpha_k - \overline{\gamma}_{k-1}\rho_k + 1}{\overline{\rho}_k - \overline{\gamma}_{k-1}}, \quad \text{for } k = 1, 2, \ldots, n-1, \\ \gamma_n &= \rho_n. \end{aligned} \tag{2.4}$$

Proof. By (2.1), we get

$$\rho_k\left(1 - \overline{\gamma}_k \rho_{k+1}\right) = \gamma_k - \rho_{k+1}, \tag{2.5}$$

then

$$\rho_{k+1}\left(\overline{\rho}_k - \overline{\gamma}_k\right) = \gamma_k \overline{\rho}_k - 1. \tag{2.6}$$

Substituting the above formula into (2.3), we obtain

$$\alpha_k = \overline{\gamma}_{k-1}(\rho_k - \gamma_k) + \gamma_k \overline{\rho}_k - 1. \tag{2.7}$$

Because $\rho_k \neq \gamma_{k-1}$, we have

$$\gamma_k = \frac{\alpha_k - \overline{\gamma}_{k-1}\rho_k + 1}{\overline{\rho}_k - \overline{\gamma}_{k-1}}, \quad k = 1, 2, \ldots, n-1. \tag{2.8}$$

□

Lemma 2.4. *Let $x \in \mathbb{C}$ with $|x| = 1$ and $\varphi_k(\lambda)$ be the characteristic polynomials of $\widetilde{H}_k$, then*

$$\varphi_k(x)\overline{x}^k = \rho_k \overline{\varphi}_k(x), \quad k = 1, 2, \ldots, n. \tag{2.9}$$

Proof. It is easy to verify that

$$\begin{aligned}
\varphi_k(x)\overline{x}^k &= \left(x - \lambda_1^{(k)}\right)\left(x - \lambda_2^{(k)}\right)\cdots\left(x - \lambda_k^{(k)}\right)\overline{x}^k \\
&= \left(1 - \overline{x}\lambda_1^{(k)}\right)\left(1 - \overline{x}\lambda_1^{(k)}\right)\cdots\left(1 - \overline{x}\lambda_k^{(k)}\right) \\
&= \lambda_1^{(k)}\lambda_2^{(k)}\cdots\lambda_k^{(k)}\left(\overline{\lambda}_1^{(k)} - \overline{x}\right)\left(\overline{\lambda}_2^{(k)} - \overline{x}\right)\cdots\left(\overline{\lambda}_k^{(k)} - \overline{x}\right) \\
&= (-1)^k \det\left(\widetilde{H}_k\right)\overline{\varphi}_k(x) \\
&= (-1)^k(-1)^k \rho_k \overline{\varphi}_k(x) \\
&= \rho_k \overline{\varphi}_k(x).
\end{aligned} \tag{2.10}$$

□

3. The Solution of Problem 1

We now consider the solvability conditions of Problem 1 and give the following theorem.

Theorem 3.1. *For $2n-1$ given real number $\theta_1^{(k)}, \theta_k^{(k)} \in (-\pi, \pi]$ $(k = 1, 2, \ldots, n)$, there is a unique $H(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathscr{H}_n$ such that $\lambda_1^{(k)} = \exp(\theta_1^{(k)})$, $\lambda_k^{(k)} = \exp(\lambda_k^{(k)})$ are, respectively, the minimal and the maximal eigenvalues of $\widetilde{H}_k$ $(k = 1, 2, \ldots, n)$, if and only if*

$$-\pi < \theta_1^{(n)} < \theta_1^{(n-1)} < \cdots < \theta_1^{(2)} < \theta_1^{(1)} < \theta_2^{(2)} < \cdots < \theta_{n-1}^{(n-1)} < \theta_n^{(n)} \leq \pi. \tag{3.1}$$

Proof. Sufficiency. Notice that

$$-\pi < \theta_1^{(n)} < \theta_1^{(n-1)} < \cdots < \theta_1^{(2)} < \theta_1^{(1)} < \theta_2^{(2)} < \cdots < \theta_{n-1}^{(n-1)} < \theta_n^{(n)} \leq \pi. \tag{3.2}$$

By Lemma 2.1 we have that, if $\lambda_1^{(k)}$, $\lambda_k^{(k)}$ are the eigenvalues of $\widetilde{H}_k$, they must be the minimal and the maximal eigenvalues of $\widetilde{H}_k$, respectively. So Problem 1 having a solution is equivalent to that the following equations:

$$\begin{aligned} \varphi_k\left(\lambda_1^{(k)}\right) &= 0, \\ \varphi_k\left(\lambda_k^{(k)}\right) &= 0, \end{aligned} \tag{3.3}$$

having solutions α_{k-1}, ρ_k satisfying $|\rho_k| = 1$ for all $k = 1, 2, \ldots, n$.

For $j = 1$, we get $\rho_1 = \lambda_1^{(1)} = \exp(i\theta_1^{(1)})$, so $|\rho_1| = 1$.

For $2 \le j \le n$, by Lemma 2.1, from (2.2) and (3.3), we have

$$\begin{aligned} \left(\lambda_1^{(k)} + \rho_k\overline{\rho}_{k-1}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right) - \alpha_{k-1}\lambda_1^{(k)}\varphi_{k-2}\left(\lambda_1^{(k)}\right) &= 0, \\ \left(\lambda_k^{(k)} + \rho_k\overline{\rho}_{k-1}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) - \alpha_{k-1}\lambda_k^{(k)}\varphi_{k-2}\left(\lambda_k^{(k)}\right) &= 0. \end{aligned} \tag{3.4}$$

Then

$$\begin{aligned} \alpha_{k-1}\lambda_1^{(k)}\varphi_{k-2}\left(\lambda_1^{(k)}\right) - \rho_k\overline{\rho}_{k-1}\varphi_{k-1}\left(\lambda_1^{(k)}\right) &= \lambda_1^{(k)}\varphi_{k-1}\left(\lambda_1^{(k)}\right), \\ \alpha_{k-1}\lambda_k^{(k)}\varphi_{k-2}\left(\lambda_k^{(k)}\right) - \rho_k\overline{\rho}_{k-1}\varphi_{k-1}\left(\lambda_k^{(k)}\right) &= \lambda_k^{(k)}\varphi_{k-1}\left(\lambda_k^{(k)}\right). \end{aligned} \tag{3.5}$$

Let $m_k \equiv \lambda_1^{(k)}\varphi_{k-2}(\lambda_1^{(k)})\varphi_{k-1}(\lambda_k^{(k)}) - \lambda_k^{(k)}\varphi_{k-2}(\lambda_k^{(k)})\varphi_{k-1}(\lambda_1^{(k)})$, we now show that $m_k \neq 0$ by contradiction.

Assume that $m_k = 0$. Multiplying the first and second equation of (3.5) by $\varphi_{k-1}(\lambda_k^{(k)})$, $\varphi_{k-1}(\lambda_1^{(k)})$, respectively, we get

$$\left(\lambda_1^{(k)} - \lambda_k^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) = 0, \tag{3.6}$$

so we obtain $\lambda_1^{(k)} = \lambda_k^{(k)}$ by $\varphi_{k-1}(\lambda_1^{(k)}) \neq 0$ and $\varphi_{k-1}(\lambda_k^{(k)}) \neq 0$. This is a contradiction with $\lambda_1^{(k)} \neq \lambda_k^{(k)}$, therefore, $m_k \neq 0$. By $|\rho_{k-1}| = 1$, we get $-\overline{\rho}_{k-1}m_j \neq 0$. Then (3.5) have the unique solution

$$\alpha_{k-1} = \frac{\overline{\rho}_{k-1}\left(\lambda_k^{(k)} - \lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right)}{-\overline{\rho}_{k-1}m_k}, \tag{3.7}$$

$$\rho_k = \frac{\lambda_1^{(k)}\lambda_k^{(k)}\left(\varphi_{k-2}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) - \varphi_{k-2}\left(\lambda_k^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\right)}{-\overline{\rho}_{k-1}m_k}. \tag{3.8}$$

We show $|\rho_k| = 1$ by induction. By $\rho_1 = \exp(i\theta_1^{(1)})$, so $|\rho_1| = 1$. Assume that $|\rho_j| = 1$, for $j = 1, 2, \ldots, k-1$.

By (3.8), $\lambda_1^{(k)} \neq 0$, and $\lambda_k^{(k)} \neq 0$, we have

$$\begin{aligned}
\rho_k &= \frac{\lambda_1^{(k)}\lambda_k^{(k)}\left(\varphi_{k-2}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) - \varphi_{k-2}\left(\lambda_k^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\right)\left(\overline{\lambda}_1^{(k)}\right)^{k-1}\left(\overline{\lambda}_k^{(k)}\right)^{k-1}}{-\overline{\rho}_{k-1}\left(\lambda_1^{(k)}\varphi_{k-2}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) - \lambda_k^{(k)}\varphi_{k-2}\left(\lambda_k^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\right)\left(\overline{\lambda}_1^{(k)}\right)^{k-1}\left(\overline{\lambda}_k^{(k)}\right)^{k-1}} \\
&= \frac{\left(\overline{\lambda}_1^{(k)}\right)^{k-2}\left(\overline{\lambda}_k^{(k)}\right)^{k-2}\left(\varphi_{k-2}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) - \varphi_{k-2}\left(\lambda_k^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\right)}{-\overline{\rho}_{k-1}\rho_{k-2}\rho_{k-1}\left(\overline{\varphi}_{k-2}\left(\lambda_1^{(k)}\right)\overline{\varphi}_{k-1}\left(\lambda_k^{(k)}\right) - \overline{\varphi}_{k-2}\left(\lambda_k^{(k)}\right)\overline{\varphi}_{k-1}\left(\lambda_1^{(k)}\right)\right)} \qquad (3.9)\\
&= \frac{\left(\overline{\lambda}_1^{(k)}\right)^{k-2}\left(\overline{\lambda}_k^{(k)}\right)^{k-2}\left(\varphi_{k-2}\left(\lambda_1^{(k)}\right)\varphi_{k-1}\left(\lambda_k^{(k)}\right) - \varphi_{k-2}\left(\lambda_k^{(k)}\right)\varphi_{k-1}\left(\lambda_1^{(k)}\right)\right)}{\left|\rho_{k-1}\right|^2\rho_{k-2}\left(\overline{\varphi}_{k-2}\left(\lambda_1^{(k)}\right)\overline{\varphi}_{k-1}\left(\lambda_k^{(k)}\right) - \overline{\varphi}_{k-2}\left(\lambda_k^{(k)}\right)\overline{\varphi}_{k-1}\left(\lambda_1^{(k)}\right)\right)},
\end{aligned}$$

so $|\rho_k| = 1$.

Now we have α_k $(k = 1, 2, \ldots, n-1)$ and ρ_k $(k = 1, 2, \ldots, n)$, by Lemma 2.3, we can get γ_k, for $k = 1, 2, \ldots, n$. Then we obtain the $n\times n$ unitary Hessenberg matrix $H = H(\gamma_1, \gamma_2, \ldots, \gamma_n)$.

Necessity. Suppose that Problem 1 has a unique solution, that is, $\lambda_1^{(k)} = \exp(\theta_1^{(k)})$, $\lambda_k^{(k)} = \exp(\lambda_k^{(k)})$ are, respectively, the minimal and the maximal eigenvalues of $\widetilde{H}_k$ $(k = 1, 2, \ldots, n)$, using Lemma 2.3, we get

$$-\pi < \theta_1^{(n)} < \theta_1^{(n-1)} < \cdots < \theta_1^{(2)} < \theta_1^{(1)} < \theta_2^{(2)} < \cdots < \theta_{n-1}^{(n-1)} < \theta_n^{(n)} \leq \pi. \qquad (3.10)$$

□

Remark 3.2. Assume that η_0 is not the eigenvalue of H, we define

$$\begin{aligned}
\rho_n &= \gamma_n, \\
\rho_k &= \frac{\gamma_k + \overline{\eta}_0\rho_{k+1}}{1 + \overline{\eta}_0\overline{\gamma}_k\rho_{k+1}}, \quad k = n-1, n-2, \ldots, 1.
\end{aligned} \qquad (3.11)$$

Then Lemmas 2.1 and 2.2 still hold true.

4. Algorithm and Example

Based on the above analysis, it is natural that we should propose the following algorithm for solving Problem 1.

Algorithm 4.1. Input $-\pi < \theta_1^{(n)} < \theta_1^{(n-1)} < \cdots < \theta_1^{(2)} < \theta_1^{(1)} < \theta_2^{(2)} < \cdots < \theta_n^{(n)} \leq \pi$;
Output H_n;

(1) Set $\rho_1 = \exp(i\theta_1^{(1)})$;

(2) Compute α_{k-1} and ρ_k by (3.7) and (3.8) for $k = 2, 3, \ldots, n$;

(3) Set $\gamma_0 = 1$;

(4) Compute γ_k by (2.4) for $k = 1, 2, \ldots, n-1$;

(5) Set $\gamma_n = \rho_n$.

We present an example to illustrate this algorithm.

Example 4.2. Let $n = 5$, given $\theta_1^{(1)} = \pi/6$; $\theta_1^{(2)} = -\pi/8$, $\theta_2^{(2)} = \pi/4$; $\theta_1^{(3)} = -\pi/4$, $\theta_3^{(3)} = \pi/3$; $\theta_1^{(4)} = -\pi/3$, $\theta_4^{(4)} = \pi/2$; $\theta_1^{(5)} = -\pi/2$, $\theta_5^{(5)} = 2\pi/3$. By $\lambda_j^{(k)} = \exp(i\theta_j^{(k)})$, we get

$$
\begin{gathered}
\lambda_1^{(1)} = 0.8660 + 0.5000i; \\
\lambda_1^{(2)} = 0.9239 - 0.3827i, \qquad \lambda_2^{(2)} = 0.7071 + 0.7071i; \\
\lambda_1^{(3)} = 0.7071 - 0.7071i, \qquad \lambda_3^{(3)} = 0.5000 + 0.8660i; \\
\lambda_1^{(4)} = 0.5000 - 0.8660i, \qquad \lambda_4^{(4)} = 0.0000 + 1.0000i; \\
\lambda_1^{(5)} = 0.0000 - 1.0000i, \qquad \lambda_5^{(5)} = -0.5000 + 0.8660i.
\end{gathered}
\tag{4.1}
$$

Using Algorithm 4.1, we obtain $\{\rho_i\}_{i=1}^5$, $\{\alpha_i\}_{i=1}^4$, $\{\gamma_i\}_{i=1}^5$ listed in Table 1. The unitary Hessenberg matrix is given as follows:

$$
H = \begin{pmatrix}
0.7588+0.4471i & -0.3354-0.1185i & 0.1490+0.0810i & 0.0045-0.0151i & 0.1169+0.2346i \\
0.4736 & 0.6493-0.1269i & -0.3152+0.0109i & 0.0071+0.0283i & -0.4088-0.2656i \\
0 & 0.6601 & 0.4024+0.0643i & -0.0020-0.0377i & 0.4526+0.4382i \\
0 & 0 & 0.8401 & 0.0068+0.0317i & -0.4436-0.3106i \\
0 & 0 & 0 & 0.9982 & 0.0438-0.0407i
\end{pmatrix}.
\tag{4.2}
$$

We recompute the spectrum of $\widetilde{H}_k$ $(k = 1, 2, \ldots, n)$, and get

$$
\begin{aligned}
\Lambda\left(\widetilde{H}_1\right) &= \left(\underline{0.8660 + 0.5000i}\right), \\
\Lambda\left(\widetilde{H}_2\right) &= \left(\underline{0.9239 - 0.3827i}, \underline{0.7071 + 0.7071i}\right), \\
\Lambda\left(\widetilde{H}_3\right) &= \left(\underline{0.7071 - 0.7071i}, 0.9013 + 0.4332i, \underline{0.5000 + 0.8660i}\right), \\
\Lambda\left(\widetilde{H}_4\right) &= \left(\underline{0.5000 - 0.8660i}, 0.9964 - 0.0850i, 0.7329 + 0.6803i, \underline{0.0000 + 1.0000i}\right), \\
\Lambda\left(\widetilde{H}_5\right) &= \left(\underline{0.0000-1.0000i}, 0.7937-0.6083i, 0.9411+0.3381i, 0.6262+0.7796i, \underline{-0.5000+0.8660i}\right).
\end{aligned}
\tag{4.3}
$$

Table 1: $\{\rho_i\}_{i=1}^5, \{\alpha_i\}_{i=1}^4, \{\gamma_i\}_{i=1}^5$.

i	ρ_i	α_i	γ_i
1	$-0.8660 - 0.5000i$	$-0.2265 - 0.0451i$	$-0.7588 - 0.4471i$
2	$0.9239 + 0.3827i$	$-0.4727 - 0.0442i$	$0.7083 + 0.2501i$
3	$-0.7584 - 0.6517i$	$-0.7675 - 0.3218i$	$-0.4766 - 0.2591i$
4	$0.3747 + 0.9271i$	$-1.3652 - 0.2759i$	$-0.0169 + 0.0574i$
5	$-0.4458 - 0.8951i$	—	$-0.4458 - 0.8951i$

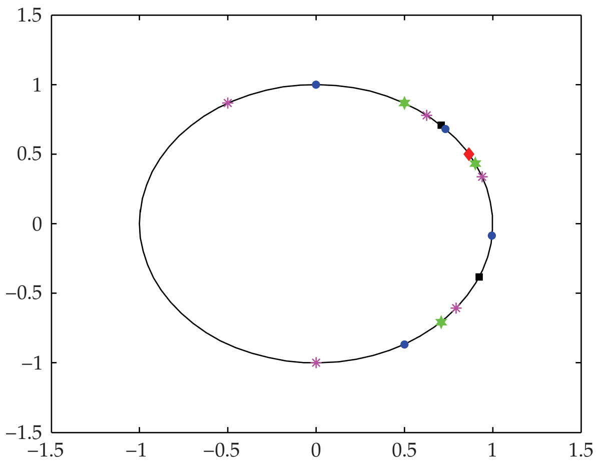

Figure 1: The eigenvalues of $\widetilde{H}_k$.

These obtained data show that Algorithm 4.1 is quite efficient, Figure 1 illustrates the eigenvalues of $\widetilde{H}_k$ $(k = 1, 2, \ldots, 5)$.

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