

## Research Article

# Characterizing Growth and Form of Fractal Cities with Allometric Scaling Exponents

**Yanguang Chen**

*Department of Geography, College of Urban and Environmental Sciences, Peking University, Beijing 100871, China*

Correspondence should be addressed to Yanguang Chen, chenyg@pku.edu.cn

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Fractal growth is a kind of allometric growth, and the allometric scaling exponents can be employed to describe growing fractal phenomena such as cities. The spatial features of the regular fractals can be characterized by fractal dimension. However, for the real systems with statistical fractality, it is incomplete to measure the structure of scaling invariance only by fractal dimension. Sometimes, we need to know the ratio of different dimensions rather than the fractal dimensions themselves. A fractal-dimension ratio can make an allometric scaling exponent (ASE). As compared with fractal dimension, ASEs have three advantages. First, the values of ASEs are easy to be estimated in practice; second, ASEs can reflect the dynamical characters of system's evolution; third, the analysis of ASEs can be made through prefractal structure with limited scale. Therefore, the ASEs based on fractal dimensions are more functional than fractal dimensions for real fractal systems. In this paper, the definition and calculation method of ASEs are illustrated by starting from mathematical fractals, and, then, China's cities are taken as examples to show how to apply ASEs to depiction of growth and form of fractal cities.

## 1. Introduction

Dimension is a measurement of space, and measurement is the basic link between mathematical models and empirical research. So, dimension is a necessary measurement for spatial analysis. Studying geographical spatial phenomena of scaling invariance such as cities and systems of cities has highlighted the value of fractal dimension [1–6]. However, there are two problems in practical work. On the one hand, sometimes it is difficult for us to determine the numerical value of fractal dimension for some realistic systems, but it is fairly easy to calculate the ratio of different fractal parameters. On the other hand, in many cases, it is enough to reveal the system's information by the fractal-dimension ratios and it is unnecessary to compute fractal dimension further [7]. The ratio of different dimensions of a fractal can constitute an allometric coefficient under certain conditions. As a parameter

of scale-free systems, the allometric coefficient is in fact a scaling exponent, and the fractal-dimension ratio can be called allometric scaling exponent (ASE).

The use of ASEs for the simple regular fractals in the mathematical world is not very noticeable. But for the quasifractals or random fractals in the real world, the function of ASEs should be viewed with special respect. A city can be regarded as an evolutive fractal. The land-use patterns, spatial form, and internal structure of a city can be modeled and simulated with ideas from fractal geometry [2, 8–14]. There are many kinds of fractal dimensions which can be employed to characterize urban form and structure. The fractal parameters in the most common use include grid dimension, radial dimension, and boundary dimension. Based on the box-counting method, we can estimate the values of boundary dimension and grid dimension at the same time and calculate their ratios to get the ASEs. The scaling exponent can reflect the geographical information that we need. However, it is not convenient to evaluate the fractal dimensions of the complicated systems. In practice, we can directly estimate the ASE values by skipping the calculation of fractal dimensions, and thus gain our ends of spatial analysis.

This work is devoted to discussing how to characterize growth and form of fractal cities with ASEs. The remaining part of this paper is structured as follows. Section 2 presents the basic classification of allometric relations and brings to light the mathematical relationships between allometric scaling and fractal dimension. Section 3 gives the definition of ASE based on regular fractals, including self-similar and self-affine fractals. The concept of allometry is extended becomingly. Section 4 shows the function and use of ASE in urban studies through examples and generalizes this scaling exponent from form to function. Finally, this paper is concluded with a brief summary.

## 2. Allometric Growth and Fractals

### 2.1. The Law of Allometric Growth

The law of allometric growth originated from biological sciences [15, 16], and allometric analysis is introduced to social science by Naroll and Von Bertalanffy [17, 18]. From then on, the formulation gradually becomes a law of the urban geography, describing the relationship observed to be invariable between urban area and population for all cases in which the specified conditions are met [2, 16, 19, 20]. In biology, the allometric rule was defined formally as follows [21, page 247]. “The rate of relative growth of an organ is a constant fraction of the rate of relative growth of the total organism.” Actually it can also be restated in a broad sense as follows [7]. “The rate of relative growth of a part of a system is a constant fraction of the rate of relative growth of the whole or another part of the system.” Where urban systems are concerned, allometric growth can be divided into two types: longitudinal allometry and cross allometry. The cross allometry is also called transversal allometry, which can be divided into two types, too. One is based on the rank-size distribution of cities, and the other is based on hierarchy of cities or cascade structure of urban systems. The relationships among the three types of allometric growth are as follows:

$$\text{Allometric growth} \begin{cases} \text{Longitudinal allometry based on time } (t), \\ \text{Cross allometry} \begin{cases} \text{Cross-sectional allometry based on rank } (k), \\ \text{Hierarchical allometry based on class } (m). \end{cases} \end{cases} \quad (2.1)$$

The allometric equation usually takes the form of power law. Let  $x$  and  $y$  as functions of the variable  $v$  represent two measures of a system. If the relation between  $x$  and  $y$  follows the allometric scaling law, then we have

$$y(v) \propto x(v)^b, \quad (2.2)$$

where  $v$  refers to the variable such as time ( $t$ ), rank ( $k$ ), and class ( $m$ ) and  $b$  refers to the scaling exponent. If we research the process of urban evolvement in different years, then we will get two time series,  $x(t)$  and  $y(t)$ , which represent a city's population, area or perimeter (boundary length), and so forth, in the  $t$ th year. Thus, we have an equation of longitudinal allometry:

$$y(t) \propto x(t)^b. \quad (2.3)$$

If we study  $n$  cities in a region, then we should put these cities in order by ranking them according to population size. Let  $v = k = 1, 2, \dots, n$  be city rank. Then, we will get an equation of cross-sectional allometry in the following form:

$$y(k) \propto x(k)^b. \quad (2.4)$$

Further, we can put these cities into a hierarchy with cascade structure in terms of city size [22]. Let  $v = m = 1, 2, \dots, M$  be city class, and  $M$  represents the bottom class of the hierarchy. Then, we have a hierarchical allometric equation such as

$$y(m) \propto x(m)^b. \quad (2.5)$$

The three kinds of allometric relations can be applied to urban studies. If  $x$  refers to the urban population and  $y$  refers to the corresponding urban area, then the above equations reflect the well-known allometric relations between the urban area and population [2, 16, 23]. If  $x$  represents urban area and  $y$  denotes the corresponding urban boundary length, then the above equations suggest the allometric scaling relations between the urban area and perimeter. The concept of allometric growth reminds us of the scaling relation between urban area and population at first. As a matter of fact, the geometrical relation between urban area and perimeter is a more typical allometric relation. If an urban area is compared to the volume of an animal, then the corresponding urban perimeter can be compared to the animal's epidermal area, and the urban population within the boundary can be compared to the animal's body weight.

The allometric relations between urban area and population are more familiar to geographers, but this paper attaches more importance to the scaling relationship between urban area and perimeter. A problem arises about defining urban boundary. Since urban form can be treated as a fractal body like Fournier dusts [14], where is an unambiguous urban boundary? Since an urban boundary can be treated as a fractal line [2], how can we figure out the length of the boundary? The key is to utilize the resolution of remote sensing (RS) image and scaling range of statistical self-similarity. For given resolution of an RS image or digital map, we can find a continuous close curve around a city figure, and this curve can be defined as urban boundary. Although a city can be theoretically treated as a fractal,

the cities in the real world are actually prefractals rather than proper fractals. There is usually a scaling range on the log-log plot of estimating fractal dimension. The lower limit of the scale-free range suggests an urban boundary, by which we can calculate a length of the prefractal geographical line. A lot of methods and approaches have been developed for defining urban boundaries and estimating the fractal dimension of the boundaries [24, 25]. It is not difficult to constitute the allometric scaling relation between urban perimeter and area.

## 2.2. Fractal Properties of Allometric Growth

Scaling and dimensional analysis actually proceeded from physics, starting with Newton. Nowadays, allometric analysis, scaling analysis, and dimensional analysis have reached the same goal in theoretical exploration by different routes of development [26, 27]. Allometric scaling relation is essentially a kind of geometrical measure relation. According to the principle of dimensional consistency in mathematics, one measure  $x$  is in proportion to another measure  $y$  only when the two measures,  $x$  and  $y$ , have the same dimension as one another [16]. Otherwise, the two measures should be transformed into another two measures with identical dimension. Let the dimension of  $x$  be  $D_x$  and let that of  $y$  be  $D_y$ . Then, we have

$$y(v)^{1/D_y} \propto x(v)^{1/D_x}, \quad (2.6)$$

or

$$y(v) \propto x(v)^{D_y/D_x}. \quad (2.7)$$

Comparing (2.2) with (2.7) shows that

$$b = \frac{D_y}{D_x}. \quad (2.8)$$

This suggests that the scaling exponent,  $b$ , is just the ratio of one dimension,  $D_y$ , to the other,  $D_x$ .

Allometric growth, including longitudinal allometry and cross allometry, can be divided into five types in terms of  $b$  value. This kind of classification of allometry will help us describe growing fractals efficiently. When  $b > 1$ , that is,  $D_y > D_x$ , positive allometry results, and this implies that  $y$  increases at a faster rate than  $x$ . When  $0 < b < 1$ , that is,  $D_y < D_x$ , negative allometry results, and this suggests that  $y$  increases at a slower rate. When  $b = 1$ , that is,  $D_y = D_x$ , isometry results, and this implies that the two variables,  $x$  and  $y$ , increase in linear proportions with respect to one another [16]. When  $b < 0$ , inverse allometry results. When  $b = 0$ , no allometry or isometry results, and this value corresponds to constant proportionality. Isometry is a special case of allometry, and it is originally defined as follows: "the growth rates in different parts of a growing organism are the same." For allometric scaling, a small change in a system can lead to an enormous and disproportionate increase in the size of subsystems. In contrast, isometric scaling means that growth does not lead to any change in geometry of a system.

In the next section we will reveal the allometric scaling relations in fractal bodies. To give a better explanation to allometric scaling of fractals, we should draw a mathematical

analogy between self-similar structure and urban cascade structure. For an urban hierarchy with cascade structure, we can use three exponential functions based on hierarchical structure to characterize it. These three exponential functions can be changed into a set of power functions [28, 29]. For a fractal, we can take a set of power laws to describe it, and the set of power laws can also be transformed into a set of exponential laws based on the steps of fractal generation [22]. In practice, the exponential function for continuous distribution can be substituted with geometric series in the discrete framework.

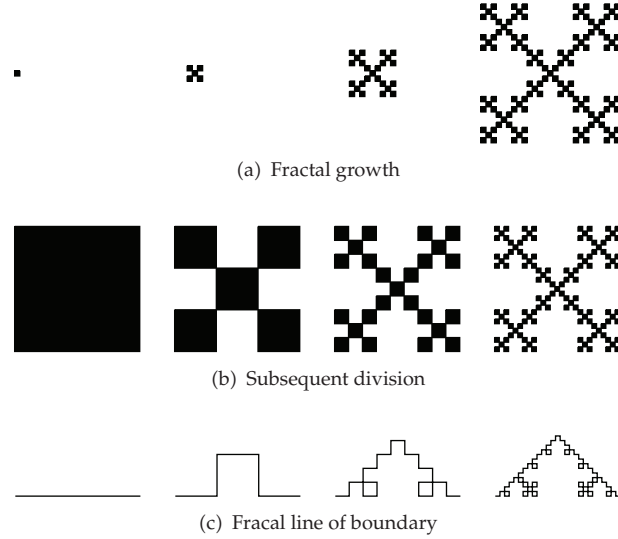
The relation between fractal geometry and Euclidean geometry is of “duality”. For a Euclidean geometrical body, the dimension is known (0, 1, 2, or 3), but its direct measures such as length, area, and volume are unknown without calculation. On the contrary, for a fractal geometrical body, the direct measures, including length, area, and volume, are always known, but its dimension required to be calculated. For example, the dimension of a rectangle is  $d = 2$ , which can be known by common sense. However, the area of the rectangle cannot be known before the side lengths are measured. In contrast, the area of the Sierpinski gasket is zero and its inner boundary length is infinite, which can be known before implementing computation. However, the dimension of the gasket could not be known without any measuring and calculation. For the regular fractals in mathematics, the concepts of length, area, and volume begin to make little sense. It is dimension that reflects the quality and quantity of fractal bodies. Of course, the dual relation is valid only for the proper fractals. As for the prefractals or the statistical fractals, the conventional measure is still serviceable. Precisely because of this, we can investigate the allometric scaling relation of cities by means of length, area, and population size among others.

However, there are no any real fractals in the real world. The so-called fractals in natural and human systems are of scaling invariance only within certain scale range. They are actually prefractals or quasifractals in the statistical sense. Therefore, we need the concepts of length, area, or even volume to build the allometric models although a fractal body has no length (e.g., it is 0 for the Cantor set, or infinite for the Koch curve) or area (it equals 0) in theory. A solution to this problem is to rely on the scale concept: we examine fractals within certain scaling range. For a fractal hierarchy with infinite classes ( $m = 1, 2, 3, \dots$ ), we can investigate the finite classes, say, the first  $M$  classes ( $m = 1, 2, 3, \dots, M$ ). Here  $m$  denotes the order of class, and  $M$  is a positive integer indicative of the last class. In this instance, it is more suitable to use ASE rather than fractal dimension to characterize the real-world fractal phenomena. The reason is that the fractal dimension calculation requires the radius of covering balls or the side length of boxes to tend towards infinitesimal in theory, but there exists no infinitesimal length scale in reality. The allometric scaling is out of this restriction—an allometric analysis can be made within a certain scaling range.

### 3. Allometric Scaling of Fractals

#### 3.1. Allometric Scaling of Self-Similar Fractals

A fractal usually suggests the allometric scaling relations in the broad sense, and this can be illustrated with several classical fractal patterns. Figure 1 shows a growing fractal, which was constructed twenty years ago [30, 31]. In urban studies, this kind of growing fractal is sometimes employed to model or even simulate urban growth by introducing chance factors [2, 23, 32]. Indeed, there is an analogy between the way a fractal develops and the way a city grows. The fractal of isotropic growth can be made through two ways: one is ceaseless cumulative process, which represents stepwise expansion of urban population (Figure 1(a));



**Figure 1:** A growing fractal and its self-similar boundary line (the first four steps are obtained by Vicsek [31], Longley et al. [23], Frankhauser, [33]).

the other is endless subdivision process, which represents gradual aggregation of regional population (Figure 1(b)). Finally, these two fractal evolutions reach the same goal by different routes. The similarity dimension of this fractal is

$$D_a = -\frac{\ln(N_{m+1}/N_m)}{\ln(s_{m+1}/s_m)} = \frac{\ln 5}{\ln 3} \approx 1.465, \quad (3.1)$$

where  $N_m = 5^{m-1}$ ,  $s_m = 3^{1-m}$ ,  $D_a$  denotes the fractal dimension,  $m$  is the order indicating operation step, and  $N_m$  refers to the number of self-similar parts with length scale  $s_m$  in the  $m$ th step needed to cover the whole structure. For this regular fractal without overlapped parts, similarity dimension equals the Hausdorff dimension and the box dimension.

The boundary of the growing fractal is a kind of the quadratic Koch curve (Figure 1(c)), and its dimension is

$$D_l = -\frac{\ln(N_{m+1}/N_m)}{\ln(s_{m+1}/s_m)} = \frac{\ln 5}{\ln 3} \approx 1.465, \quad (3.2)$$

in which  $D_l$  denotes the dimension of fractal boundary; the remaining notation is the same as in the foregoing formula. We can see that the dimension of the growing fractal form equals that of the growing fractal curve in this special case [31].

For the growing fractal of infinite accumulative process displayed in Figure 1(a), the longitudinal allometry is the same as the cross allometry. We can use a geometrical series to describe the number, area, and perimeter of the self-similar copies of step  $m$  in the form

$$f_m = f_1 5^{m-1}, \quad (3.3)$$

where  $m$  denotes the step number,  $f_m$  represents the number of self-similar parts ( $f_m = N_m$ ), the area ( $f_m = A_m$ ), or the perimeter ( $f_m = L_m$ ) of the fractal copies in the  $m$ th step, and  $f_1 = 1$  is a parameter. Then, the allometric scaling relation between area and perimeter is

$$L_m \propto A_m^b = A_m^{D_l/D_a}. \quad (3.4)$$

The ratio of boundary dimension  $D_l$  to form dimension  $D_a$  is 1; that is,  $b = D_l/D_a = 1$ , which indicates an isometric scaling relation. The parameter  $b$  refers to the scaling exponent. Actually it is very easy to describe the fractal growth in Figure 1(a) using an allometric relation. The longitudinal allometry is the same as the cross allometry, and the scaling exponent equals unity.

For the fractal growth of continuous subdivision process illustrated by Figure 1(b), the case is different. First of all, let us examine the longitudinal allometry. We need three equations to describe the number, area, and perimeter of self-similar copies in the  $m$ th step, and the results are as follows:

$$N_m = N_1 5^{m-1}, \quad (3.5)$$

$$A_m = A_1 \left(\frac{5}{9}\right)^{m-1}, \quad (3.6)$$

$$L_m = L_1 \left(\frac{5}{3}\right)^{m-1}. \quad (3.7)$$

Obviously, if  $m$  approaches to infinity, both the number of fractal copies and the length of fractal-boundary will diverge, while the area of fractal body will tend to zero.

Where the growing process is concerned, the scaling relations between fractal-part number and boundary length can be derived by taking the logarithm of (3.5) and (3.7) and by eliminating  $m - 1$ , and the result is

$$N_m \propto L_m^a = L_m^{\ln(5)/\ln(5/3)}. \quad (3.8)$$

The scaling exponent  $a = \ln(5)/\ln(5/3) \approx 3.151$ . The scaling of fractal-part number versus boundary length is a positive allometry. Conversely, the scaling of fractal-boundary length versus fractal-part number is a negative allometry. The allometric scaling relation between the fractal-boundary length and fractal-part area is in the following form:

$$L_m \propto A_m^b = A_m^{\ln(5/3)/\ln(5/9)}. \quad (3.9)$$

The scaling exponent  $b = \ln(5/3)/\ln(5/9) \approx -0.869$ . The scaling phenomenon with an exponent  $b < 0$  is so-called inverse allometric growth.

Next, we should investigate the cross allometry. In fact, the cross allometry based on Figure 1(b) is the same as that based on Figure 1(a). Suppose that the area of each fractal copy



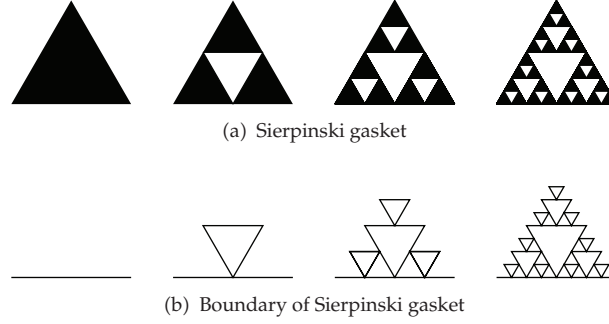


Figure 2: The Sierpinski gasket and its interior boundary lines (the first four steps).

is one unit. Then, the area and boundary length of fractal copies in the  $m$ th step of Figure 1(b) can be formulated as

$$A_m = A_1 5^{m-1}, \quad (3.10)$$

$$L_m = L_1 5^{m-1}. \quad (3.11)$$

Thus, we have

$$L_m \propto A_m^{\ln(5)/\ln(5)} = A_m^{[\ln(5)/\ln(3)]/[\ln(5)/\ln(3)]} = A_m^{D_l/D_a}. \quad (3.12)$$

This suggests that the cross allometry based on Figure 1(b) is an isometry because the scaling exponent is  $b = D_l/D_a = 1$ . Indeed, (3.10) to (3.12) hold for Figure 1(a), but for the inner mapping procedure of Figure 1(b), the surface of the elements changes at each step when going on with iteration.

Another well-known example of fractals is the Sierpinski gasket which is displayed in Figure 2(a) [5], and its interior border curve is also a fractal line, which is shown in Figure 2(b). If we introduce a chance factor indicating randomness, then the interior boundary can be used to simulate random walk curve. The fractal dimension of the Sierpinski gasket is

$$D_a = -\frac{\ln(N_{m+1}/N_m)}{\ln(s_{m+1}/s_m)} = \frac{\ln 3}{\ln 2} \approx 1.585, \quad (3.13)$$

where  $N_m = 3^{m-1}$  and  $s_m = 2^{1-m}$ . If we use the box-counting method to evaluate the fractal dimension of its interior boundary, then we have minimum box number such as

$$N_m = 3N_{m-1} + 2^{m-1}, \quad (3.14)$$

where  $m = 1, 2, 3, \dots$  and  $N_0 = 0$ . By recurrence, we get

$$N_m = \sum_{j=0}^{m-1} (3^{m-1-j} 2^j) = 3^{m-1} \sum_{j=0}^{m-1} \left[ \left( \frac{2}{3} \right)^j \right] \xrightarrow{m \rightarrow \infty} 3^{m-1} \frac{1}{1-2/3} = 3^m, \quad (3.15)$$



where  $j = 1, 2, \dots, m-1$ . This suggests that, when  $m$  becomes large enough,  $N_m$  will approach  $3^m$ . So, the box dimension of the interior boundary is

$$D_l = -\frac{\ln N_m}{\ln s_m} = \frac{m \ln 3}{(m-1) \ln 2} \xrightarrow{m \rightarrow \infty} \frac{\ln 3}{\ln 2} \approx 1.585. \quad (3.16)$$

It is easy to understand the cross allometry of the Sierpinski gasket. For simplicity, let us consider the result of step  $m$  by ignoring its growing process. The pattern is in fact a prefractal. Assuming that the area of the smallest fractal part in the geometrical figure is one unit, we can get an allometric scaling relation between the fractal-part area and the interior boundary length as

$$L_m \propto A_m^{\ln(3)/\ln(3)} = A_m^{[\ln(3)/\ln(2)]/[\ln(3)/\ln(2)]} = A_m^{D_l/D_a}. \quad (3.17)$$

This suggests that, for the cross allometric relation, the scaling exponent is  $b = D_l/D_a = 1$ , indicating an isometric relation.

The longitudinal allometry of the gasket is somewhat different. In the  $m$ th step, the number, area, and interior boundary of the self-similar copies can be formulated as follows:

$$\begin{aligned} N_m &= N_1 3^{m-1}, \\ A_m &= A_1 \left(\frac{3}{4}\right)^{m-1}, \\ L_m &= L_1 \left(\frac{5}{2}\right)^{m-1}. \end{aligned} \quad (3.18)$$

So the allometric scaling relation between fractal-copy number and interior boundary length can be expressed as

$$N_m \propto L_m^a = L_m^{\ln(3)/\ln(5/2)}. \quad (3.19)$$

Thus, we have a scaling exponent  $a = \ln(3)/\ln(5/2) \approx 1.199$ . This is a kind of positive allometry. The relations between the interior boundary length and the fractal-copy area can be written as

$$L_m \propto A_m^b = A_m^{\ln(3/2)/\ln(3/4)}. \quad (3.20)$$

Therefore,  $b = \ln(3/2)/\ln(3/4) \approx -3.185$ , and this is a kind of inverse allometry.

Generally speaking, we can describe a fractal in two ways based on allometric relations: one is longitudinal allometry, and the other is cross allometry. For the longitudinal

allometry indicative of growing process, the measures of number, area, and length can be expressed with three exponential equations as follows:

$$\begin{aligned} N_m &= N_1 r_n^{m-1}, \\ A_m &= A_1 r_a^{1-m}, \\ L_m &= L_1 r_l^{m-1}, \end{aligned} \quad (3.21)$$

in which the parameters  $r_n = N_{m+1}/N_m$ ,  $r_a = A_m/A_{m+1}$ , and  $r_l = L_{m+1}/L_m$  represent number ratio, area ratio, and length ratio, respectively. The proportionality coefficients  $N_1$ ,  $A_1$ , and  $L_1$  are constant (commonly we have  $N_1 = 1$ ,  $A_1 = 1$ , and  $L_1 = 1$ ). Then we derive three power laws or negative power laws from (3.21) [29]. In practice, we can make use of any two of the following three power laws:

$$\begin{aligned} N_m &= \left( N_1 A_1^{\ln r_n / \ln r_a} \right) A_m^{-\ln r_n / \ln r_a} \propto A_m^{-a}, \\ L_m &= \left( L_1 A_1^{\ln r_l / \ln r_a} \right) A_m^{-\ln r_l / \ln r_a} \propto A_m^{-b}, \\ N_m &= \left( N_1 L_1^{-\ln r_n / \ln r_l} \right) L_m^{\ln r_n / \ln r_l} \propto L_m^c. \end{aligned} \quad (3.22)$$

The parameters

$$a = \frac{\ln r_n}{\ln r_a}, \quad b = \frac{\ln r_l}{\ln r_a}, \quad c = \frac{\ln r_n}{\ln r_l} \quad (3.23)$$

represent different ASEs.

For the cross allometry indicating hierarchical structure, we can derive an area-length scaling relation. If  $m$  value is limited and the area of the basic fractal parts is one unit, then the area measure ( $A_m$ ) and the number measure ( $N_m$ ) can be mathematically regarded as equivalent to one another. Thus, the area or number and length will be formulated as

$$\begin{aligned} A_m &= A_1 r_a^{m-1}, \\ L_m &= L_1 r_l^{m-1}. \end{aligned} \quad (3.24)$$

Based on (3.24), an allometric scaling relation between the area and the length is

$$L_m = \left( L_1 A_1^{-\ln r_l / \ln r_a} \right) A_m^{\ln r_l / \ln r_a} \propto A_m^{D_l/D_a}. \quad (3.25)$$

Here the scaling exponent is defined by  $b = \ln r_l / \ln r_a = D_l/D_a$ .

For the same scaling relation between area and length, there exists a difference between the longitudinal allometry and the cross allometry. Sometimes the longitudinal allometry is an inverse allometry, while the cross allometry is always a positive allometry or a negative allometry. The allometric scaling relations are very useful for us to characterize the random

fractals. In fractal theory, if  $m$  is large enough, then the length  $L_m$  approaches infinity while the area  $A_m$  tends to zero. However, in allometric analysis, we usually take  $m = 1, 2, \dots, M$ , and  $M$  is a finite number. In this instance, both area  $A_m$  and length  $L_m$  are limited values. This suggests that a fractal concept is defined under an extreme condition, while the concept of allometric growth is defined within certain scale range.

In geographical studies, the scaling exponents have been associated with allometry and dimension in the context of fractal properties of cities for many years. A comparison of relations between urban area and border length, between urban population and radius, between urban border length and radius, and so forth was discussed by Longley et al. [34]. In the novel paper on form following function, Batty and Kim [35] once discussed several relationships among scaling law, allometry, and urban form. Frankhauser [3] tackled the relations between urban perimeter and area as well as surface classes. Imre and Bogaert [36] presented that the scaling exponent of the urban area-perimeter relation is just ratio of the fractal dimension of urban boundary to that of urban form. More recently, Benguigui et al. [11] and Thomas et al. [14] considered also the relation between perimeter and area for cities by relating the scaling exponents with fractal dimension. The innovative point of this paper rests with that all these relations will be examined within the theoretical framework of hierarchy of cities.

### 3.2. Allometric Scaling of Self-Affine Fractals

The self-affine fractals can also be described with allometric scaling equations. If the way of isotropic growth is turned into the way of anisotropic growth, then the self-similar growing fractal in Figure 1(a) will become a self-affine growing fractal displayed in Figure 3 [30, 31]. To characterize this kind of self-affine fractals, we need the following two equations:

$$\begin{aligned} N(S) &= S_h^{D_h}, \\ N(S) &= S_v^{D_v}, \end{aligned} \quad (3.26)$$

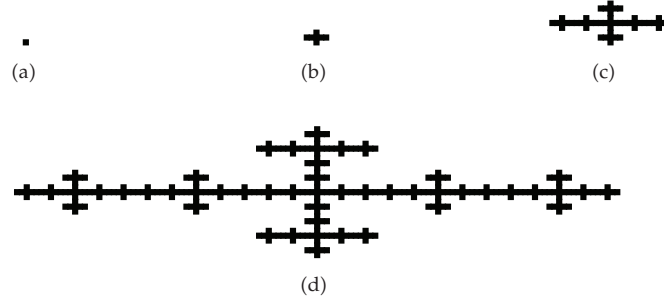
where  $N$  is the number of self-affine copies in a certain step,  $S_h$  refers to the length scale of horizontal direction,  $S_v$  refers to the length scale of vertical direction,  $D_h$  denotes the fractal dimension based on horizontal measurement, and  $D_v$  represents the fractal dimension based on vertical measurement. Obviously,  $N = 7^{m-1}$ ,  $S_h = 5^{m-1}$ , and  $S_v = 3^{m-1}$ , in which  $m = 1, 2, 3, \dots$ . Thus, we have two fractal dimension values

$$D_h = \frac{\ln N(S)}{\ln S_h} = \frac{\ln 7}{\ln 5} = 1.209, \quad D_v = \frac{\ln N(S)}{\ln S_v} = \frac{\ln 7}{\ln 3} = 1.771. \quad (3.27)$$

From (3.26), an allometric scaling relation can be derived as follows:

$$S_h = S_v^{D_v/D_h} = S_v^{\ln(5)/\ln(3)} = S_v^\alpha. \quad (3.28)$$

The scaling exponent of the two growing directions is  $\alpha = D_v/D_h = 1.465$  [37]. It is evident that, for the growing fractals, isotropy indicates isometry (Figure 1(a)), while anisotropy suggests allometry of different directions (Figure 3).



**Figure 3:** A sketch map of the self-affine growing fractal (the first four steps are obtained by Vicsek [31]).

The allometric scaling relations between fractal-part area and boundary length can be characterized from the perspective of fractal growth. The area ( $A_m$ ) and length ( $L_m$ ) in the  $m$ th step can be formulated as

$$A_m = A_1 7^{m-1}, \quad (3.29)$$

$$L_m = L_1 7^{m-1}. \quad (3.30)$$

For simplicity, only one side of boundary length is considered in (3.30). The result implies that the scaling relation between area and length is an isometry; that is,  $L_m \propto A_m$ . But in the real world, the case is more complicated, and the area-length scaling may suggest an allometric growth.

The relation between area and perimeter of the self-affine fractal can be expressed by a linear equation. Based on the fractal structure in Figure 3, the perimeter  $P_m$  can be formulated as

$$P_m = 2L_m + 2. \quad (3.31)$$

Thus, we have

$$P_m = \left( \frac{2L_1}{A_1} \right) A_m + 2. \quad (3.32)$$

The area-perimeter relation of the fractal in question is an isometry. This suggests that linear relation sometimes represents a kind of isometry.

## 4. Empirical Analysis

### 4.1. The Scaling Relation between Urban Area and Boundary

The scaling relation between urban area and perimeter bears an analogy with the allometric relation of fractal growth. Let us take China's cities as an example to illuminate this question. The processed data came from the study by Wang et al. [38], who estimated the fractal dimension of the boundaries of 31 megacities in China—a megacity is the city with a

population of more than 1,000,000. The approach to evaluation of fractal dimension is grid coverage method, which is similar to the box-counting method. In fact, the original data of Wang et al. [38] are from the Institute of Geographical Sciences and Natural Resources Research (IGSNRR), Chinese Academy of Science (CAS), China. By means of remote-sensing data and Geographical Information System (GIS), IGSNRR built the national resources and environmental database based on the maps with a scale of 1:100,000. The database mainly includes the land-use data in 2000, which is inputted in 2000, and data in 1990, which is inputted in 2001. By using the results from the study by Wang et al. [38], we can explain the allometric scaling relations between urban area and perimeter.

In the digital map, we can use a set of grid consisting of uniform squares to cover the urban figure. Changing the side length of squares of the grid  $s$ , the number of the squares occupied by urban boundary,  $L(s)$ , and the number of the squares occupied by urban figure,  $N(s)$ , will vary correspondingly. The urban area  $A(s)$  can be defined by square number,  $N(s)$ , that is,  $A(s) = N(s)$ , and the boundary length is represented by the number  $L(s)$ . If the relation between urban area and perimeter follows the allometric scaling law, then, according to (3.25), we have

$$\ln L(s) = C + \frac{D_l}{D_a} \ln A(s) = C + b \ln A(s), \quad (4.1)$$

where  $C$  is a constant,  $D_a$  refers to the fractal dimensions of urban form, and  $D_l$  to the fractal dimensions of urban boundary. Thus, the ASE value is given by  $b = D_l/D_a$ . Wang et al. [38] changed the side length of squares of the grid for 19 times, and the minimum area of the square is  $200 \times 200 \text{ m}^2$ . The scaling exponents of the 31 megacities in China are evaluated by using the advanced language of GIS software (Table 1). Assuming that the dimension of urban form  $D_a = 2$ , Wang et al. [38] estimated the fractal dimension of urban boundary, and the result is  $D_l = D_a b = 2b$  (see "boundary dimension 1" in Table 1). The information of spatiotemporal evolution of China's cities from 1990 to 2000 can be revealed through ASE values.

The true dimension of urban form, however, is expected to be less than 2; namely  $D_a < 2$ . Theoretically, a city boundary has an analogy with the triadic Koch curve, which has a fractal dimension  $D_l \approx 1.262$ . Empirical analyses and computer simulation have shown that the fractal dimension of urban boundary is close to the Koch curve's dimension on the average [25, 39]. On the other hand, urban growth and form can be modeled by diffusion-limited aggregation (DLA) model and dielectric breakdown model (DBM), and the average fractal dimension  $D_a$  is close to 1.701 [2, 8]. The DBM-simulation process has been well illustrated by Batty [9]. By using the value  $D_a = 1.701$ , we can reevaluate the boundary dimension of the 31 cities in China (see "boundary dimension 2" in Table 1). Therefore, the scaling exponent of urban area-perimeter relation is expected to be  $b \approx 1.262/1.701 \approx 0.742$ . Actually, the average value of ASEs of the 31 megacities is about 0.742 in 1990, and around 0.727 in 2000. In the sense of statistical average, the results approach the theoretical expectation on the whole.

A question may be put as follows: what are the real values of the boundary dimension and form dimension of China's cities? We cannot know them by the data from the study by Wang et al. [38]. For example, the boundary dimension of Beijing city in 2000 may be about 1.444, may be about 1.228, and may be other numerical values. We cannot make sure. However, we know the approximate value of ASE; that is,  $b \approx 0.722$  (Table 1). According to (2.8), the higher value of ASE suggests higher boundary dimension or lower form dimension.

**Table 1:** The fractal dimension values of urban boundaries and ASEs of the area-perimeter relations of 31 megacities in China in 1990 and 2000.

| City         | 1990                     |                                 |                                  | 2000                     |                                 |                                  |
|--------------|--------------------------|---------------------------------|----------------------------------|--------------------------|---------------------------------|----------------------------------|
|              | Scaling exponent ( $b$ ) | Boundary dimension 1 ( $D_1$ )* | Boundary dimension 2 ( $D_1$ )** | Scaling exponent ( $b$ ) | Boundary dimension 1 ( $D_1$ )* | Boundary dimension 2 ( $D_1$ )** |
| Anshan       | 0.735                    | 1.469                           | 1.250                            | 0.690                    | 1.380                           | 1.174                            |
| Beijing      | 0.751                    | 1.502                           | 1.277                            | 0.722                    | 1.444                           | 1.228                            |
| Changchun    | 0.702                    | 1.404                           | 1.194                            | 0.701                    | 1.401                           | 1.192                            |
| Changsha     | 0.766                    | 1.532                           | 1.303                            | 0.763                    | 1.526                           | 1.298                            |
| Chengdu      | 0.838                    | 1.676                           | 1.425                            | 0.837                    | 1.674                           | 1.424                            |
| Chongqing    | 0.753                    | 1.505                           | 1.281                            | 0.723                    | 1.446                           | 1.230                            |
| Dalian       | 0.745                    | 1.489                           | 1.267                            | 0.737                    | 1.474                           | 1.254                            |
| Fushun       | 0.706                    | 1.411                           | 1.201                            | 0.683                    | 1.366                           | 1.162                            |
| Guangzhou    | 0.702                    | 1.403                           | 1.194                            | 0.772                    | 1.544                           | 1.313                            |
| Guiyang      | 0.874                    | 1.748                           | 1.487                            | 0.871                    | 1.742                           | 1.482                            |
| Hangzhou     | 0.800                    | 1.599                           | 1.361                            | 0.783                    | 1.565                           | 1.332                            |
| Harbin       | 0.685                    | 1.369                           | 1.165                            | 0.654                    | 1.307                           | 1.112                            |
| Jilin        | 0.712                    | 1.424                           | 1.211                            | 0.716                    | 1.432                           | 1.218                            |
| Jinan        | 0.717                    | 1.433                           | 1.220                            | 0.732                    | 1.463                           | 1.245                            |
| Kunming      | 0.794                    | 1.588                           | 1.351                            | 0.736                    | 1.472                           | 1.252                            |
| Lanzhou      | 0.741                    | 1.482                           | 1.260                            | 0.736                    | 1.471                           | 1.252                            |
| Nanchang     | 0.727                    | 1.454                           | 1.237                            | 0.751                    | 1.502                           | 1.277                            |
| Nanjing      | 0.785                    | 1.569                           | 1.335                            | 0.747                    | 1.494                           | 1.271                            |
| Qingdao      | 0.689                    | 1.377                           | 1.172                            | 0.653                    | 1.305                           | 1.111                            |
| Qiqihaer     | 0.678                    | 1.355                           | 1.153                            | 0.670                    | 1.340                           | 1.140                            |
| Shanghai     | 0.741                    | 1.481                           | 1.260                            | 0.711                    | 1.422                           | 1.209                            |
| Shenyang     | 0.650                    | 1.300                           | 1.106                            | 0.639                    | 1.278                           | 1.087                            |
| Shijiazhuang | 0.786                    | 1.571                           | 1.337                            | 0.733                    | 1.466                           | 1.247                            |
| Taiyuan      | 0.777                    | 1.554                           | 1.322                            | 0.769                    | 1.538                           | 1.308                            |
| Tangshan     | 0.750                    | 1.500                           | 1.276                            | 0.728                    | 1.456                           | 1.238                            |
| Tianjin      | 0.688                    | 1.376                           | 1.170                            | 0.678                    | 1.356                           | 1.153                            |
| Urumchi      | 0.724                    | 1.447                           | 1.232                            | 0.721                    | 1.441                           | 1.226                            |
| Wuhan        | 0.738                    | 1.475                           | 1.255                            | 0.747                    | 1.494                           | 1.271                            |
| Xian         | 0.731                    | 1.461                           | 1.243                            | 0.683                    | 1.366                           | 1.162                            |
| Zhengzhou    | 0.753                    | 1.506                           | 1.281                            | 0.713                    | 1.426                           | 1.213                            |
| Zibo         | 0.763                    | 1.525                           | 1.298                            | 0.747                    | 1.493                           | 1.271                            |
| Average      | 0.742                    | 1.483                           | 1.262                            | 0.727                    | 1.454                           | 1.237                            |

\*The scaling exponent and the first kind of fractal dimension values (boundary dimension 1) were estimated and provided by Dr. Xinsheng Wang, who assumed that the dimension of urban form is  $D_a = 2$  (see [38]). \*\*The second kind of fractal dimension values (boundary dimension 2) was estimated through the scaling exponent values by this paper's author, who assumed that the dimension of urban form is  $D_a = 1.701$ .

Where geographical space is concerned, the ASE values of Northern China's cities are higher than those of Southern China's cities in the mass. The landform of Northern China is mainly plain, while Southern China is principally of mountainous terrain. Generally speaking, the form dimensions of the cities in plain are higher than those of cities in mountainous region. For boundary dimension, the opposite is true. The cities with the highest ASE values are in northeastern plain of China, such as Shenyang and Harbin, while the cities with the lowest ASE values are in southwestern mountainous region, such as Guiyang and Chengdu. Where geographical evolution is concerned, as a whole, the ASE values in 2000 are lower than those in 1990. As we know, the boundary dimension of a city becomes lower and lower over time [25]. For the form dimension, the opposite is true [10]. This suggests that the ASE values of urban boundary and form tend to descend with the lapse of time. The function and use of a spatial measure can be thrown out by comparison and relation. ASE is just the result of comparing or relating one fractal dimension with the other fractal dimension.

#### ***4.2. The Scaling Relations between Urban Area and Population***

The allometric scaling analysis of urban form can be generalized to the relation between urban area and population. The allometric model of urban area-population scaling can be used to predict regional population growth [40]. Let us take China's system of cities as example to illustrate the pattern of allometric relation. We will take the national capital, Beijing, as example to make a longitudinal allometric analysis and employ the system of cities in China as another example to make a cross allometric analysis. The original data came from the Ministry of Housing and Urban-Rural Development of China. The two basic measures, urban population and built-up area, are used as variables. For Chinese, the "built-up area" is also called as "surface area of built district". In fact, the term "urban area" is a concept of administrative sense in China. There is no certain relationship between urban area and urban landscape; therefore, the concept of urban area cannot reflect urban form effectively. The built-up area in Chinese is similar to the concept of urbanized area in the western world. Perhaps the former is smaller than the latter. In short, it is built-up area rather than urban area that is suitable for us to make an allometric analysis for China's cities. From now on, "urban area" will be used to mean "built-up area" in the context.

First of all, we carry out a longitudinal allometric analysis of Beijing's growth. The data are urban population and built-up area from 1991 to 2005 (Table 2). Through the log-log plot we can find that city area and population from 1991 to 2004 satisfy the allometric scaling relation in the mass (Figure 4). However, the data point of 2005 is an exceptional value by reason of demography (see the appendix). So far, there has been no strict definition for cities, and both urban population and urban area are varied frequently in China. The alteration is sometimes caused by administrative factors rather than urban growth itself. From 1991 to 2000, the land use of Beijing is grimly restricted with urban policy. From 2001 to 2003, urban region of the city was enlarged suddenly by governmental behavior instead of urban natural growth. Since 2004, urban land use was restricted again so that Beijing's area is not proportional to its population size. If the abnormal variation is on the small side, then it could not influence the appearance of the statistical law of urban evolvement. However, if the change is too large, then the scaling relation will be broken. The abovementioned outliers will appear if administrative factors disturb urban development violently.

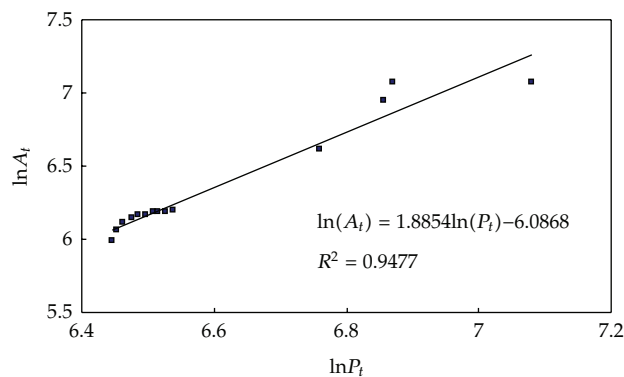
Excluding the data of 2005 from our consideration according to the value of double standard error, we can make a regression analysis by using the data from 1991 to 2004. On the whole, the process of urban growth conforms to the allometric scaling law to some



**Table 2:** The urban area and population of Beijing from 1991 to 2005.

| Year    | Original data                   |  | Logarithmic value |           | Standardization       |                       |
|---------|---------------------------------|--|-------------------|-----------|-----------------------|-----------------------|
|         | Urban population $P_t$ (10,000) | Built-up area $A_t$ (km <sup>2</sup> ) | $\ln P_t$         | $\ln A_t$ | $\text{Sdz}(\ln P_t)$ | $\text{Sdz}(\ln A_t)$ |
| 1991    | 629.6                           | 397.4                                  | 6.445             | 5.985     | -0.772                | -1.028                |
| 1992    | 634.7                           | 429.4                                  | 6.453             | 6.062     | -0.742                | -0.843                |
| 1993    | 640.8                           | 454.1                                  | 6.463             | 6.118     | -0.707                | -0.709                |
| 1994    | 649.6                           | 467.0                                  | 6.476             | 6.146     | -0.657                | -0.642                |
| 1995    | 656.1                           | 476.8                                  | 6.486             | 6.167     | -0.620                | -0.592                |
| 1996    | 662.8                           | 476.8                                  | 6.496             | 6.167     | -0.582                | -0.592                |
| 1997    | 670.0                           | 488.1                                  | 6.507             | 6.191     | -0.542                | -0.536                |
| 1998    | 675.3                           | 488.3                                  | 6.515             | 6.191     | -0.513                | -0.535                |
| 1999    | 682.4                           | 488.3                                  | 6.526             | 6.191     | -0.474                | -0.535                |
| 2000    | 690.9                           | 490.1                                  | 6.538             | 6.195     | -0.429                | -0.526                |
| 2001    | 861.4                           | 747.8                                  | 6.759             | 6.617     | 0.387                 | 0.485                 |
| 2002    | 949.7                           | 1043.5                                 | 6.856             | 6.950     | 0.748                 | 1.282                 |
| 2003    | 962.7                           | 1180.1                                 | 6.870             | 7.073     | 0.799                 | 1.577                 |
| 2004    | 1187.0                          | 1182.3                                 | 7.079             | 7.075     | 1.573                 | 1.581                 |
| 2005    | 1538.0                          | 1200.0                                 | 7.338             | 7.090     | <b>2.532</b>          | 1.617                 |
| Average | 806.1                           | 667.3                                  | 6.654             | 6.415     | 0.000                 | 0.000                 |
| Stdev   | 260.3                           | 313.4                                  | 0.270             | 0.418     | 1.000                 | 1.000                 |

Original data source: The Ministry of Housing and Urban-Rural Development of China and 1991–2005 *Statistic Annals of China's Urban Construction*.

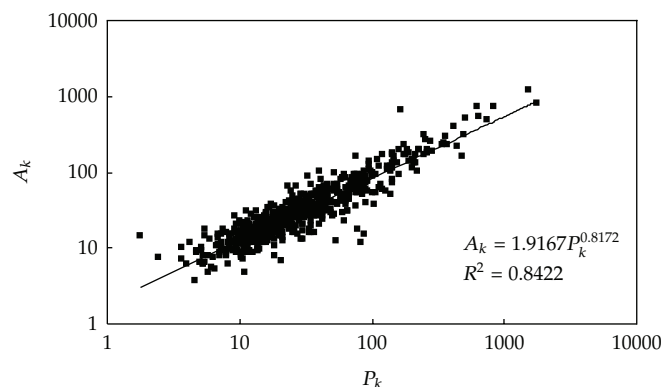


**Figure 4:** The allometric scaling relation between the built-up area and urban population of Beijing (1991–2004).

extent (Figure 4). A least squares calculation of 14-year data yields the following scaling model of longitudinal allometry:

$$A_t = 0.002P_t^{1.885}, \quad (4.2)$$

where  $t$  denotes year ( $t = 1991, 1992, \dots, 2004$ ) and  $A_t$  and  $P_t$  are built-up area and urban population of year  $t$ , respectively. The scaling exponent is estimated as about  $b \approx 1.885$ , and

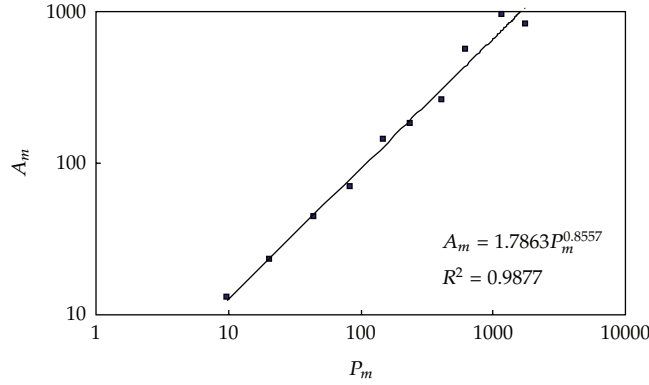


**Figure 5:** The allometric scaling relation between built-up area and urban population of China's cities (2005).

the goodness of fit is  $R^2 \approx 0.948$ . Compared with the urban population, the city area expanded in the mode of positive allometry. That is to say, if the populations of Beijing increase a unit, the city area will increase more than a unit. In other words, the urban area grows quicker than the urban population. This is an allometric mode of wasteful land use, and the scaling exponent suggests that the urban expansion of Beijing should be restricted by taking strong economic measures.

It will not be surprising if some readers doubt the result from the Beijing case. The effect of fitting Beijing's dataset to (2.3) is not very satisfying. The application of the scaling law to Beijing is mainly based on an apriori idea rather than some statistical evidences. The apriori idea is that urban systems should follow but sometimes offend the law of allometric growth [41]. In fact, human systems are different from physical systems to a degree. The empirical laws on physical systems are of spatiotemporally translational symmetry, while the empirical laws on human systems are not of translational symmetry in both space and time. For physical systems, a counterexample or exceptional case is enough to overthrow an empirical law. However, for human systems, a few counterexamples or exceptional cases are not enough to overrule any empirical law which is supported by many observational data. Just because of this, August Lösch, the well-known German economist, once said that if a mathematical model, supported by many cases does not agree with reality, it may be reality rather than the mathematical model that is wrong (This opinion refers to a letter from Professor Michael Woldenberg at State University of New York (2004)) [41]. It is hard to clarify this notion in a few lines of words. Many empirical evidences from other cities lend support to the scaling relation between urban area and population (see, e.g., [2, 16, 37, 40]). If a city, say, Beijing, fails to follow this law, it is the city instead of the scaling law that is wrong. If so, the urban man-land relation should be improved according to the allometric scaling law. The meaning of Beijing case lies in three aspects. First, it gives an approximate scaling exponent for our understanding of Beijing's growth. Second, it reveals the problem of Beijing's development to be resolved in the future. Third, it suggests that we should develop an urban theory by the ideas from fractals and allometric growth for urban planning and spatial optimization of cities.

Next, let us make a cross-sectional allometric analysis based on the rank-size relationships. For simplicity, only the allometric pattern in 2005 is shown (Figure 5). This year, there were 660 cities which were approved officially in China. We rank the population



**Figure 6:** The allometric scaling relation between the average built-up area and urban population of China's hierarchy of cities (2005).

size of these cities from the largest to the smallest and put urban area in order coinciding with population size. As a result, we have an allometric scaling relation such as

$$A_k = 1.917P_k^{0.817}, \quad (4.3)$$

where  $k$  denotes city rank by population ( $k = 1, 2, \dots, 660$ ),  $A_k$  refers to the area of the city of rank  $k$ , and  $P_k$  refers to the city population of rank  $k$ . The scaling exponent is estimated as  $b \approx 0.817$ , and the goodness of fitting is  $R^2 \approx 0.842$ . According to our rule of sorting order, we have an inequality  $P_k \geq P_{k+1}$  to a certainty. However, another inequation,  $A_k \geq A_{k+1}$ , will not necessarily come into existence. The reason is that a city that has more population does not imply that it has larger built-up area.

Finally, we can make an allometric analysis based on hierarchy of cities with cascade structure. Putting the 660 cities in order by population size, we can classify them in a top-down way in terms of the  $2^n$  principle of cities: the first class has one city—the city of rank 1, the second class has two cities—the cities of ranks 2 and 3, the third class has four cities—the cities of ranks 4, 5, 6, and 7, and so on. Evidently, the  $m$ th class will have  $2^{m-1}$  cities [22]. Thus, the 660 cities can be divided into ten classes, and the city number in the last class is expected to be  $N_{10} = 2^9 = 512$  according to the theoretical rule. However, the cities in the last class are less developed, and we have only 149 cities  $660 - (2^0 + 2^1 + 2^2 + \dots + 2^8) = 149$ . Moreover, the sizes of the cities in the tenth class are smaller than what is expected theoretically. Therefore, the bottom class is a lame-duck class [42]. A least squares computation gives the following allometric scaling model:

$$A_m = 1.786P_m^{0.856}, \quad (4.4)$$

in which  $m$  denotes the order of class ( $m = 1, 2, \dots, 10$ ),  $A_m$  refers to the average area of the cities of order  $m$ , and  $P_m$  refers to the average population size of the cities in the  $m$ th class. The scaling exponent is estimated as  $b \approx 0.856$ , and the goodness of fitting is  $R^2 \approx 0.987$  (Figure 6).

What is the expected value of the scaling exponent of urban area and population relation? Through spectral analysis, we can learn that the dimension of the urban population

is  $D_p = 2$ . On the other hand, simulation experiment analyses showed that the expected fractal dimension of the urban form is  $D_a \approx 1.701$  [8, 9]. Therefore, ASE is expected to be  $b = D_a/D_p \approx 1.701/2 \approx 0.85$ . ASE of China's cities in 2005 is close to this value.

The allometric scaling pattern based on the rank-size distribution is equivalent in theory to the allometric scaling relation based on hierarchical structure. The former differs from the latter to some extent in empirical analysis, but the difference between the two is not very significant. One gives the scaling exponent  $b \approx 0.817$ , and the other yields  $b \approx 0.856$ . As far as our example is concerned, the scaling exponent based on hierarchical system is closer to the theoretical expectation. After all, the two scaling models belong to the negative allometry ( $b < 1$ ). This suggests that, by and large, the land use of China's cities is comparatively reasonable.

Further, by using the least squares calculation, we can easily fit the data to the scaling relation between the urban number and population as well as the scaling relation between the urban number and area. Of course, the lame-duck class should be removed as an outlier from the regression analysis because the number of cities in this class is too few to support the scaling relation. Based on the hierarchy of cities without the tenth class, the modeling results are as follows:

$$\begin{aligned} N_m &= 14784.254P_m^{-1.262}, \\ N_m &= 28133.543A_m^{-1.435}. \end{aligned} \tag{4.5}$$

The coefficients of determination are  $R^2 \approx 0.995$  and  $R^2 \approx 0.975$ , respectively. This implies that the fractal dimension of hierarchy of cities is  $D \approx 1.262$  by population measure or  $D \approx 1.435$  by area measure.

In light of the nature of allometric growth, we can classify the geographical space into three types. The allometric patterns reflected by Figures 1, 2, and 3 as well as Table 1 correspond to the real space (R-space); Figure 4 illustrates a longitudinal allometry which corresponds to the phase space (P-space); the cross allometry reflected by Figures 5 and 6 corresponds to the order space (O-space). The real space is the conventional concept of geographical space, while the phase space and the order space belong to the generalized space, a kind of abstract space [7]. In the 2-dimension real space, the fractal dimension, especially the box-counting dimension, of urban form is generally less than 2; in the generalized space, however, the fractal dimension of urban form cannot be confined by the dimension of the Euclidean space in which the urban form exists.

In addition, there are some inherent relationships among fractal structure, allometric relation and self-organized networks. In the process of measuring the fractal dimension of the form displayed in Figures 1, 2, and 3, the linear size  $s_m$  can be replaced by length scale  $d_m$ , which denotes the distance between two centers of immediate fractal copies of order  $m$ . Thus, we have a network-based definition of fractal dimension  $D = -\ln(N_{m+1}/N_m) / \ln(d_m/d_{m+1})$  [29]. In fact, fractal geometry, allometric concepts, and network science are being slowly integrated into a new theory, which can be employed to explain the evolvement and development of cities [43]. As space is limited, many questions are pending further discussion in future studies.

## 5. Conclusion

A fractal growth is actually an allometric growth; the process of allometric growth is always involved with a number of fractal dimension relations. For a simple regular fractal, say, the Koch curve, one fractal dimension is enough to characterize its geometrical property. However, for a random fractal, especially, for a prefractal phenomenon, for example, a city, it is not sufficient to characterize its form and structure with only one fractional dimension. We should employ a set of fractal parameters including various fractal dimensions, the ratios of fractal dimensions, among others, to describe the complicated systems of scaling invariance. The ratio of one fractal dimension to the other related fractal dimension can constitute an ASE discussed above. Now, the main conclusions in this paper can be drawn and summarized as follows.

First, if the form of the growing phenomena such as cities is self-similar, then the boundaries of the phenomena will be of self-similarity also. The geometric relationship between the boundary length and the whole form is always an allometric scaling relation. The allometric relation can be described from two angles of view. One is the longitudinal allometry, and the other is the cross allometry. The former reflects the progressively evolutive process of the fractal growing from an initiator, while the latter reflects the hierarchy with cascade structure corresponding to the growing process. The method of allometric scaling analysis can be applied not only to the isotropic growing phenomena indicating self-similar fractals, but also to the anisotropic growths indicative of self-affine fractals. For the self-affine patterns, there exists an allometric scaling relation between the parts in different directions.

Second, the mathematical description of the allometric growth rests with two aspects: one is various fractal dimensions, and the other is ASEs. The fractal characterization is a static method, laying emphasis on the best result by assuming the linear size of fractal elements to approach to zero. Theoretically, if the linear size of fractal measure becomes infinitesimal, then the fractal dimension value will approach a real constant. By contrast with fractal dimension, ASE lays stress on an evolutive process or a kind of spatial relations. The linear size of yard measure for estimating ASEs does not necessarily tend towards infinitesimal. As long as the scaling range is long enough, the result will be satisfying. This is significant for urban studies because the self-similarity of cities is valid only within certain scale limits. By means of allometric analysis we can reveal the regularity and complexity of urban evolvement and structure efficiently.

Third, the fractal studies can be generalized from real space to the abstract space in terms of allometric growth. All of the fractals that can be directly exhibited by maps or pictures are fractal in actual space. However, there are lots of fractals which cannot be immediately represented by graphics. The scaling invariance of this kind of fractals can be indirectly revealed with mathematical transformation and log-log plots. The majority of these fractals often comes from the generalized space. Urban form and boundaries belongs to the real space, but the scaling relation between urban area and population belong to the generalized space. It is difficult to evaluate the dimensions of the fractals in the abstract space, but it is easy to estimate the ratio of different fractal dimensions based on the generalized space. In many cases, what we want to know is just the fractal dimension ratios rather than the fractal dimensions themselves. Through allometric analyses we can directly calculate the ratio of fractal dimensions and thus obtain ASEs; thereby we can research the structure and functions of fractal systems.

## Appendix

### How to Reveal the Outliers of an Allometric Scaling Relation

The city of Beijing is taken as an example to show how to reveal the outliers of an allometric scaling relation. The allometric growth is in fact based on exponential growth theoretically. Suppose that both urban area and population increase exponentially with the passage of time. Taking the logarithm of urban population  $P_t$  and built-up area  $A_t$ , we get the logarithmic values of the two measures,  $\ln P_t$  and  $\ln A_t$ . Then, we can standardize the logarithmic measures by z-score, and the results are represented by  $Sdz(\ln P_t)$  and  $Sdz(\ln A_t)$  in Table 2. If one or two of the standardized data are greater than double-standard error, 2, then the values can be regarded as outliers on the statistical significance of  $\alpha = 0.05$ . Clearly, the data point in 2005 is an exceptional value.

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## References

- [1] M. Batty, *Cities and Complexity: Understanding Cities with Cellular Automata*, MIT Press, Cambridge, Mass, USA, 2005.
- [2] M. Batty and P. A. Longley, *Fractal Cities: A Geometry of Form and Function*, Academic Press, London, UK, 1994.
- [3] P. Frankhauser, *La Fractalité des Structures Urbaines*, Economica, Paris, France, 1994.
- [4] G. Haag, "The rank-size distribution of settlements as a dynamic multifractal phenomenon," *Chaos, Solitons and Fractals*, vol. 4, no. 4, pp. 519–534, 1994.
- [5] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman and Company, San Francisco, Calif, USA, 1982.
- [6] R. White and G. Engelen, "Urban systems dynamics and cellular automata: fractal structures between order and chaos," *Chaos, Solitons and Fractals*, vol. 4, no. 4, pp. 563–583, 1994.
- [7] Y. Chen and S. Jiang, "An analytical process of the spatio-temporal evolution of urban systems based on allometric and fractal ideas," *Chaos, Solitons and Fractals*, vol. 39, no. 1, pp. 49–64, 2009.
- [8] M. Batty, "Generating urban forms from diffusive growth," *Environment & Planning A*, vol. 23, no. 4, pp. 511–544, 1991.
- [9] M. Batty, "Cities as fractals: simulating growth and form," in *Fractals and Chaos*, A. J. Crilly, R. A. Earnshaw, and H. Jones, Eds., pp. 43–69, Springer, New York, NY, USA, 1991.
- [10] L. Benguigui, D. Czamanski, M. Marinov, and Y. Portugali, "When and where is a city fractal?" *Environment and Planning B*, vol. 27, no. 4, pp. 507–519, 2000.
- [11] L. Benguigui, E. Blumenfeld-Lieberthal, and D. Czamanski, "The dynamics of the Tel Aviv morphology," *Environment and Planning B*, vol. 33, no. 2, pp. 269–284, 2006.
- [12] M.-L. De Keersmaecker, P. Frankhauser, and I. Thomas, "Using fractal dimensions for characterizing intra-urban diversity: the example of Brussels," *Geographical Analysis*, vol. 35, no. 4, pp. 310–328, 2003.
- [13] I. Thomas, P. Frankhauser, and M.-L. De Keersmaecker, "Fractal dimension versus density of built-up surfaces in the periphery of Brussels," *Papers in Regional Science*, vol. 86, no. 2, pp. 287–308, 2007.
- [14] I. Thomas, P. Frankhauser, and C. Biernacki, "The morphology of built-up landscapes in Wallonia (Belgium): a classification using fractal indices," *Landscape and Urban Planning*, vol. 84, no. 2, pp. 99–115, 2008.



- [15] J. Gayon, "History of the concept of allometry," *American Zoologist*, vol. 40, no. 5, pp. 748–758, 2000.
- [16] Y. Lee, "An allometric analysis of the US urban system: 1960–80," *Environment & Planning A*, vol. 21, no. 4, pp. 463–476, 1989.
- [17] R. S. Naroll and L. von Bertalanffy, "The principle of allometry in biology and social sciences," *General Systems Yearbook*, vol. 1, Part II, pp. 76–89, 1956.
- [18] R. S. Naroll and L. von Bertalanffy, "The principle of allometry in biology and social sciences," *Ekistics*, vol. 36, no. 215, pp. 244–252, 1973.
- [19] S. Nordbeck, "Urban allometric growth," *Annaler B*, vol. 53, no. 1, pp. 54–67, 1971.
- [20] W. R. Tobler, "Satellite confirmation of settlement size coefficients," *Area*, vol. 1, no. 3, pp. 30–34, 1969.
- [21] M. J. Beckmann, "City hierarchies and distribution of city sizes," *Economic Development and Cultural Change*, vol. 6, no. 3, pp. 243–248, 1958.
- [22] Y. Chen and Y. Zhou, "Multi-fractal measures of city-size distributions based on the three-parameter Zipf model," *Chaos, Solitons and Fractals*, vol. 22, no. 4, pp. 793–805, 2004.
- [23] P. A. Longley, M. Batty, and J. Shepherd, "The size, shape and dimension of urban settlements," *Transactions of the Institute of British Geographers*, vol. 16, no. 1, pp. 75–94, 1991.
- [24] P. A. Longley and M. Batty, "On the fractal measurement of geographical boundaries," *Geographical Analysis*, vol. 21, no. 1, pp. 47–67, 1989.
- [25] P. A. Longley and M. Batty, "Fractal measurement and line generalization," *Computers and Geosciences*, vol. 15, no. 2, pp. 167–183, 1989.
- [26] J.-H. He and J.-F. Liu, "Allometric scaling laws in biology and physics," *Chaos, Solitons and Fractals*, vol. 41, no. 4, pp. 1836–1838, 2009.
- [27] B. J. West, "Comments on the renormalization group, scaling and measures of complexity," *Chaos, Solitons and Fractals*, vol. 20, no. 1, pp. 33–44, 2004.
- [28] Y. Chen and Y. Zhou, "Reinterpreting central place networks using ideas from fractals and self-organized criticality," *Environment and Planning B*, vol. 33, no. 3, pp. 345–364, 2006.
- [29] Y. Chen and Y. Zhou, "Scaling laws and indications of self-organized criticality in urban systems," *Chaos, Solitons and Fractals*, vol. 35, no. 1, pp. 85–98, 2008.
- [30] R. Jullien and R. Botet, *Aggregation and Fractal Aggregates*, World Scientific, Teaneck, NJ, USA, 1987.
- [31] T. Vicsek, *Fractal Growth Phenomena*, World Scientific, Singapore, 1989.
- [32] R. White and G. Engelen, "Cellular automata and fractal urban form: a cellular modelling approach to the evolution of urban land-use patterns," *Environment & Planning A*, vol. 25, no. 8, pp. 1175–1199, 1993.
- [33] P. Frankhauser, "The fractal approach: a new tool for the spatial analysis of urban agglomerations," *Population*, vol. 10, no. 1, pp. 205–240, 1998.
- [34] P. Longley, M. Batty, J. Shepherd, and G. Sadler, "Do green belts change the shape of urban areas? A preliminary analysis of the settlement geography of South East England," *Regional Studies*, vol. 26, no. 5, pp. 437–452, 1992.
- [35] M. Batty and K. S. Kim, "Form follows function: reformulating urban population density functions," *Urban Studies*, vol. 29, no. 7, pp. 1043–1069, 1992.
- [36] A. R. Imre and J. Bogaert, "The fractal dimension as a measure of the quality of habitats," *Acta Biotheoretica*, vol. 52, no. 1, pp. 41–56, 2004.
- [37] Y. Chen and J. Lin, "Modeling the self-affine structure and optimization conditions of city systems using the idea from fractals," *Chaos, Solitons and Fractals*, vol. 41, no. 2, pp. 615–629, 2009.
- [38] X. Wang, J. Liu, D. Zhuang, and L. Wang, "Spatial-temporal changes of urban spatial morphology in China," *Acta Geographica Sinica*, vol. 60, no. 3, pp. 392–400, 2005.
- [39] M. Batty and P. A. Longley, "The morphology of urban land use," *Environment & Planning B*, vol. 15, no. 4, pp. 461–488, 1988.
- [40] C. P. Lo and R. Welch, "Chinese urban population estimates," *Annals of the Association of American Geographers*, vol. 67, no. 2, pp. 246–253, 1977.
- [41] Y. G. Chen, *Fractal Urban Systems: Scaling, Symmetry, and Spatial Complexity*, Scientific Press, Beijing, China, 2008.
- [42] K. Davis, "World urbanization: 1950–1970," in *Systems of Cities*, I. S. Bourne and J. W. Simons, Eds., pp. 92–100, Oxford University Press, New York, NY, USA, 1978.
- [43] M. Batty, "The size, scale, and shape of cities," *Science*, vol. 319, no. 5864, pp. 769–771, 2008.