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Research Article

Simple-Zero and Double-Zero Singularities of a Kaldor-Kalecki Model of Business Cycles with Delay

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We study the Kaldor-Kalecki model of business cycles with delay in both the gross product and the capital stock. Simple-zero and double-zero singularities are investigated when bifurcation parameters change near certain critical values. By performing center manifold reduction, the normal forms on the center manifold are derived to obtain the bifurcation diagrams of the model such as Hopf, homoclinic and double limit cycle bifurcations. Some examples are given to confirm the theoretical results.

1. Introduction

In the last decade, the study of delayed differential equations that arose in business cycles has received much attention. The first model of business cycles can be traced back to Kaldor [1] who used a system of ordinary differential equations to study business cycles in 1940 by proposing nonlinear investment and saving functions so that the system may have cyclic behaviors or limit cycles, which are important from the point of view of economics. Kalecki [2] introduced the idea that there is a time delay for investment before a business decision. Krawiec and Szydłowski [3–5] incorporated the idea of Kalecki into the model of Kaldor by proposing the following Kaldor-Kalecki model of business cycles:

$$\begin{aligned} \frac{dY(t)}{dt} &= \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} &= I(Y(t-\tau),K(t)) - qK(t), \end{aligned} \tag{1.1}$$

where Y is the gross product, K is the capital stock, $\alpha > 0$ is the adjustment coefficient in the goods market, $q \in (0,1)$ is the depreciation rate of capital stock, $I(Y,K)$ and $S(Y,K)$

are investment and saving functions, and $\tau \geq 0$ is a time lag representing delay for the investment due to the past investment decision. This model has been studied extensively by many authors; see [6–11]. Several authors also discussed similar models [12–14] and established the existence of limit cycles.

Considering that past investment decisions [6] also influence the change in the capital stock, Kaddar and Talibi Alaoui [15] extended the model (1.1) by imposing delays in both the gross product and capital stock. Thus adding the same delay to the capital stock K in the investment function $I(Y, K)$ of the second equation of Sys. (1.1) leads to the following Kaldor-Kalecki model of business cycles:

$$\begin{aligned} \frac{dY(t)}{dt} &= \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{dK(t)}{dt} &= I(Y(t-\tau), K(t-\tau)) - qK(t). \end{aligned} \tag{1.2}$$

As in [3]; also see [10, 16, 17], using the following saving and investment functions S and I, respectively,

$$S(Y, K) = \gamma Y, \quad I(Y, K) = I(Y) - \beta K, \tag{1.3}$$

where $\beta > 0$ and $\gamma \in (0, 1)$ are constants, we obtain the following system:

$$\begin{aligned} \frac{dY(t)}{dt} &= \alpha\left[I(Y(t)) - \beta K(t) - \gamma Y(t)\right], \\ \frac{dK(t)}{dt} &= I(Y(t-\tau)) - \beta K(t-\tau) - qK(t). \end{aligned} \tag{1.4}$$

Kaddar and Talibi Alaoui [15] studied the characteristic equation of the linear part of Sys. (1.4) at an equilibrium point and used the delay τ as a bifurcation parameter to show that the Hopf bifurcation may occur under some conditions as τ passes some critical values. However, they did not obtain the stability of the bifurcating limit cycles and the direction of the Hopf bifurcation. Wang and Wu [18] further studied Sys. (1.4) and gave a more detailed discussion of the distribution of the eigenvalues of the characteristic equation which has a pair of purely imaginary roots. They derived the normal forms on the center manifold for sys. (1.4) to give the direction of the Hopf bifurcation and the stability of the bifurcating limit cycles for some critical values of τ.

However, under certain conditions, the characteristic equation of the linear part of Sys. (1.4) may have a simple-zero root, a double-zero root, or a simple zero root and a pair of purely imaginary roots. In this paper, simple-zero (fold) and double-zero (Bogdanov-Takens) singularities for Sys. (1.4) and their corresponding dynamical behaviors are investigated by using k and τ as bifurcation parameters (where k is defined in Section 2). The discussion of zero-Hopf singularity will be addressed in a coming paper.

The rest of this manuscript is organized as follows. In Section 2, a detailed presentation is given for the distribution of eigenvalues of the linear part of Sys. (1.4) at an equilibrium point in the (k, τ)-parameter space. In Section 3, the theory of center manifold reduction for general delayed differential equations (DDEs) is briefly introduced. In Sections 4 and 5, center manifold reduction is performed for Sys. (1.4); and hence, the normal forms for simple-zero and double-zero singularities are obtained on the center manifold, respectively. In Section 6, the normal forms for the double-zero singularity are used to predict the bifurcation diagrams such as Hopf, homoclinic, and double limit cycle bifurcations for the original Sys. of (1.4). Finally in Section 7, some numerical simulations are presented to confirm the theoretical results.

2. Distribution of Eigenvalues

Throughout the rest of this paper, we assume that

$$\alpha, \beta > 0, \quad q, \gamma \in (0,1), \text{ and } I(s) \text{ is a nonlinear } C^4 \text{ function}, \tag{2.1}$$

and that (Y^*, K^*) is an equilibrium point of Sys. (1.4). Let $I^* = I(Y^*)$, $u_1 = Y - Y^*$, $u_2 = K - K^*$, and $i(s) = I(s + Y^*) - I^*$. Then Sys. (1.4) can be transformed as

$$\begin{aligned} \frac{du_1(t)}{dt} &= \alpha\left[i(u_1(t)) - \beta u_2(t) - \gamma u_1(t)\right], \\ \frac{du_2(t)}{dt} &= i(u_1(t-\tau)) - \beta u_2(t-\tau) - q u_2(t). \end{aligned} \tag{2.2}$$

Let the Taylor expansion of i at 0 be

$$i(u) = ku + i^{(2)}u^2 + i^{(3)}u^3 + \mathcal{O}\left(|u|^4\right), \tag{2.3}$$

where

$$k = i'(0) = I'(Y^*), \qquad i^{(2)} = \frac{1}{2}i''(0) = \frac{1}{2}I''(Y^*), \qquad i^{(3)} = \frac{1}{3!}i'''(0) = \frac{1}{3!}I'''(Y^*). \tag{2.4}$$

The linear part of Sys. (2.2) at $(0,0)$ is

$$\begin{aligned} \frac{du_1(t)}{dt} &= \alpha\left[(k-\gamma)u_1(t) - \beta u_2(t)\right], \\ \frac{du_2(t)}{dt} &= ku_1(t-\tau) - \beta u_2(t-\tau) - q u_2(t), \end{aligned} \tag{2.5}$$

and the corresponding characteristic equation is

$$\Delta(\lambda) \equiv \lambda^2 + A\lambda + B + (\beta\lambda + C)e^{-\lambda\tau} = 0, \tag{2.6}$$

where

$$A = q - \alpha(k-\gamma), \qquad B = -\alpha q(k-\gamma), \qquad C = \alpha\beta\gamma. \tag{2.7}$$

For $\tau = 0$, (2.6) becomes

$$\lambda^2 + (A+\beta)\lambda + B + C = 0. \tag{2.8}$$

Define

$$k^* = \frac{\beta\gamma}{q} + \gamma, \qquad k^{**} = \frac{q+\beta}{\alpha} + \gamma. \tag{2.9}$$

Theorem 2.1. *Let $\tau = 0$. If $k < \min\{k^*, k^{**}\}$, then all roots of (2.8) have negative real parts, and hence (Y^*, K^*) is asymptotically stable. If $k > \min\{k^*, k^{**}\}$, then (2.8) has a positive root and a negative root, and hence, (Y^*, K^*) is unstable.*

Now assume $\tau > 0$. Clearly $\Delta(0) = 0$ if and only if $k = k^*$. Next we always assume that $k = k^*$. It is easy to attain

$$\begin{aligned} \Delta'(\lambda) &= 2\lambda + q - \frac{\alpha\beta\gamma}{q} + \beta e^{-\lambda\tau} - (\beta\lambda + C)\tau e^{-\lambda\tau}, \\ \Delta''(\lambda) &= 2 - 2\beta\tau e^{-\lambda\tau} + \beta\tau^2\lambda e^{-\lambda\tau} + C\tau^2 e^{-\lambda\tau}. \end{aligned} \tag{2.10}$$

Define $\tau^* = (q^2 + q\beta - \alpha\beta\gamma)/\alpha\beta\gamma q$. Then we have that,

$$\Delta'(0) = \frac{\alpha\beta\gamma}{q}(\tau^* - \tau), \qquad \Delta''(0)\big|_{\tau=\tau^*} = \frac{q^4 - \beta^2 q^2 + \alpha^2\beta^2\gamma^2}{\alpha\beta\gamma q^2}. \tag{2.11}$$

Define

$$f(x) = x^2 + \beta x - \alpha\beta\gamma, \qquad g(x) = x^2 - \beta^2 x + \alpha^2\beta^2\gamma^2. \tag{2.12}$$

Hence if $f(q) \le 0$, $\tau^* \le 0$, and hence $\Delta'(0) < 0$, and if $f(q) > 0$, $\tau^* > 0$, and hence $\Delta'(0) = 0$ if and only if $\tau = \tau^*$. Also $\Delta''(0)|_{\tau=\tau^*} \ne 0$ if and only if $g(q^2) \ne 0$. Thus we obtain the following result.

Lemma 2.2. *Suppose that $k = k^*$. Then the following are considered.*

(i) *If $\tau^* \le 0$, then (2.6) has a simple root 0 for all $\tau > 0$.*

(ii) *Let $\tau^* > 0$. Then the following are given.*

  (a) *Equation (2.6) has a simple root 0 if and only if $\tau \ne \tau^*$,*

  (b) *Equation (2.6) has a double root 0 if and only if $\tau = \tau^*$ and $g(q^2) \ne 0$.*

Let ωi $(\omega > 0)$ be a purely imaginary root of (2.6). After plugging it into (2.6) and separating the real and imaginary parts, we have that

$$\begin{aligned} \omega^2 + \alpha\beta\gamma &= \alpha\beta\gamma\cos(\omega\tau) + \beta\omega\sin(\omega\tau), \\ \frac{q^2 - \alpha\beta\gamma}{q}\omega &= \alpha\beta\gamma\sin(\omega\tau) - \beta\omega\cos(\omega\tau). \end{aligned} \tag{2.13}$$

Adding squares of two equations yields

$$\omega^2 + \frac{g(q^2)}{q^2} = 0. \tag{2.14}$$

Then (2.14) has a nonzero solution if and only if $g(q^2) < 0$ and does not have a nonzero solution if and only if $g(q^2) \geq 0$. If $g(q^2) < 0$, from (2.14), we solve ω as follows:

$$\omega = \omega_0 \equiv \frac{1}{q}\sqrt{-g(q^2)}, \tag{2.15}$$

and from (2.13), we solve $\cos(\omega_0\tau)$, $\sin(\omega_0\tau)$ as:

$$\begin{aligned} \cos(\omega_0\tau) &= \frac{-q^2\omega_0^2 + \alpha\beta\gamma\omega_0^2 + q\alpha\gamma(\alpha\beta\gamma + \omega_0^2)}{q\beta(\alpha^2\gamma^2 + \omega_0^2)} \equiv a, \\ \sin(\omega_0\tau) &= \frac{q^2\alpha\gamma\omega_0 - \alpha^2\beta\gamma^2\omega_0 + q\alpha\beta\gamma\omega_0 + q\omega_0^3}{q\beta(\alpha^2\gamma^2 + \omega_0^2)} \equiv b. \end{aligned} \tag{2.16}$$

Define

$$\delta = \begin{cases} \arccos a, & \text{if } b \geq 0, \\ 2\pi - \arccos a, & \text{if } b < 0. \end{cases} \tag{2.17}$$

From (2.16), we obtain

$$\tau = \tau_j \equiv \frac{1}{\omega_0}(\delta + 2j\pi), \quad j = 0, 1, 2, \ldots. \tag{2.18}$$

Clearly if $\beta > 2\alpha\gamma$, then $g(x) = 0$ has two positive roots, and if $\beta \leq 2\alpha\gamma$, then $g(x) \geq 0$. Now, under $k = k^*$, we impose the following conditions:

(H1) $\beta \leq 2\alpha\gamma$, $\tau^* \leq 0$,

(H2) $\beta \leq 2\alpha\gamma$, $\tau^* > 0$, $\tau \neq \tau^*$,

(H3) $\beta \leq 2\alpha\gamma$, $\tau^* > 0$, $\tau = \tau^*$,

(H4) $\beta > 2\alpha\gamma$, $\tau^* > 0$, $\tau \neq \tau^*$, $g(q^2) \geq 0$, (H5) $\beta > 2\alpha\gamma$, $\tau^* > 0$, $\tau \neq \tau^*$, $g(q^2) < 0$,

(H6) $\beta > 2\alpha\gamma$, $\tau^* > 0$, $\tau = \tau^*$, $g(q^2) \geq 0$,

(H7) $\beta > 2\alpha\gamma$, $\tau^* > 0$, $\tau = \tau^*$, $g(q^2) < 0$.

Based on Lemma 2.2, we have the following result.

Lemma 2.3. *Suppose that $k = k^*$ and $0 < q < 1$. Then the following are obtained.*

(i) *Under one of the conditions (H1), (H2), and (H4),* (2.6) *has a simple zero root and does not have other roots in the imaginary axis.*

(ii) *Under the condition (H5),* (2.6) *has a simple zero root and a pair of purely imaginary roots $\pm\omega_0 i$ in the imaginary axis if $\tau = \tau_j$, $j = 0, 1, 2, \ldots$.*

(iii) *Under one of the conditions (H3) and (H6), then* (2.6) *has a double root 0 and does not have other roots in the imaginary axis.*

(iv) *Under the condition (H7),* (2.6) *has a double zero root and a pair of purely imaginary roots $\pm\omega_0 i$ in the imaginary axis if $\tau^* = \tau_j$ for some j.*

Now we use the roots of $f(x) = 0$, $g(x) = 0$ to give a more detailed discussion for the roots of (2.6). Define

$$
\begin{aligned}
q_0 &= \frac{1}{2}\left(-\beta + \sqrt{\beta^2 + 4\alpha\beta\gamma}\right),\\
q_1 &= \frac{1}{2}\left(\beta^2 - \sqrt{\beta^4 - 4\alpha^2\beta^2\gamma^2}\right),\\
q_2 &= \frac{1}{2}\left(\beta^2 + \sqrt{\beta^4 - 4\alpha^2\beta^2\gamma^2}\right).
\end{aligned}
\tag{2.19}
$$

Clearly q_0 is the positive root of $f(x) = 0$ and q_1, q_2 are two positive roots of $g(x) = 0$ if $\beta > 2\alpha\gamma$. Note that $f(x) \leq 0$ if $0 < x \leq q_0$, and $f(x) > 0$ if $x > q_0$, $g(x) \geq 0$ if $0 < x \leq q_1$, or $x \geq q_2$, then $g(x) < 0$ if $q_1 < x < q_2$. Also note that as well as if $\beta > 2\alpha\gamma$, $q_0^2 < q_1$. In fact it is based on the following calculation:

$$
\begin{aligned}
q_1 - q_0^2 &= \frac{1}{2}\left(\beta^2 - \sqrt{\beta^4 - 4\alpha^2\beta^2\gamma^2}\right) - \frac{1}{4}\left(-\beta + \sqrt{\beta^2 + 4\alpha\beta\gamma}\right)^2\\
&= \frac{\beta}{2}\left(\sqrt{\beta^2 + 4\alpha\beta\gamma} - \sqrt{\beta^2 - 4\alpha^2\gamma^2} - 2\alpha\gamma\right)\\
&= \frac{2\alpha\beta\gamma\left(\beta - \sqrt{\beta^2 - 4\alpha^2\gamma^2}\right)}{\sqrt{\beta^2 + 4\alpha\beta\gamma} + \sqrt{\beta^2 - 4\alpha^2\gamma^2} + 2\alpha\gamma}\\
&> 0.
\end{aligned}
\tag{2.20}
$$

Thus for $\beta > 2\alpha\gamma$, we always have $q_0 < \sqrt{q_1} < \sqrt{q_2}$. Noting that $q \in (0, 1)$, we have the following result.

Lemma 2.4. *Let $\beta > 2\alpha\gamma$. Then the following are given.*

(i) *Suppose that $q_0 \geq 1$. Then for $0 < q < 1$, then* (2.6) *has a simple zero root and does not have roots in the imaginary axis.*

(ii) *Suppose that $q_0 < 1 \leq \sqrt{q_1} < \sqrt{q_2}$. If $0 < q \leq q_0$, then* (2.6) *has a simple zero root and does not have roots in the imaginary axis. And if $q_0 < q < 1$,* (2.6) *has a double zero root and does not have roots in the imaginary axis.*

(iii) *Suppose that $q_0 < \sqrt{q_1} < 1 < \sqrt{q_2}$. If $0 < q \leq q_0$, then* (2.6) *has a simple zero root and does not have roots in the imaginary axis. If $q_0 < q \leq \sqrt{q_1}$, then* (2.6) *has a double zero root and does not have roots in the imaginary axis. And if $\sqrt{q_1} < q < 1$, then* (2.6) *has a double zero root and has a pair of purely imaginary roots.*

(iv) *Suppose that $\sqrt{q_2} \geq 1$. Then if $0 < q \leq q_0$, then* (2.6) *has a simple zero root and does not have roots in the imaginary axis. If $q_0 < q \leq \sqrt{q_1}$, then* (2.6) *has a double zero root and does not have roots in the imaginary axis. If $\sqrt{q_1} < q < \sqrt{q_2}$, then* (2.6) *has a double zero root and has a pair of purely imaginary roots when $\tau^* = \tau_j$ for some j. And if $\sqrt{q_2} \leq q < 1$,* (2.6) *has a double zero root and does not have a pair of purely imaginary roots.*

Define $\lambda(\tau) = \sigma(\tau) + i\omega(\tau)$ to be the root of (2.6) such that $\sigma(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. Then we have the following result.

Lemma 2.5. *Suppose that $k = k^*$ and $g(q^2) < 0$. Then $\sigma'(\tau_j) > 0$.*

Proof. Differentiating (2.6) with respect to τ yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{[2\lambda + q - \alpha(k-\gamma)]e^{\lambda\tau} + \beta}{\lambda\beta(\lambda + \alpha\gamma)} - \frac{\tau}{\lambda}, \tag{2.21}$$

and a simple calculation gives

$$\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j} = \frac{\alpha^2\beta^2\gamma^2 + q^2(-\beta^2 + q^2 + 2\omega_0^2)}{\beta^2 q^2(\alpha^2\gamma^2 + \omega_0^2)} = \frac{q^2\beta^2 - \alpha^2\beta^2\gamma^2 - q^4}{\beta^2 q^2(\alpha^2\gamma^2 + \omega_0^2)}, \tag{2.22}$$

which gives

$$\operatorname{Sign}\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_j} = \operatorname{Sign}\left(-g\left(q^2\right)\right) = 1, \tag{2.23}$$

thus completing the proof. □

Next we discuss the distribution of other roots of (2.6). We need the following lemma due to Ruan and Wei [19].

Lemma 2.6. *Consider the exponential polynomial*

$$P\left(\lambda, e^{-\lambda\tau}\right) = p(\lambda) + q(\lambda)e^{-\lambda\tau}, \tag{2.24}$$

where p, q are real polynomials such that $\deg(q) < \deg(p)$ *and* $\tau \geq 0$. *As* τ *varies, the sum of the order of zeros of* $P(\lambda, e^{-\lambda\tau})$ *on the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

Lemma 2.7. *Let* $k = k^*$ *and* $\tau > 0$. *Then, the following are obtained.*

(i) *If* $q > q_0$, *then all roots of* (2.6) *except 0 and purely imaginary roots have negative real parts,*

(ii) *If* $0 < q \leq q_0$, *then* (2.6) *has at least one positive root.*

Proof. Note that, for $\tau = 0$, if $q > q_0$ or $q^2 + q\beta > \alpha\beta\gamma$, $\Delta(\lambda) = 0$ has a zero root and a negative root. Using Lemmas 2.2 and 2.6, we obtain claim (i). For $\tau = 0$, $\Delta(\lambda) = 0$ has a zero root and a positive root if $0 < q \leq q_0$ or $q^2 + q\beta \leq \alpha\beta\gamma$. For $\tau > 0$, let

$$f(\lambda) = \frac{\Delta(\lambda)}{\lambda} = \lambda + A + \beta e^{-\lambda\tau} + \frac{B + Ce^{-\lambda\tau}}{\lambda}. \tag{2.25}$$

Also noting that $B + C = 0$ when $k = k^*$, we have that

$$\lim_{\lambda \to 0^+} f(\lambda) = A + \beta - C\tau = \frac{1}{q}\left[\left(q^2 + q\beta - \alpha\beta\gamma\right) - \alpha\beta\gamma\tau\right] < 0, \tag{2.26}$$

and $\lim_{\lambda \to \infty} f(\lambda) = \infty$. This proves the second part of the lemma and completes the proof of the lemma. □

3. Center Manifold Reduction

In this section, we briefly summarize the theory of center manifold reduction for general DDEs. The material is mainly taken from [20, 21]. Consider the following DDE:

$$\frac{dx}{dt} = L(\mu)x_t + G(x_t, \mu), \tag{3.1}$$

where $x \in C([-\tau, 0], \mathbb{R}^n)$, $\mu \in \mathbb{R}^p$. This equation is equivalent to

$$\frac{dx}{dt} = L(\mu)x_t + G(x_t, \mu), \quad \frac{d\mu}{dt} = 0, \tag{3.2}$$

which can be written as

$$\frac{dX}{dt} = \mathcal{L}X_t + F(X_t), \tag{3.3}$$

where $X = (x, \mu)^T$, $F(X_t) = (G(x_t), 0)^T$, and $\mathcal{L} = \text{diag}(L, 0)$. Define $X \in C := C([-\tau, 0], \mathbb{R}^{n+p})$ with supreme norm and $X_t \in C$ is defined by $X_t(\theta) = X(t+\theta)$, $-\tau \leq \theta \leq 0$; $\mathcal{L} : C \to L(\mathbb{R}^{n+p})$ is a bounded linear operator; and $F : C \to C$ is a C^k $(k \geq 2)$ function with $F(0) = 0$, $DF(0) = 0$. Consider the following linear system:

$$\dot{X}(t) = \mathcal{L}X_t. \tag{3.4}$$

Since $\mathcal{L}$ is a bounded linear operator, then $\mathcal{L}$ can be represented by a Riemann-Stieltjes integral

$$\mathcal{L}\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta), \quad \forall\varphi \in C, \tag{3.5}$$

by the Riesz representation theorem, where $\eta(\theta)$ $(\theta \in [-\tau, 0])$ is an $(n+p) \times (n+p)$ matrix function of bounded variation. Let $\mathcal{A}_0$ be the infinitesimal generator for the solution semigroup defined by Sys. (3.4) such that

$$\mathcal{A}_0\varphi = \dot{\varphi}, \quad D(\mathcal{A}_0) = \left\{\varphi \in C^1([-\tau, 0], \mathbb{R}^{n+p}) : \dot{\varphi}(0) = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta)\right\}. \tag{3.6}$$

Define the bilinear form between C and $C^* = C([0, \tau], \mathbb{R}^{(n+p)*})$ (where $\mathbb{R}^{(n+p)*}$ is the space of all row $(n+p)$-vectors) by

$$\langle\psi, \varphi\rangle = \psi(0)\varphi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi, \quad \forall\psi \in C^*, \ \forall\varphi \in C. \tag{3.7}$$

The adjoint of $\mathcal{A}_0$ is defined by $\mathcal{A}_0^*$ as

$$\mathcal{A}_0^*\psi = -\dot{\psi}, \quad D(\mathcal{A}_0^*) = \left\{\varphi \in C^1\left([0, \tau], \mathbb{R}^{(n+p)*}\right) : \dot{\psi}(0) = -\int_{-\tau}^{0} \psi(-\theta)d\eta(\theta)\right\}. \tag{3.8}$$

In our setting, (3.3) has p trivial components. Assume that the characteristic equation of (3.3) has eigenvalue zero with multiplicity $2p$ and all other eigenvalues have negative real parts. Then $\mathcal{L}$ has a generalized eigenspace P which is invariant under the flow (3.4). Let P^* be the space adjoint with P in C^*. Then C can be decomposed as $C = P \oplus Q$ where $Q = \{\varphi \in C : \langle\psi, \varphi\rangle = 0, \forall\psi \in P^*\}$. Choose the bases Φ and Ψ for P and P^*, respectively, such that

$$\langle\Psi, \Phi\rangle = I, \quad \dot{\Phi} = \Phi J, \quad \dot{\Psi} = -J\Psi, \tag{3.9}$$

where J is Jordan matrix associated with the eigenvalue 0.

To consider Sys. (3.3), we need to enlarge the space C to the following BC:

$$BC = \left\{\varphi : [-\tau, 0] \to \mathbb{R}^{n+p} : \varphi \text{ is continuous on } [-\tau, 0), \ \exists \lim_{\theta \to 0^-} \varphi(\theta) \in \mathbb{R}^{n+p}\right\}. \tag{3.10}$$

The elements of BC can be expressed as $\psi = \varphi + X_0\alpha$ with $\varphi \in C,\ \alpha \in \mathbb{R}^{n+p}$, and

$$X_0(\theta) = \begin{cases} 0, & -\tau \le \theta < 0, \\ I, & \theta = 0, \end{cases} \tag{3.11}$$

where I is the $n \times n$ identity matrix. Define the projection $\pi : BC \to P$ by

$$\pi\left(\varphi + X_0\alpha\right) = \Phi\left[\left(\Psi, \varphi\right) + \Psi(0)\alpha\right]. \tag{3.12}$$

Then the enlarged phase space BC can be decomposed as $BC = P \oplus \ker \pi$. Let $X = \Phi x + y$ with $x \in \mathbb{R}^{2p}$ and $y \in Q^1 = \{\varphi \in Q : \dot{\varphi} \in C\}$. Then (3.3) can be decomposed as

$$\begin{aligned} \dot{x} &= Jx + \Psi(0)F\left(\Phi x + y\right), \\ \dot{y} &= \mathcal{A}_{Q^1}y + (I - \pi)\ X_0F\left(\Psi x + y\right), \end{aligned} \tag{3.13}$$

where $\mathcal{A}$ is an extension of the infinitesimal generator $\mathcal{A}_0$ from C^1 to BC, defined by

$$\mathcal{A}_0\varphi = \dot{\varphi} + X_0\left[L\varphi - \dot{\varphi}(0)\right] = \begin{cases} \dot{\varphi}, & -1 \le \theta < 0, \\ \displaystyle\int_{-\tau}^{0} d\eta(t)\varphi(t), & \theta = 0, \end{cases} \tag{3.14}$$

for $\varphi \in C^1$ and its adjoint by $\mathcal{A}^*$ is defined by

$$\mathcal{A}^*\psi = \begin{cases} -\dot{\psi}, & 0 < s \le \theta, \\ \displaystyle\int_{-\tau}^{0} \psi(-\theta)d\eta(\theta), & s = 0, \end{cases} \tag{3.15}$$

for $\psi \in C^{1*}$. Let $F(v) = \sum_{j\ge2}(1/j!)F_j(v)$. Then Sys. (3.13) becomes

$$\begin{aligned} \dot{x} &= Jx + \sum_{j\ge2}\frac{1}{j!}f_j^1(x, y), \\ \dot{y} &= \mathcal{A}_{Q^1}y + \sum_{j\ge2}\frac{1}{j!}f_j^2(x, y), \end{aligned} \tag{3.16}$$

where

$$f_j^1(x, y) = \Psi(0)F_j\left(\Phi x + y\right), \qquad f_j^2(x, y) = (I - \pi)X_0F_j\left(\Phi x + y\right). \tag{3.17}$$

On the center manifold, (3.16) can be approximated as

$$\dot{x} = Jx + \sum_{j\geq 2} \frac{1}{j!} f_j^1(x,0). \tag{3.18}$$

4. Simple-Zero Singularity

In this section, we assume that the condition (H2) holds. From the definition of τ^*, we know that $\tau^* > 0$ if and only if $q > q_0$. Therefore (H2) is equivalent to

$$k = k^*, \quad q > q_0, \quad \tau > 0, \quad \tau \neq \tau^*. \tag{4.1}$$

From (ii) of Lemma 2.4 and (ii) of Lemma 2.7, we know that, at $(0,0)$, the characteristic equation of the linear part of Sys. (2.5) has a simple zero root and the rest of roots have negative parts. We treat k as a bifurcation parameter near k^*.

Set $C := C([-\tau,0],\mathbb{R}^3)$, $C^* := C([0,\tau],\mathbb{R}^{3*})$. Let $\mu = k - k^*$. Then Sys. (2.5) can be rewritten as

$$\begin{aligned}
\frac{du_1}{dt} &= \alpha\left[\frac{\beta\gamma}{q}u_1(0) - \beta u_2(0) + \mu u_1(0) + i^{(2)}u_1^2(0) + i^{(0)}u_1^3(t)\right] + \mathcal{O}\left(|\mu||u|^2 + |u|^4\right),\\
\frac{du_2}{dt} &= k^* u_1(-\tau) - qu_2(t) + \mu u_1(-\tau) - \beta u_2(-\tau) + i^{(2)}u_1^2(-\tau) + i^{(3)}u_1^3(-\tau) + \mathcal{O}\left(|\mu||u|^2 + |u|^4\right),\\
\frac{d\mu}{dt} &= 0.
\end{aligned} \tag{4.2}$$

The linearization of Sys. (4.2) at $(0,0,0)$ is

$$\begin{aligned}
\frac{du_1}{dt} &= \frac{\alpha\beta\gamma}{q}u_1(0) - \alpha\beta u_2(0),\\
\frac{du_2}{dt} &= k^* u_1(-\tau) - qu_2(0) - \beta u_2(-\tau),\\
\frac{d\mu}{dt} &= 0.
\end{aligned} \tag{4.3}$$

Let $\eta(\theta) = \mathbb{A}\delta(\theta) + \mathbb{B}\delta(\theta+\tau)$ where

$$\mathbb{A} = \begin{pmatrix} \frac{\alpha\beta\gamma}{q} & -\alpha\beta & 0 \\ 0 & -q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 \\ k^* & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.4}$$

Let $X = (u_1, u_2, \mu)^T$ and

$$F(X_t) = \begin{pmatrix} \alpha\mu u_1(0) + \alpha i^{(2)} u_1^2(0) + \alpha i^{(3)} u_1^3(0) + \mathcal{O}\left(|\mu||u|^2 + |u|^4\right) \\ \mu u_1(-\tau) + i^{(2)} u_1^2(-\tau) + i^{(3)} u_1^3(-\tau) + \mathcal{O}\left(|\mu||u|^2 + |u|^4\right) \\ 0 \end{pmatrix}. \tag{4.5}$$

Define

$$L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta), \quad \forall \varphi \in C. \tag{4.6}$$

Then Sys. (4.2) becomes

$$\dot{X}(t) = LX_t + F(X_t). \tag{4.7}$$

From (3.7), the bilinear form can be expressed as

$$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) + \int_{-\tau}^{0} \psi(\xi + \tau)\mathbb{B}\varphi(\xi)d\xi. \tag{4.8}$$

It is not hard to see that the infinitesimal generator $\mathcal{A} : C^1 \to BC$ is given by

$$\mathcal{A}\varphi = \dot{\varphi} + X_0\left[L\varphi - \dot{\varphi}(0)\right] = \begin{cases} \dot{\varphi}, & -\tau \le \theta < 0, \\ \mathbb{A}\varphi(0) + \mathbb{B}\varphi(-\tau), & \theta = 0, \end{cases} \tag{4.9}$$

for $\varphi \in C^1$ and its adjoint $\mathcal{A}^*$ by

$$\mathcal{A}^*\psi = \begin{cases} -\dot{\psi}, & 0 < s \le \theta, \\ \psi(0)\mathbb{A} + \psi(\tau)\mathbb{B}, & s = 0, \end{cases} \tag{4.10}$$

for $\psi \in C^{1*}$.

Next we obtain the bases for the center space P and its adjoint space P^*, respectively. Let $\mathcal{A}\varphi = 0$ for $\varphi \in C^1$, that is,

$$\dot{\varphi}(\theta) = 0 \quad \text{for } -\tau \le \theta < 0, \quad \mathbb{A}\varphi(0) + \mathbb{B}\varphi(-\tau) = 0 \quad \text{for } \theta = 0. \tag{4.11}$$

then we know that φ is a constant vector $(a_1, a_2, a_3)^T \in \mathbb{R}^3 \setminus \{0\}$ such that

$$(\mathbb{A} + \mathbb{B})(a_1, a_2, a_3)^T = 0. \tag{4.12}$$

Then we have two linearly independent solutions $\varphi_1 = (q, \gamma, 0)^T$, $\varphi_2 = (0, 0, 1)^T$ which are bases for the center space P. Let $\Phi = (\varphi_1, \varphi_2)$.

Similarly, let $\mathscr{A}^*\psi = 0$ for $\psi \in C^{1*}$, that is,

$$-\dot{\psi}(s) = 0 \quad \text{for } 0 < s \leq \tau, \quad \psi(0)\mathbb{A} + \psi(\tau)\mathbb{B} = 0 \quad \text{for } s = 0, \tag{4.13}$$

then we know that ψ is a constant vector $(b_1, b_2, b_3) \in \mathbb{R}^{3*} \setminus \{0\}$ such that

$$(b_1, b_2, b_3)(\mathbb{A} + \mathbb{B}) = 0. \tag{4.14}$$

From this we have two linearly independent solutions $\psi_1 = (-(q+\beta), \alpha\beta, 0)$ and $\psi_2 = (0, 0, 1)$ which are bases for the center space P^*. Let $\Psi = (r\psi_1, \psi_2)^T$ with r being determined such that $\langle \psi_1, \varphi_1 \rangle = 1$. In fact

$$r = \frac{1}{q\alpha\beta\gamma(\tau - \tau^*)}. \tag{4.15}$$

Clearly r is well defined since $\tau - \tau^* \neq 0$. It is not hard to check that $\dot{\Phi} = \Phi J$, $\dot{\Psi} = -J\Psi$ and $\langle \Psi, \Phi \rangle = I$, where $J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Let $u = \Phi x + y$. Then Sys. (4.2) can be decomposed as

$$\begin{aligned} \dot{x} &= \Psi F(\Phi x + y), \\ \dot{y} &= A_{Q^1} y + (I - \pi) X_0 F(\Phi x + y). \end{aligned} \tag{4.16}$$

Write $x = (x_1, \mu)$. Note that

$$\Psi(0) F(\Phi x) = \begin{pmatrix} -r\alpha q^2 \left(\mu x_1 + q i^{(2)} x_1^2 + q^2 i^{(3)} x_1^3\right) \\ 0 \end{pmatrix} + \text{h.o.t.}. \tag{4.17}$$

Here h.o.t. represents higher-order terms. Thus, for sufficiently small μ, on the center manifold, if $i^{(2)} \neq 0$, then Sys. (4.2) becomes

$$\dot{x}_1 = -r\alpha q^2 \mu x_1 - r\alpha q^3 i^{(2)} x_1^2 + \text{h.o.t.},$$
$$\dot{\mu} = 0. \tag{4.18}$$

If $i^{(2)} = 0$ and $i^{(3)} \neq 0$, then Sys. (4.2) can be transformed into the following form:

$$\dot{x}_1 = -r\alpha q^2 \mu x_1 - r\alpha q^4 i^{(3)} x_1^3 + \text{h.o.t.},$$
$$\dot{\mu} = 0. \tag{4.19}$$

Thus we have the following results.

Theorem 4.1. *Let μ be small. Then consider what follows.*

(i) *Suppose that $\mu = 0$. Then if $i^{(2)} \neq 0$, the equilibrium (Y^*, K^*) is unstable, and if $i^{(2)} = 0$ and $i^{(3)} \neq 0$, then the equilibrium (Y^*, K^*) is asymptotically stable for $(\tau - \tau^*) i^{(3)} > 0$ and unstable if $(\tau - \tau^*) i^{(3)} < 0$.*

(ii) *The equilibrium (Y^*, K^*) is asymptotically stable if $(\tau - \tau^*)\mu > 0$ and unstable if $(\tau - \tau^*)\mu < 0$.*

(iii) *At (Y^*, K^*, k^*), Sys.* (1.4) *undergoes a transcritical bifurcation if $i^{(2)} \neq 0$ and a pitchfork bifurcation if $i^{(2)} = 0$ and $i^{(3)} \neq 0$.*

5. Double-Zero Singularity

In this section, we assume that one of the conditions (H3) and (H6) holds and $g(q^2) > 0$, or equivalently, as

$$k = k^*, \quad \tau = \tau^*, \quad q > q_0, \quad g\left(q^2\right) > 0. \tag{5.1}$$

From Section 2, we can see that, at $(0,0)$, the characteristic equation of Sys. (2.5) has a double root 0 and all other roots have negative real parts if $k = k^*$ and $\tau = \tau^*$. We treat (k, τ) as a bifurcation parameter near (k^*, τ^*).

By scaling $t \to t/\tau$, Sys. (2.2) can be written as

$$\begin{aligned}\frac{du_1(t)}{dt} &= \alpha\tau(k - \gamma)u_1(t) - \alpha\beta\tau u_2(t) + \alpha\tau i^{(2)} u_1^2(t) \\ &\quad + \tau i^{(3)} u_1^3(t) + \mathcal{O}\left(|u_1|^4\right), \\ \frac{du_2(t)}{dt} &= \tau k u_1(t-1) - (q + \beta)\tau u_2(t) + \tau i^{(2)} u_1^2(t-1) \\ &\quad + \tau i^{(3)} u_1^3(t-1) + \mathcal{O}\left(|u_1|^4\right).\end{aligned} \tag{5.2}$$

Let $C := C([-1,0],\mathbb{R}^4), C^* := C([0,1],\mathbb{R}^{4*})$. Let $\mu_1 = k - k^*$, $\mu_2 = \tau - \tau^*$. Then on C we have

$$\begin{aligned}
\frac{du_1}{dt} &= \alpha\bigg[\frac{\beta\gamma}{q}\tau^* u_1(0) - \beta\tau^* u_2(0) + \tau^*\mu_1 u_1(0) \\
&\quad + \frac{\beta\gamma}{q}\mu_2 u_1(0) - \beta\mu_2 u_2(0) + \tau^* i^{(2)} u_1^2(0) \\
&\quad + i^{(2)}\mu_2 u_1^2(0) + i^{(3)}\tau^* u_1^3(0) + i^{(3)}\mu_2 u_1^3(0)\bigg] + \mathcal{O}\Big(|\mu|^2|u| + |\mu||u|^4\Big), \\
\frac{du_2}{dt} &= \frac{\beta\gamma}{q}\tau^* u_1(t-1) - q\tau^* u_2(t) - \beta\tau^* u_2(-1) \\
&\quad + \tau^*\mu_1 u_1(-1) + \frac{\beta\gamma}{q}\mu_2 u_1(-1) - q\mu_2 u_2(0) - \beta\mu_2 u_2(-1) \\
&\quad + i^{(2)}\tau^* u_1^2(-1) + i^{(2)}\mu_2 u_1^2(-1) \\
&\quad + i^{(3)}\tau^* u_1^3(-1) + i^{(3)}\mu_2 u_1^3(-1) + \mathcal{O}\Big(|\mu|^2|u| + |\mu||u|^4\Big), \\
&\qquad \frac{d\mu_1}{dt} = 0, \qquad \frac{d\mu_2}{dt} = 0.
\end{aligned} \tag{5.3}$$

The linearization of Sys. (5.3) at $(0,0,0,0)$ is

$$\begin{aligned}
\frac{du_1(t)}{dt} &= \frac{\alpha\beta\gamma}{q}\tau^* u_1(0) - \alpha\beta\tau^* u_2(0), \\
\frac{du_2(t)}{dt} &= k^*\tau^* u_1(-1) - q\tau^* u_2(0) - \beta\tau^* u_2(-1), \\
\frac{d\mu_1}{dt} &= 0, \qquad \frac{d\mu_2}{dt} = 0.
\end{aligned} \tag{5.4}$$

Let

$$\eta(\theta) = \mathbb{A}\delta(\theta) + \mathbb{B}\delta(\theta+1), \tag{5.5}$$

where

$$\mathbb{A} = \begin{pmatrix} \frac{\alpha\beta\gamma}{q}\tau^* & -\alpha\beta\tau^* & 0 & 0 \\ 0 & -q\tau^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ k^*\tau^* & -\beta\tau^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.6}$$

Define

$$\mathcal{L}\varphi = \int_{-1}^{0} d\eta(\theta)\varphi(\theta), \quad \forall \varphi \in C. \tag{5.7}$$

Let $C^1 = C^1([-1,0], \mathbb{R}^4)$. Let $X = (u_1, u_2, \mu_1, \mu_2)^T$ and $F(X_t) = (F^1, F^2, 0, 0)^T$ where

$$\begin{aligned} F^1 &= \alpha\left[\tau^*\mu_1 u_1(0) + \frac{\beta\gamma}{q}\mu_2 u_1(0) - \beta\mu_2 u_2(0) + i^{(2)}\tau^* u_1^2(0) + i^{(3)}\tau^* u_1^3(0)\right] \\ &\quad + \mathcal{O}\left(|\mu|^2|u| + |\mu||u|^4\right), \\ F^2 &= \tau^*\mu_1 u_1(-1) + \frac{\beta\gamma}{q}\mu_2 u_1(-1) - q\mu_2 u_2(0) - \beta\mu_2 u_2(-1) \\ &\quad + i^{(2)}\tau^* u_1^2(0) + i^{(3)}\tau^* u_1^3(-1) + \mathcal{O}\left(|\mu|^2|u| + |\mu||u|^4\right). \end{aligned} \tag{5.8}$$

Then Sys. (5.3) can be transformed into

$$\dot{X}(t) = \mathcal{L}X_t + F(X_t). \tag{5.9}$$

Let $C^* = C([0,1], \mathbb{R}^{4*})$. From (3.7), the bilinear inner product between C and C^* can be expressed by

$$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) + \int_{-1}^{0} \psi(\xi+1)\mathbb{B}\varphi(\xi)d\xi, \tag{5.10}$$

for $\varphi \in C$ and $\psi \in C^*$. As in Section 4, the infinitesimal generator $\mathcal{A} : C^1 \to BC$ associated with $\mathcal{L}$ is given by

$$\mathcal{A}\varphi = \dot{\varphi} + X_0\left[\mathcal{L}\varphi - \dot{\varphi}(0)\right] = \begin{cases} \dot{\varphi}, & -1 \le \theta < 0, \\ \mathbb{A}\varphi(0) + \mathbb{B}\varphi(-1), & \theta = 0, \end{cases} \tag{5.11}$$

for $\varphi \in C^1$ and its adjoint by

$$\mathcal{A}^*\psi = \begin{cases} -\dot{\psi}, & 0 < s \le 1, \\ \psi(0)\mathbb{A} + \psi(1)\mathbb{B}, & s = 0, \end{cases} \tag{5.12}$$

for $\psi \in C^{1*}$. From Section 2, we know that 0 is an eigenvalue of $\mathcal{A}$ and $\mathcal{A}^*$ with multiplicity 4. Now we compute eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$ associated with 0, respectively.

Next we obtain the bases for the center space P and its adjoint space P^*, respectively. Let $\mathcal{A}\varphi = 0$ for $\varphi \in C^1$. This means that

$$\dot{\varphi}(\theta) = 0 \quad \text{for } -1 \le \theta < 0, \quad \mathbb{A}\varphi(0) + \mathbb{B}\varphi(-1) = 0 \quad \text{for } \theta = 0. \tag{5.13}$$

From this we obtain that $\varphi(\theta) = \varphi^0$ is a constant vector in $\mathbb{R}^4$ satisfying

$$(\mathbb{A} + \mathbb{B})\varphi^0 = 0. \tag{5.14}$$

This equation has three linearly independent solutions: $a_1 = (q, \gamma, 0, 0)^T$, $a_3 = (0, 0, 1, 0)^T$, $a_4 = (0, 0, 0, 1)^T$. Let φ_1^0 be one of those. Suppose that $\mathcal{A}a_2 = \varphi_1^0$ for $a_2 \in C^1$, namely,

$$\dot{a}_2(\theta) = \varphi_1^0 \quad \text{for } -1 \le \theta < 0, \quad \mathbb{A}a_2(0) + \mathbb{B}a_2(-1) = \varphi_1^0 \quad \text{for } \theta = 0. \tag{5.15}$$

This implies that there is a constant vector φ_2^0 in $\mathbb{R}^4$ such that $a_2(\theta) = \varphi_1^0\theta + \varphi_2^0$ and

$$\mathcal{L}\left(\varphi_1^0\theta + \varphi_2^0\right) = \varphi_1^0. \tag{5.16}$$

Since

$$\mathcal{L}\left(\varphi_1^0\theta + \varphi_2^0\right) = \mathcal{L}\left(\varphi_1^0\theta\right) + \mathcal{L}\left(\varphi_2^0\right) = -B\varphi_1^0 + (\mathbb{A} + \mathbb{B})\varphi_2^0, \tag{5.17}$$

we have that

$$(\mathbb{A} + \mathbb{B})\varphi_2^0 = (I + \mathbb{B})\varphi_1^0. \tag{5.18}$$

It is easy to see that (5.18) has no solution if φ_1^0 is either a_3 or a_4. For $\varphi_1^0 = a_1$, setting $\varphi_2^0 = (0, l, 0, 0)^T$ in (5.18), we obtain

$$l = -\frac{q^2\gamma}{q^2 + q\beta - \alpha\beta\gamma}, \tag{5.19}$$

and hence $a_2(\theta) = (\theta q, l + \gamma\theta, 0, 0)^T$. Thus we obtain bases a_1, a_2, a_3, a_4 for the center space P. Let $\Phi = (a_1, a_2, a_3, a_4)$. Then we have that $\dot{\Phi} = \Phi J$ where $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Similarly, let $\mathcal{A}^*\psi_2 = 0$ for $\psi_2 \in C^{1*}$, that is,

$$-\dot{\psi}_2(s) = 0 \quad \text{for } 0 < s \le 1, \quad \psi_2(0)\mathbb{A} + \psi_2(-1)\mathbb{B} = 0 \quad \text{for } s = 0, \tag{5.20}$$

which means that $\psi_2(s) = \psi_2^0$ is a constant vector $\psi_2^0 \in \mathbb{R}^{4*} \setminus \{0\}$ satisfying

$$\psi_2^0(\mathbb{A} + \mathbb{B}) = 0. \tag{5.21}$$

This equation has three linearly independent solutions: $b_2 = (m(q+\beta), -m\alpha\beta, 0, 0)$, $b_3 = (0,0,1,0)$, $b_4 = (0,0,0,1)$. Asserting that $\langle b_2, a_2\rangle = 1$ gives

$$m = \frac{2\left(q^2 + q\beta - \alpha\beta\gamma\right)}{q^4 - q^2\beta^2 + \alpha^2\beta^2\gamma^2}. \tag{5.22}$$

Let ψ_2^0 be one of b_2, b_3, b_4. Suppose $\mathcal{A}^* b_1 = \psi_2^0$, that is,

$$-\dot{b}_1(s) = \psi_2^0 \quad \text{for } 0 < s \le 1, \quad b_1(0)\mathbb{A} + b_1(1)\mathbb{B} = \psi_2^0 \quad \text{for } s = 0, \tag{5.23}$$

which implies that there is $\psi_1^0 \in \mathbb{R}^{4*}$ such that $b_1(s) = -\psi_2^0 s + \psi_1^0$ satisfying

$$\mathcal{L}^*\left(-\psi_2^0 s + \psi_1^0\right) = \psi_2^0. \tag{5.24}$$

Since

$$\mathcal{L}^*\left(-\psi_2^0 s + \psi_1^0\right) = -\mathcal{L}^*\left(\psi_2^0 s\right) + \mathcal{L}^*\left(\psi_1^0\right) = -\psi_2^0\mathbb{B} + \psi_1^0(\mathbb{A} + \mathbb{B}), \tag{5.25}$$

we have

$$\psi_1^0(\mathbb{A} + \mathbb{B}) = \psi_2^0(I + \mathbb{B}). \tag{5.26}$$

It is not hard to check that (5.26) has no solution if $\psi_2^0 = b_3$ or b_4. Letting $\psi_2^0 = b_2$, setting $\psi_1^0 = (n_1, n_2, 0, 0)$ in (5.26) and using $\langle b_1, a_2\rangle = 0$, we can get n_1 and n_2:

$$\begin{aligned} n_1 &= \frac{2(q+\beta)\left(q^6 - 3q^5\beta - 3q^4\beta^2 + q^3\beta^3 + 3q^2\alpha^2\beta^2\gamma^2 - 3q\alpha^2\beta^3\gamma^2 + 2\alpha^3\beta^3\gamma^3\right)}{3\left(q^4 - q^2\beta^2 + \alpha^2\beta^2\gamma^2\right)^2}, \\ n_2 &= \frac{2\alpha\beta\left(-q^2 + 2q\beta + \alpha\beta\gamma\right)\left(q^2 + q\beta - \alpha\beta\gamma\right)^2}{3\left(q^4 - q^2\beta^2 + \alpha^2\beta^2\gamma^2\right)^2}. \end{aligned} \tag{5.27}$$

Hence

$$b_1(s) = \psi_1^0 - s\psi_2^0 = \left(-m(q+\beta)s + n_1, m\alpha\beta s + n_2, 0, 0\right). \tag{5.28}$$

Then b_1, b_2, b_3, b_4 are bases of the center space P^*. Let $\Psi = (b_1, b_2, b_3, b_4)^T$. Then $\langle \Psi, \Phi\rangle = I$, $\dot{\Phi} = \Phi J$ and $\dot{\Psi} = -J\Psi$.

Let $u = \Phi x + y$, namely,

$$\begin{aligned} u_1(\theta) &= qx_1 + q\theta x_2 + y_1(\theta), \\ u_2(\theta) &= \gamma x_1 + (l + \gamma\theta)x_2 + y_2(\theta). \end{aligned} \tag{5.29}$$

Then Sys. (5.9) can be decomposed as

$$\begin{aligned}\dot{x} &= Jx + \Psi(0)F(\Phi x + y),\\ \dot{y} &= A_{Q^1}y + (I - \pi)X_0F(\Phi x + y).\end{aligned} \tag{5.30}$$

Write $x = (x_1, x_2, \mu_1, \mu_2)$. Then, on the center manifold, Sys. (5.30) becomes

$$\begin{aligned}\dot{x}_1 &= (a_{11}\mu_1 + a_{21}\mu_2)x_1 + (a_{21}\mu_1 + a_{22}\mu_2)x_2\\ &\quad + \alpha n_1\tau^*\left(i^{(2)}q^2x_1^2 + i^{(3)}q^3\tau^*x_1^3\right) + n_2\tau^*\\ &\quad \times \left[i^{(2)}q^2(x_1 - x_2)^2 + i^{(3)}q^3\tau^*(x_1 - x_2)^3\right] + \text{h.o.t.},\\ \dot{x}_2 &= (b_{11}\mu_1 + b_{21}\mu_2)x_1 + (b_{12}\mu_1 + b_{22}\mu_2)x_2\\ &\quad + m\alpha(q + \beta)\tau^*\left(i^{(2)}q^2x_1^2 + i^{(3)}q^3\tau^*x_1^3\right) + m\alpha\beta\tau^*\\ &\quad \times \left[i^{(2)}q^2(x_1 - x_2)^2 + i^{(3)}q^3\tau^*(x_1 - x_2)^3\right] + \text{h.o.t.},\\ \dot{\mu}_1 &= 0, \qquad \dot{\mu}_2 = 0,\end{aligned} \tag{5.31}$$

where

$$\begin{gathered}a_{11} = q\tau^*(\alpha n_1 + n_2), \qquad a_{21} = -\alpha((q + \beta)\gamma - qk^*)n_1 + (-(2q + \beta)\gamma + qk^*)n_2,\\ a_{12} = -qn_2\tau^*, \qquad a_{22} = -(l\alpha\beta n_1 + ((q + \beta)(l - \gamma) + qk^*)n_2),\\ b_{11} = mq^2\alpha\tau^*, \qquad b_{21} = mq\alpha\beta\gamma, \qquad b_{12} = mq\alpha\beta\tau^*, \qquad b_{22} = m\alpha\beta(-(q + \beta)\gamma + qk^*).\end{gathered} \tag{5.32}$$

Next we use techniques of nonlinear transformations in [22] to transform Sys. (5.31) into normal forms. If $i^{(2)} \neq 0$, then up to the second order, Sys. (5.31) can be written as

$$\begin{aligned}\dot{x}_1 &= (a_{11}\mu_1 + a_{21}\mu_2)x_1 + (a_{21}\mu_1 + a_{22}\mu_2)x_2\\ &\quad + \alpha n_1\tau^*i^{(2)}q^2x_1^2 + n_2\tau^*i^{(2)}q^2(x_1 - x_2)^2 + \text{h.o.t.},\\ \dot{x}_2 &= (b_{11}\mu_1 + b_{21}\mu_2)x_1 + (b_{12}\mu_1 + b_{22}\mu_2)x_2\\ &\quad + m\alpha(q + \beta)\tau^*i^{(2)}q^2x_1^2 + m\alpha\beta\tau^*i^{(2)}q^2(x_1 - x_2)^2 + \text{h.o.t.},\\ \dot{\mu}_1 &= 0, \qquad \dot{\mu}_2 = 0.\end{aligned} \tag{5.33}$$

This system can be transformed into the following normal form:

$$\begin{aligned}\dot{x}_1 &= x_2 + \text{h.o.t.},\\ \dot{x}_2 &= \rho_1x_1 + \rho_2x_2 + a_1x_1^2 + b_1x_1x_2 + \text{h.o.t.},\end{aligned} \tag{5.34}$$

where

$$\rho_1 = b_{11}\mu_1 + b_{21}\mu_2, \qquad \rho_2 = (a_{11}+b_{12})\mu_1 + (a_{21}+b_{22})\mu_2,$$
$$a_1 = mq^3\alpha\tau^* i^{(2)}, \qquad b_1 = 2q^2(m\alpha\beta + \alpha n_1 + n_2)\tau^* i^{(2)}. \tag{5.35}$$

Since

$$\begin{aligned} \left|\frac{\partial\rho}{\partial\mu}\right| &= \det\begin{pmatrix} \dfrac{\partial\rho_1}{\partial\mu_1} & \dfrac{\partial\rho_1}{\partial\mu_2} \\ \dfrac{\partial\rho_2}{\partial\mu_1} & \dfrac{\partial\rho_2}{\partial\mu_2} \end{pmatrix} \\ &= -mq^2\alpha\gamma\Big(m\alpha\beta^2 + \alpha\beta n_1 + (q+\beta)n_2\Big)\tau^* \\ &= -\frac{4q^3\alpha^3\beta^2\gamma^2\tau^*(q^2+q\beta-\alpha\beta\gamma)}{(q^4-q^2\beta^2+\alpha^2\beta^2\gamma^2)^2} \neq 0, \end{aligned} \tag{5.36}$$

we have that $(\mu_1,\mu_2) \to (\rho_1,\rho_2)$ is regular and hence the transversality condition holds.

If $i^{(2)} = 0$ and $i^{(3)} \neq 0$, then up to the third order, Sys. (5.31) becomes

$$\begin{aligned} \dot{x}_1 &= (a_{11}\mu_1 + a_{21}\mu_2)x_1 + (a_{21}\mu_1 + a_{22}\mu_2)x_2 \\ &\quad + \alpha n_1\tau^* i^{(3)}q^3\tau^* x_1^3 + n_2\tau^* i^{(3)}q^3\tau^*(x_1-x_2)^3 + \text{h.o.t.}, \\ \dot{x}_2 &= (b_{11}\mu_1 + b_{21}\mu_2)x_1 + (b_{12}\mu_1 + b_{22}\mu_2)x_2 \\ &\quad + m\alpha(q+\beta)\tau^* i^{(3)}q^3\tau^* x_1^3 + m\alpha\beta\tau^* i^{(3)}q^3\tau^*(x_1-x_2)^3 + \text{h.o.t.}, \\ \dot{\mu}_1 &= 0, \qquad \dot{\mu}_2 = 0. \end{aligned} \tag{5.37}$$

This system can be transformed into the following normal form:

$$\begin{aligned} \dot{x}_1 &= x_2 + \text{h.o.t.}, \\ \dot{x}_2 &= \rho_1 x_1 + \rho_2 x_2 + a_2 x_1^3 + b_2 x_1^2 x_2 + \text{h.o.t.}, \end{aligned} \tag{5.38}$$

where $a_2 = mq^4\alpha\tau^* i^{(3)}$, $b_2 = 3q^3(m\alpha\beta + \alpha n_1 + n_2)\tau^* i^{(3)}$.

6. Bifurcation Diagrams

In this section, we will use the truncated systems (5.34) and (5.38) to obtain bifurcation diagrams of Sys. (5.3).

First, we consider the truncated system of (5.34):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \rho_1 x_1 + \rho_2 x_2 + a_1 x_1^2 + b_1 x_1 x_2, \end{aligned} \tag{6.1}$$

where a_2 and b_2 are in Section 5. Note that $(a_1, b_1) \to (-a_1, -b_1)$ under the transformation $(x_1, x_2) \to (-x_1, -x_2)$. We may assume that $i^{(2)} > 0$. After the change of coordinates

$$x_1 = \frac{a_1}{b_1^2}\left(\xi_1 - \frac{\rho_1}{b_1}\right), \qquad x_2 \longrightarrow s\frac{a_1^2}{b_1^3}\xi_2, \qquad t \longrightarrow \left|\frac{b_1}{a_1}\right|\tau, \tag{6.2}$$

we have (still using x_1, x_2 for simplicity) that

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \nu_1 + \nu_2 x_1 + x_1^2 + s x_1 x_2, \end{aligned} \tag{6.3}$$

where $\nu_1 = -(b_1^2/a_1^3)(b_1\rho_1 - a_1\rho_2)$, $\nu_2 = (b_1/a_1^2)(b_1\rho_1 - 2a_1\rho_1)$ and $s = \text{sign}\ a_1b_1 = \pm 1$. Simple calculation shows that $s = \text{sign}(\delta)$ where

$$\delta = q^6 - 3q^4\beta^2 - 2q^3\beta^3 + 3q^2\alpha^2\beta^2\gamma^2 + 2\alpha^3\beta^3\gamma^3. \tag{6.4}$$

Now take $s = -1$, namely $\delta < 0$. The complete bifurcation diagrams of Sys. (6.3) can be found in [22]. Here, we just briefly list some results. For (ν_1, ν_2) small enough, consider the following.

(i) Sys. (6.3) undergoes a fold bifurcation when (ν_1, ν_2) is on the curves

$$T^+ = \left\{(\nu_1, \nu_2) : 4\nu_1 - \nu_2^2 = 0,\ \nu_2 > 0\right\}, \qquad T^- = \left\{(\nu_1, \nu_2) : 4\nu_1 - \nu_2^2 = 0,\ \nu_2 < 0\right\}. \tag{6.5}$$

(ii) Sys. (6.3) undergoes a Hopf bifurcation when (ν_1, ν_2) is on the half-line

$$H = \{(\nu_1, \nu_2) : \nu_1 = 0,\ \nu_2 < 0\}, \tag{6.6}$$

and the Hopf bifurcation gives rise to a stable limit cycle.

(iii) Sys. (6.3) undergoes a homoclinic loop bifurcation when (ν_1, ν_2) is on the curve

$$P = \left\{(\nu_1, \nu_2) : \nu_1 = -\frac{6}{25}\nu_2^2,\ \nu_2 < 0\right\}. \tag{6.7}$$

Moreover, when (ν_1, ν_2) is in the region between the curves H and P, Sys. (6.1) has a unique stable periodic orbit.

For $s = 1$, under the transformation $t \to -t$, $x_1 \to -x_1$, we can get Sys. (6.12) whose parametric portrait remains as it was but the cycle becomes unstable. Applying the above results and using the expressions of ν_1, ν_2, we obtain the following result regarding Sys. (5.3).

Theorem 6.1. *Suppose that $i^{(2)} > 0$ and $\delta < 0$. For sufficiently small μ_1, μ_2, consider the following*

(i) *Sys.* (5.3) *undergoes a fold bifurcation in the half-lines*

$$\overline{T}_+ = \left\{ (\mu_1, \mu_2) : \mu_1 = -\frac{\beta\gamma}{q\tau^*}\mu_2,\ \mu_2 < 0 \right\}, \qquad \overline{T}_- = \left\{ (\mu_1, \mu_2) : \mu_1 = -\frac{\beta\gamma}{q\tau^*}\mu_2,\ \mu_2 > 0 \right\}. \tag{6.8}$$

(ii) *Sys.* (5.3) *undergoes a Hopf bifurcation on the curve*

$$\overline{H} = \left\{ (\mu_1, \mu_2) : \mu_1 = -\frac{\gamma(2m\alpha\beta^2 + 2\alpha\beta n_1 + (q + 2\beta)n_2)}{q(m\alpha\beta + \alpha n_1 + n_2)\tau^*}\mu_2 + \mathcal{O}\left(\mu_2^2\right),\ \mu_2 > 0 \right\}. \tag{6.9}$$

(iii) *Sys.* (5.3) *undergoes a saddle of homoclinic bifurcation on the curve*

$$\overline{P} = \left\{ (\mu_1, \mu_2) : \mu_1 = -\frac{\gamma(12m\alpha\beta^2 + 2\alpha\beta n_1 + (7q + 12\beta)n_2)}{5q(m\alpha\beta + \alpha n_1 + n_2)\tau^*}\mu_2 + \mathcal{O}\left(\mu_2^{3/2}\right),\ \mu_2 > 0 \right\}. \tag{6.10}$$

Moreover, if (μ_1, μ_2) is in the region between the curves $\overline{H}$ and $\overline{P}$, Sys. (5.3) *has a unique stable periodic orbit.*

Next, we consider the truncated system of (5.38):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \rho_1 x_1 + \rho_2 x_2 + a_2 x_1^3 + b_2 x_1^2 x_2, \end{aligned} \tag{6.11}$$

where a_2, b_2 are in Section 5. The bifurcation diagrams of this system are more complicated and interesting. We must consider two cases.

Case 1 ($i^{(3)} > 0$ so that $a_2 > 0$). We can assume $b_2 < 0$ which is equivalent to $\delta < 0$ in (6.4). Then Sys. (6.11) can be transformed as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \varepsilon_1 x_1 + \varepsilon_2 x_2 + x_1^3 - x_1^2 x_2, \end{aligned} \tag{6.12}$$

where

$$\varepsilon_1 = \left(\frac{b_2}{a_2}\right)^2 \rho_1, \qquad \varepsilon_2 = -\frac{b_2}{a_2}\rho_2. \tag{6.13}$$

The complete bifurcation diagrams of Sys. (6.12) can be found, for example, in [22–24]. For $b_2 > 0$, under the transformation $t \to -t$ and $x_1 \to -x_1$, we can get Sys. (6.12). Here, we briefly list some results: for small ε_1, ε_2 as follows.

(i) When $(\varepsilon_1, \varepsilon_2)$ is in the line

$$T_1 = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_1 = 0,\ \varepsilon_2 \in \mathbb{R}\}, \tag{6.14}$$

Sys. (6.12) undergoes a pitchfork bifurcation.

(ii) Sys. (6.12) undergoes a stable Hopf bifurcation for the trivial equilibrium point on the half-line

$$H_1 = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = 0,\ \varepsilon_1 < 0\}; \tag{6.15}$$

(iii) On the curve

$$C = \left\{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = -\frac{1}{5}\varepsilon_1 + \mathcal{O}\left(|\varepsilon_1|^{3/2}\right),\ \varepsilon_1 < 0\right\}, \tag{6.16}$$

Sys. (6.12) undergoes a heteroclinic bifurcation. Moreover, if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves H_1 and C then Sys. (6.12) has a unique stable periodic orbit.

Applying the above results and using the expressions of ρ_1, ρ_2, ε_1, ε_2, we obtain the following theorem regarding Sys. (5.3).

Theorem 6.2. *Suppose that $i^{(2)} = 0$, $i^{(3)} > 0$ and $\delta < 0$. For sufficiently small μ_1, μ_2, the following are given.*

(i) *Sys.* (5.3) *undergoes a pitchfork bifurcation in the line*

$$\overline{T}_1 = \left\{(\mu_1, \mu_2) : \mu_1 = -\frac{\beta\gamma}{q\tau^*}\mu_2,\ \mu_2 \in \mathbb{R}\right\}. \tag{6.17}$$

(ii) *Sys.* (5.3) *undergoes a stable Hopf bifurcation in the half-line*

$$\overline{H}_1 = \left\{(\mu_1, \mu_2) : \mu_1 = \frac{\gamma n_2}{\tau^*(m\alpha\beta + \alpha n_1 + n_2)}\mu_2,\ \mu_2 > 0\right\}. \tag{6.18}$$

(iii) *Sys.* (5.3) *undergoes a branch of homoclinic bifurcation on the curve*

$$\overline{C} = \left\{(\mu_1, \mu_2) : \mu_1 = \frac{\gamma(3m\alpha\beta^2 + 3\alpha\beta n_1 + (5q + 3\beta)n_2)}{2q(m\alpha\beta + \alpha n_1 + n_2)\tau^*}\mu_2 + \mathcal{O}\left(|\mu_2|^{3/2}\right),\ \mu_2 > 0\right\}. \tag{6.19}$$

Moreover, if (μ_1, μ_2) is in the region between the curves $\overline{H}_1$ and $\overline{C}$, Sys. (5.3) *has a unique stable periodic orbit.*

Case 2 ($i^{(3)} < 0$ so that $a_2 < 0$). We assume $b_2 < 0$, namely $\delta > 0$. After rescaling, then Sys. (6.11) can be transformed as

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \varepsilon_1 x_1 + \varepsilon_2 x_2 - x_1^3 - x_1^2 x_2, \tag{6.20}$$

where

$$\varepsilon_1 = \left(\frac{b_1}{a_1}\right)^2 \rho_1, \qquad \varepsilon_2 = \frac{b_1}{a_1}\rho_2. \tag{6.21}$$

For $b_2 > 0$, under the transformation $t \to -t, x_1 \to -x_1$, we can get Sys. (6.20). The complete bifurcation diagrams of this system can be found, for example, in [22–24]. Here, we briefly list some results: for small ε_1, ε_2, as follows.

(i) When $(\varepsilon_1, \varepsilon_2)$ is in the line

$$T_2 = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_1 = 0,\ \varepsilon_2 \in \mathbb{R}\}, \tag{6.22}$$

Sys. (6.20) undergoes a pitchfork bifurcation.

(ii) When $(\varepsilon_1, \varepsilon_2)$ is in the half-line

$$H_2 = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = \varepsilon_1,\ \varepsilon_1 > 0\}, \tag{6.23}$$

Sys. (6.20) undergoes a stable Hopf bifurcation at $E_{1,2}$ and the bifurcation is subcritical.

(iii) When $(\varepsilon_1, \varepsilon_2)$ is on the curve

$$C' = \left\{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = \frac{4}{5}\varepsilon_1 + \mathcal{O}\left(|\varepsilon_1|^{3/2}\right),\ \varepsilon_1 > 0\right\}, \tag{6.24}$$

Sys. (6.20) has a unique homoclinic orbit connecting E_1 and E_2 and two homoclinic orbits simultaneously at E_0. Moreover, if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves H_2 and C', Sys. (6.20) has three limit periodic orbits: a "large" one and two "small" ones.

(iv) When $(\varepsilon_1, \varepsilon_2)$ is on the curve

$$C_d = \left\{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = c\varepsilon_1 + \mathcal{O}\left(|\varepsilon_1|^{3/2}\right),\ \varepsilon_1 > 0\right\}, \tag{6.25}$$

where $c \approx 0.752$, Sys. (6.20) undergoes a double limit cycle bifurcation. Moreover, if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves C' and C_d, then Sys. (6.20) has two large limit cycles: the outer one which is stable and the inner one which is unstable, and these two cycles collide on C_d.

Applying the above results and using the expressions of ρ_1, ρ_2, ε_1, ε_2, we obtain the following theorem regarding Sys. (5.3).

Theorem 6.3. *Suppose that* $i^{(2)} = 0$, $i^{(3)} < 0$ *and* $\delta > 0$. *For sufficiently small* μ_1, μ_2, *considered the following.*

(i) *Sys.* (5.3) *undergoes a pitchfork bifurcation in the line*

$$\overline{T}_2 = \left\{ (\mu_1, \mu_2) : \mu_1 = -\frac{\beta\gamma}{q\tau^*},\ \mu_2 \in \mathbb{R} \right\}. \tag{6.26}$$

(ii) *Sys.* (5.3) *undergoes a branch of stable Hopf bifurcation on the curve*

$$\overline{H}_2 = \left\{ (\mu_1, \mu_2) : \mu_2 = -\frac{\gamma(3m\alpha\beta^2 + 3\alpha\beta n_1 + (q+3\beta)n_2)}{2q(m\alpha\beta + \alpha n_1 + n_2)\tau^*}\mu_2,\ \mu_2 < 0 \right\}. \tag{6.27}$$

(iii) *Sys.* (5.3) *has two small homoclinic orbits simultaneously at* (Y^*, K^*) *and a large homoclinic orbit on the curve*

$$\overline{C}' = \left\{ (\mu_1, \mu_2) : \mu_2 = -\frac{\gamma(12m\alpha\beta^2 + 12\alpha\beta n_1 + (5q+12\beta)n_2)}{7q(m\alpha\beta + \alpha n_1 + n_2)\tau^*}\mu_2 + \mathcal{O}\left(|\mu_2|^{3/2}\right),\ \mu_2 < 0 \right\}. \tag{6.28}$$

Moreover, if (μ_1, μ_2) *is in the region between the curves* $\overline{H}_2$ *and* $\overline{C}'$, *then Sys.* (5.3) *has three limit periodic orbits: a "large" one and two "small" ones.*

(iv) *Sys.* (5.3) *undergoes a branch of a double limit cycle bifurcation on the curve*

$$\overline{C}_d = \left\{ (\mu_1, \mu_2) : \mu_2 = -\frac{\gamma(3cm\alpha\beta^2 + 3c\alpha\beta n_1 + (q+3c\beta)n_2)}{(-1+3c)q(m\alpha\beta + \alpha n_1 + n_2)\tau^*}\mu_2 + \mathcal{O}\left(|\mu_2|^{3/2}\right),\ \mu_2 < 0 \right\}, \tag{6.29}$$

where the constant $c \approx 0.752$. *Moreover, if* (μ_1, μ_2) *is in the region between the curves* $\overline{C}'$ *and* $\overline{C}_d$, *Sys.* (5.3) *has two large different limit cycles. The outer one is stable, the inner one is unstable, and these two cycles collide on* $\overline{C}_d$.

7. Numerical Simulations

In this section, we give some examples to verify the theoretical results obtained in Section 6. For simplicity, we assume that (0,0) is one of the equilibrium points.

Example 7.1. This example demonstrates the result of Theorem 6.1. Let $\alpha = 1$, $\beta = 0.8$, $\gamma = 0.5625$, $q = 0.9$. Then $k^* = 1.0625$, $\tau^* = 2.66667$. Take

$$I(s) = \tanh(ks) + 0.1s^2. \tag{7.1}$$

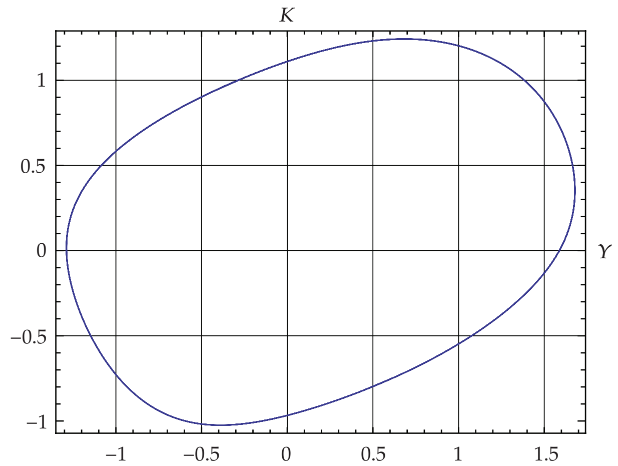

Figure 1: A stable limit cycle is generated when (μ_1, μ_2) is in the region between $\overline{H}$ and $\overline{P}$.

Then (0,0) is the trivial equilibrium point and $i^{(2)} = 0.1 > 0$ and $\delta = -0.800442 < 0$. Simple calculation shows that

$$\overline{H} = \{(\mu_1, \mu_2) : \mu_1 = -0.079918\mu_2,\ \mu_2 > 0\},$$
$$\overline{P} = \{(\mu_1, \mu_2) : \mu_1 = -0.0368852\mu_2,\ \mu_2 > 0\}. \tag{7.2}$$

Take $\mu_1 = -0.00058$, $\mu_2 = 0.01$. Then $k = k^* + \mu_1 = 1.06192, \tau = \tau^* + \mu_2 = 2.67777$, and (μ_1, μ_2) is in the region between $\overline{H}$ and $\overline{P}$. According to Theorem 6.1, Sys. (5.3) has a unique stable periodic orbit (see Figure 1).

Example 7.2. This example supports the result of Theorem 6.2. Let $\alpha = 1$, $\beta = 0.8$, $\gamma = 0.5625$, $q = 0.9$. Then $k^* = 1.0625$, $\tau^* = 2.66667$, and $\delta = -0.800442 < 0$. Take

$$I(s) = ks + 0.001s^3. \tag{7.3}$$

Then $i^{(2)} = 0$, $i^{(3)} = 0.001 < 0$, and hence, the condition of Theorem 6.2 is satisfied. After using the algorithm in Section 6, we have

$$\overline{H}_1 = \{(\mu_1, \mu_2) : \mu_1 = -0.295082\mu_2,\ \mu_2 > 0\},$$
$$\overline{C} = \left\{(\mu_1, \mu_2) : \mu_1 = -0.456455\mu_2 + \mathcal{O}\left(\mu_2^{3/2}\right),\ \mu_2 > 0\right\}. \tag{7.4}$$

Take $\mu_1 = -0.000035$, $\mu_2 = 0.0001$, and hence (μ_1, μ_2) is in the region between $\overline{H}_1$ and $\overline{C}$. Figure 2 shows that there is a limit cycle which is stable according to Theorem 6.2.

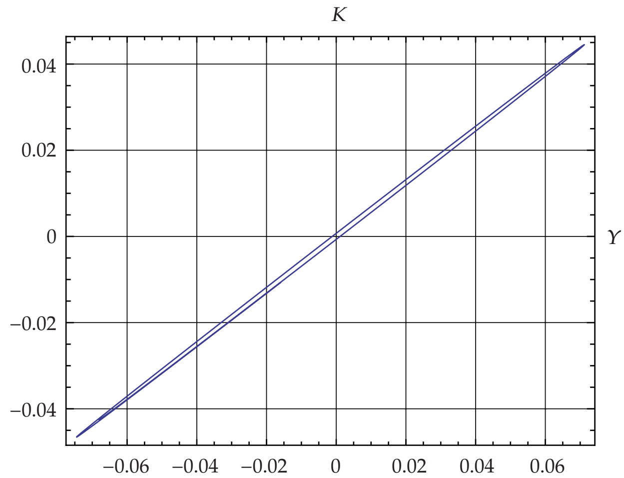

Figure 2: A stable periodic orbit is generated when (μ_1, μ_2) is located in the region between $\overline{H}$ and $\overline{P}$.

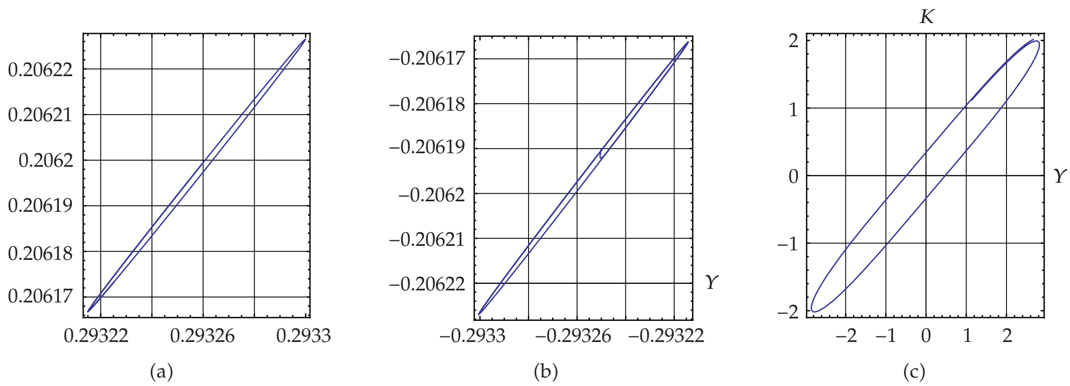

Figure 3: Three limit (two small and one large) cycles are generated when (μ_1, μ_2) is in the region between $\overline{H}_2$ and $\overline{C}'$.

Example 7.3. This example verifies the result of Theorem 6.3. Let $\alpha = 1$, $\beta = 0.5$, $\gamma = 0.5625$, $q = 0.8$. Then $k^* = 0.914063$, $\tau^* = 3.37222$ and $\delta = 0.0233 > 0$. Take

$$I(s) = ks - 0.001s^3, \tag{7.5}$$

and hence, $I''(0) = 0$, $I'''(0) < 0$. Thus the condition of Theorem 6.3 holds. Simple calculation shows

$$\begin{aligned}
\overline{H}_2 &= \{(\mu_1, \mu_2) : \mu_1 = -0.724353\mu_2,\ \mu_2 < 0\}, \\
\overline{C}' &= \left\{(\mu_1, \mu_2) : \mu_1 = -0.99011\mu_2 + \mathcal{O}\left(|\mu_2|^{3/2}\right),\ \mu_2 < 0\right\}, \\
\overline{C}_d &= \left\{(\mu_1, \mu_2) : \mu_1 = -1.09167\mu_2 + \mathcal{O}\left(|\mu_2|^{3/2}\right),\ \mu_2 < 0\right\}.
\end{aligned} \tag{7.6}$$

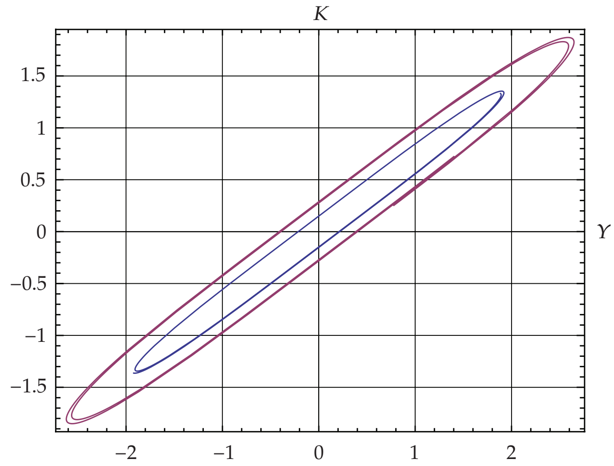

Figure 4: Two large limit cycles are generated when (μ_1, μ_2) is in the region between $\overline{C}'$ and $\overline{C}_d$.

If we take $\mu_1 = 0.0000861$, $\mu_2 = -0.0001$, then (μ_1, μ_2) is in the region between $\overline{H}_2$ and $\overline{C}'$, and hence there are two small limit cycles and a large limit cycle (Figure 3). If we take $\mu_1 = 0.0001$, $\mu_2 = -0.0001$, then (μ_1, μ_2) is in the region between $\overline{C}'$ and $\overline{C}_d$, and hence, there are two large limit cycles (Figure 4).

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