

Research Article

Strictly Decreasing Solutions of a Class of Iterative Equations on the Unit Circle

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We discuss the strictly decreasing solutions of a class of iterative equations on the unit circle \mathbb{T}^1 . The conditions for the existence, uniqueness, and stability of such solutions are presented.

1. Introduction

Let $C^0(X, X)$ be the set of all continuous self-mappings on a topological space X . Let $f \in C^0(X, X)$. For integer $n \geq 0$, define the n th iterate of f by $f^n = f \circ f^{n-1}$ and $f^0 = \text{id}$, where id denotes the identity mapping on X and \circ denotes the composition of mappings. An equation with iteration as its main operation is simply called an iterative equation. Founded on the problem of iterative roots, the problem of invariant curves and some problems originating from dynamical systems, and the iterative equation

$$\Phi(f(x), f^2(x), \dots, f^n(x)) = F(x), \quad x \in X, \quad (1.1)$$

where F and Φ are given functions and f is the unknown function, have been investigated extensively [1, 2].

Let $\mathbb{T}^1 = \{z \in \mathbb{C} : z = e^{2\pi it}, t \in \mathbb{R}\}$ denote the unit circle and $C^0(\mathbb{T}^1, \mathbb{T}^1)$ denote the set of all continuous maps from \mathbb{T}^1 to itself. Let $\mathbf{1}$ denote the point $(1, 0)$ in the complex plane \mathbb{C} so as to distinguish it from $1 \in \mathbb{R}$. The iteration on the unit circle \mathbb{T}^1 is also important. Many scholars had got a lot of results on iterative roots and iteration groups, such as [3–7]. In 2007, Zdun and Zhang [8] discussed the solutions of (1.1) on \mathbb{T}^1 , that is, the equation

$$\Phi(f(z), f^2(z), \dots, f^n(z)) = F(z), \quad z \in \mathbb{T}^1, \quad (1.2)$$

in the class of homeomorphisms

$$H_1^0(\mathbb{T}^1, \mathbb{T}^1) = \left\{ f \in C^0(\mathbb{T}^1, \mathbb{T}^1) : f(\mathbb{T}^1) = \mathbb{T}^1 \text{ homeomorphically and } f(\mathbf{1}) = \mathbf{1} \right\}. \quad (1.3)$$

In [8], the authors lift F from the circle \mathbb{T}^1 to \mathbb{R} and Φ from the n -dimensional torus \mathbb{T}^n to \mathbb{R}^n . They apply ingenious techniques of restricting and extending to these lifts so that the problem can be reduced and discussed on the compact interval $[0, 1]$. However, they only discussed the strictly increasing solutions of (1.2). In this paper, we will discuss the strictly decreasing solutions of a special form of (1.2), that is, the following equation:

$$\Phi(f(z), f^3(z), \dots, f^{2n-1}(z)) = F(z), \quad z \in \mathbb{T}^1. \quad (1.4)$$

2. Basic Definitions and Lemmas

As in [8], let $h : t \in \mathbb{R} \mapsto e^{2\pi it} \in \mathbb{T}^1$. If $v, w, z \in \mathbb{T}^1$, there exist unique $t_1, t_2 \in [0, 1)$ such that $wh(t_1) = z$ and $wh(t_2) = v$. The cyclic order on \mathbb{T}^1 is defined in [9], that is,

$$v < w < z \iff 0 < t_1 < t_2, \quad (2.1)$$

$$v \leq w \leq z \iff t_1 \leq t_2 \quad \text{or} \quad t_2 = 0. \quad (2.2)$$

Obviously, the relations $v < w < z$, $w < z < v$ and $z < v < w$ are equivalent.

Considering a nonempty subset $A \subseteq \mathbb{T}^1$, a map $f : A \rightarrow \mathbb{T}^1$ is called to be increasing (strictly increasing) if $f(v) \leq f(w) \leq f(z)$ ($f(v) < f(w) < f(z)$, resp.) for all $v, w, z \in A$ with $v < w < z$. Similarly, a map $f : A \rightarrow \mathbb{T}^1$ is called to be decreasing (strictly decreasing) if $f(z) \leq f(w) \leq f(v)$ ($f(z) < f(w) < f(v)$, resp.) for all $v, w, z \in A$ with $v < w < z$. A homeomorphism $h : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is said to be orientation preserving (orientation reversing) if it is strictly increasing (strictly decreasing, resp.).

If $v, z \in \mathbb{T}^1$ with $v \neq z$, there exist $t_v, t_z \in \mathbb{R}$ with $t_v < t_z < t_v + 1$ and $h(t_v) = v, h(t_z) = z$. Define the oriented arc

$$\overrightarrow{(v, z)} = \{w = h(t) : t \in (t_v, t_z)\}. \quad (2.3)$$

Obviously, the map f is strictly increasing (strictly decreasing) if $w \in \overrightarrow{(v, z)}$ yields $f(w) \in \overrightarrow{(f(v), f(z))}$ ($f(w) \in \overrightarrow{(f(z), f(v))}$, resp.).

2.1. The Lifts of Maps on \mathbb{T}^1

Let $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$. A continuous map $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ is said to be a lift of f if

$$h \circ \tilde{f} = f \circ h. \quad (2.4)$$

The following properties are well known.

Lemma 2.1 (see [10]). (i) Each $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ has a lift \tilde{f} . (ii) There exists a constant $k \in \mathbb{Z}$ such that each lift \tilde{f} of f satisfies $\tilde{f}(t+1) - \tilde{f}(t) = k$, for all $t \in \mathbb{R}$. The constant k is said to be the degree of f and denoted by $\deg(f)$, that is, $\deg(f) = \tilde{f}(t+1) - \tilde{f}(t)$, for all $t \in \mathbb{R}$. (iii) For $f, g \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ and $\text{id} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$, one has $\deg(g \circ f) = \deg(g) \cdot \deg(f)$ and $\deg(\text{id}) = 1$. If $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ is a homeomorphism, then $|\deg(f)| = 1$. (iv) If \tilde{f} is a lift of f , then for each $j \in \mathbb{Z}$, the map $\tilde{f} + j$ is a lift of f and every lift of f can be expressed in this form, where \mathbb{Z} denotes the set of integers.

Lemma 2.2 (see [10]). (i) If $g : [0, 1] \rightarrow \mathbb{R}$ is a continuous map such that $g(1) - g(0)$ is an integer k , then g can be extended to a map $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\tilde{g}(t+1) - \tilde{g}(t) = k, \quad \forall t \in \mathbb{R}. \quad (2.5)$$

(ii) Suppose $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, k is an integer, and $\tilde{f}(t+1) - \tilde{f}(t) = k$, for all $t \in \mathbb{R}$. Then there exists a unique $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ such that \tilde{f} is a lift of f .

A map $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ is said to be Lipschitzian if there exists a lift \tilde{f} of f satisfying

$$|\tilde{f}(t_1) - \tilde{f}(t_2)| \leq M|t_1 - t_2|, \quad \forall t_1, t_2 \in \mathbb{R} \quad (2.6)$$

for a constant $M > 0$. By (iv) of Lemma 2.1, the constant M is independent of the choice of \tilde{f} , and M is said to be a Lipschitzian constant of f .

In the sequel one will discuss the strictly increasing (strictly decreasing) maps $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ by means of their lifts.

Lemma 2.3. A lift \tilde{f} of $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ is strictly monotonic (monotonic) if and only if there exists a $t_0 \in \mathbb{R}$ such that $\tilde{f}|_{[t_0, t_0+1]}$ is strictly monotonic (monotonic).

The proof is trivial. So we omit it.

Lemma 2.4 (see [11]). A map $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ is strictly increasing (strictly decreasing) if and only if $\deg(f) = 1$ ($\deg(f) = -1$) and its lift \tilde{f} is strictly increasing (strictly decreasing) in \mathbb{R} .

Thus we get the following theorem.

Theorem 2.5. A map $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ is strictly increasing (strictly decreasing) if and only if it is an orientation-preserving (orientation-reversing) homeomorphism.

Proof. We only need to show that f has an inverse mapping. Let \tilde{f} be the lift of f . The map $f^{-1} = h \circ \tilde{f}^{-1}$ is the inverse mapping, where \tilde{f}^{-1} is the inverse function of \tilde{f} . \square

2.2. Induced Maps of Maps on \mathbb{T}^1

Firstly, we define

$$H_1^0(\mathbb{T}^1, k) = \{f \mid f \in H_1^0(\mathbb{T}^1, \mathbb{T}^1), \deg(f) = k\}, \quad (2.7)$$

where $k = 1$ or -1 . Let $h_* := h|_{(0,1)}$; the map h_* is a homeomorphism from $(0, 1)$ to $\mathbb{T}^1 \setminus \{\mathbf{1}\}$. For $f \in H_1^0(\mathbb{T}^1, -1)$, define $\tilde{f}_* := h_*^{-1} \circ f \circ h_*$ which is a self-map on $(0, 1)$. We extend \tilde{f}_* to the compact interval $I = [0, 1]$. Let

$$G(t) = \begin{cases} 1, & t = 0, \\ \tilde{f}_*(t), & t \in (0, 1), \\ 0, & t = 1. \end{cases} \quad (2.8)$$

For convenience, we call G the induced map of f , which is a self-map on $[0, 1]$.

Lemma 2.6. *The induced map G of an orientation-reversing homeomorphism $f \in H_1^0(\mathbb{T}^1, -1)$ is continuous and strictly decreasing on I . It can be extended to a lift \tilde{f} of f . This lift satisfies $\tilde{f}(0) = 1, \tilde{f}(1) = 0$ and is unique on this property.*

Proof. Obviously, G is continuous on the interval $(0, 1)$. We only need to prove $\lim_{t \rightarrow 0^+} G(t) = 1$ and $\lim_{t \rightarrow 1^-} G(t) = 0$.

Concerning the continuity at 0, we note that

$$\lim_{t \rightarrow 0^+} G(t) = \lim_{t \rightarrow 0^+} h_*^{-1} \circ f \circ h_*(t) = \lim_{t \rightarrow 0^+} h_*^{-1} \circ f(e^{2\pi it}). \quad (2.9)$$

By the continuity of f at $\mathbf{1}$, we have $\lim_{t \rightarrow 0^+} f(e^{2\pi it}) = \mathbf{1}$. Let \tilde{F} be any lift of f . By $h \circ \tilde{F} = f \circ h$, we must have $\tilde{F}(0) = k_0 \in \mathbb{Z}$, where \mathbb{Z} denotes the set of integers. By (iv) of Lemma 2.1, let \tilde{F} be a lift of f with $\tilde{F}(0) = 1$. Since $f \circ h_*(t) = h_* \circ \tilde{F}(t)$, $t \in (0, 1)$, we have

$$\lim_{t \rightarrow 0^+} h_*^{-1} \circ f \circ h_*(t) = \lim_{t \rightarrow 0^+} \tilde{F}(t) = 1. \quad (2.10)$$

We can easily verify $\lim_{t \rightarrow 1^-} G(t) = 0$ in the same way. Note that G is strictly decreasing on $(0, 1)$, so G is strictly decreasing on $[0, 1]$.

Let $t \in \mathbb{R}$, and k is the integer with $t \in [k, k + 1)$. Define $\tilde{f} := G(t - k) + k$. We can easily get that $\tilde{f} \circ h = h \circ \tilde{f}$. Suppose that \tilde{g} is another lift of f which maps $[0, 1]$ into itself and $\tilde{g}(0) = 1, \tilde{g}(1) = 0$. By (iv) of Lemma 2.1, we have $\tilde{f} = \tilde{g} + j$ for some integer j . Obviously, $j = \tilde{f}(0) - \tilde{g}(0) = 0$. Then $\tilde{f} = \tilde{g}$. \square

The following lemma is a converse of Lemma 2.6.

Lemma 2.7. *Suppose that $G \in C^0(I, I)$ is strictly decreasing and satisfies $G(0) = 1, G(1) = 0$. Then the map*

$$f := \begin{cases} h_* \circ G \circ h_*^{-1}(z), & z \in \mathbb{T}^1 \setminus \{\mathbf{1}\}, \\ \mathbf{1}, & z = \mathbf{1}. \end{cases} \quad (2.11)$$

belongs to the class $H_1^0(\mathbb{T}^1, -1)$. Moreover, G can be extended to a lift of f .

Proof. Clearly, $f(\mathbf{1}) = \mathbf{1}$, and $f|_{\mathbb{T}^1 \setminus \{\mathbf{1}\}}$ is an orientation-reversing homeomorphism of $\mathbb{T}^1 \setminus \{\mathbf{1}\}$. Then f is orientation reversing on the whole \mathbb{T}^1 . We only need to show the continuity of f at $\mathbf{1}$. Let $z \rightarrow \mathbf{1}^+$ ($z \rightarrow \mathbf{1}^-$) denote z approaches to $\mathbf{1}$ in clockwise (counter-clockwise) direction. Note that

$$\lim_{z \rightarrow \mathbf{1}^+} f(z) = \lim_{t \rightarrow 0^+} h_* \circ G \circ h_*^{-1}(e^{2\pi it}) = \lim_{t \rightarrow 0^+} h_* \circ G(t) = \mathbf{1}. \quad (2.12)$$

By the same argument, we have $\lim_{z \rightarrow \mathbf{1}^-} f(z) = \mathbf{1}$.

Given $t \in \mathbb{R}$ and integer k with $t \in [k, k+1)$, define

$$\tilde{f}(t) = G(t - k) + k. \quad (2.13)$$

One can easily show that $f \circ h = h \circ \tilde{f}$. □

2.3. Induced Maps of Maps on \mathbb{T}^n

For simplicity, let

$$\begin{aligned} \mathbb{T}^n &= \overbrace{\mathbb{T}^1 \times \dots \times \mathbb{T}^1}^n, \\ \mathbf{1}^n &= \left(\overbrace{\mathbf{1}, \dots, \mathbf{1}}^n \right). \end{aligned} \quad (2.14)$$

For $f \in H_1^0(\mathbb{T}^1, -1)$, define

$$H_f(z) := \left(f(z), f^3(z), \dots, f^{2^{n-1}}(z) \right). \quad (2.15)$$

Then (1.4) can be written as

$$\Phi \circ H_f(z) = F(z), \quad z \in \mathbb{T}^1. \quad (2.16)$$

As in [8], we also have the following two remarks.

Remark 2.8. H_f maps \mathbb{T}^1 into $(\mathbb{T}^1 \setminus \{\mathbf{1}\})^n \cup \{\mathbf{1}^n\}$.

Remark 2.9. Let $\text{Rang}(\Phi)$ denote the range of Φ . If $F \in H_1^0(\mathbb{T}^1, -1)$ and (1.4) has a solution $f \in H_1^0(\mathbb{T}^1, -1)$ on F , then $\text{Rang} \Phi = \mathbb{T}^1$ and $\Phi(\mathbf{1}^n) = \mathbf{1}$.

Let $\text{Dom}(\Phi)$ denote the domain of Φ . As in [8], $\text{Dom}(\Phi)$ is very important for (1.4). Now, we discuss the relationship of $\text{Dom}(\Phi)$ and (1.4). The first case is that the domain of Φ is \mathbb{T}^n .

2.3.1. The Case: $\text{Dom}(\Phi) = \mathbb{T}^n$

The following important and interesting lemma is proved in [8].

Lemma 2.10 (see [8]). *If $\Phi : \mathbb{T}^n \rightarrow \mathbb{T}^1$ is continuous and $\Phi(\mathbf{1}^n) = \mathbf{1}$, then there exists a unique continuous function $\tilde{\Phi} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$\Phi(h(t_1), \dots, h(t_n)) = e^{2\pi i \tilde{\Phi}(t_1, \dots, t_n)}, \quad \tilde{\Phi}(0, \dots, 0) = 0. \quad (2.17)$$

Moreover, for each $k \in \{1, \dots, n\}$, there exists an $m_k \in \mathbb{Z}$ such that

$$\tilde{\Phi}(t_1, \dots, t_k + 1, \dots, t_n) = \tilde{\Phi}(t_1, \dots, t_k, \dots, t_n) + m_k, \quad \forall t_1, \dots, t_k \in \mathbb{R}. \quad (2.18)$$

The map $\tilde{\Phi}$ is said to be the lift of Φ , and (m_1, \dots, m_n) is defined to be the degree of Φ and is denoted by $\text{deg}(\Phi)$.

Lemma 2.11. *Let \tilde{F} be the lift of F with $\tilde{F}(1) = 0$, and let $\tilde{\Phi}$ be the lift of Φ with $\tilde{\Phi}(0, \dots, 0) = 0$. Let $f \in H_1^0(\mathbb{T}^1, -1)$ be a solution of (1.4), and let \tilde{f} be a lift of f with $\tilde{f}(1) = 0$. Then (1.4) is equivalent to*

$$\tilde{\Phi}(\tilde{f}(t), \tilde{f}^3(t), \dots, \tilde{f}^{2n-1}(t)) = \tilde{F}(t), \quad t \in I. \quad (2.19)$$

Proof. Note that $f^j(h(t)) = h(\tilde{f}^j(t))$ for all $t \in \mathbb{R}$. For $z = e^{2\pi i t}$, $t \in I$ and (1.4) is equivalent to

$$\Phi(e^{2\pi i \tilde{f}(t)}, e^{2\pi i \tilde{f}^3(t)}, \dots, e^{2\pi i \tilde{f}^{2n-1}(t)}) = e^{2\pi i \tilde{F}(t)}. \quad (2.20)$$

By Lemma 2.10, $e^{2\pi i \tilde{\Phi}(\tilde{f}(t), \tilde{f}^3(t), \dots, \tilde{f}^{2n-1}(t))} = e^{2\pi i \tilde{F}(t)}$. This means that for each $t \in I$, we have

$$\tilde{\Phi}(\tilde{f}(t), \tilde{f}^3(t), \dots, \tilde{f}^{2n-1}(t)) = \tilde{F}(t) + k(t), \quad (2.21)$$

where $k(t) \in \mathbb{Z}$. Since $\tilde{f}(1) = 0$, $\tilde{F}(1) = 0$, and $\tilde{\Phi}(0, \dots, 0) = 0$, we get $k(t) = 0$. By the continuity of \tilde{f} , \tilde{F} , and $\tilde{\Phi}$, we get that $k(t) \equiv 0$. \square

Theorem 2.12. *Suppose that Φ is continuous, $\Phi(\mathbf{1}^n) = \mathbf{1}$, F is continuous, $F(\mathbf{1}) = \mathbf{1}$, and (1.4) has a solution $f \in H_1^0(\mathbb{T}^1, -1)$. Let $\text{deg}(\Phi) = (m_1, \dots, m_n)$. Then $\text{deg}(F) = -(m_1 + \dots + m_n)$.*

Proof. Let \tilde{F} be the lift of F with $\tilde{F}(1) = 0$, and let $\tilde{\Phi}$ be the lift of Φ with $\tilde{\Phi}(0, \dots, 0) = 0$. Let $f \in H_1^0(\mathbb{T}^1, -1)$ be the solution of (1.4), and let \tilde{f} be its lift with $\tilde{f}(1) = 0$. Note that

$$\tilde{f}(t+1) = \tilde{f}(t) - 1, \quad t \in \mathbb{R}. \quad (2.22)$$

Let $\tilde{H}(t) = (\tilde{f}(t), \tilde{f}^3(t), \dots, \tilde{f}^{2n-1}(t))$. Since $\text{deg}(\Phi) = (m_1, \dots, m_n)$, then by (2.18), we have

$$\tilde{\Phi}(-1, \dots, -1) = \tilde{\Phi}(0, \dots, 0) - m_1 - \dots - m_n. \quad (2.23)$$

Thus, we get that

$$\begin{aligned}
\tilde{F}(t+1) &= \tilde{\phi}(\tilde{H}(t+1)) = \tilde{\Phi}(\tilde{H}(t) + (-1, \dots, -1)) \\
&= \tilde{\Phi}(\tilde{H}(t)) - m_1 - \dots - m_n \\
&= \tilde{F}(t) - m_1 - \dots - m_n.
\end{aligned} \tag{2.24}$$

□

Definition 2.13. Suppose $\Phi : \mathbb{T}^n \rightarrow \mathbb{T}^1$ is continuous. Let $1 \leq i \leq n$, and let $w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n \in \mathbb{C}$ be fixed.

(1) One says Φ is increasing with respect to the i th variable if the map

$$\Phi(w_1, \dots, w_{i-1}, z_i, w_{i+1}, \dots, w_n) \tag{2.25}$$

is increasing.

(2) One says Φ is constant with respect to the i th variable if there is a $z_0 \in \mathbb{T}^1$ such that the map

$$\Phi(w_1, \dots, w_{i-1}, z_i, w_{i+1}, \dots, w_n) \equiv z_0. \tag{2.26}$$

Corollary 2.14. Let $F \in H_1^0(\mathbb{T}^1, -1)$, and Φ is continuous with $\Phi(\mathbf{1}^n) = \mathbf{1}$. If Φ is increasing with respect to each variable and Φ is nonconstant in at least two variables, then (1.4) has no solutions in $H_1^0(\mathbb{T}^1, -1)$.

Proof. Let $\deg(\Phi) = (m_1, \dots, m_n)$. Hence, by Lemma 2.3, we have $m_1 \geq 0, \dots, m_n \geq 0$ and $m_k = 0$ if and only if Φ is constant with respect to the k th variable. If (1.4) has a solution f in $H_1^0(\mathbb{T}^1, -1)$, then by Theorem 2.12, we get that $\deg(F) = -(m_1 + \dots + m_n) \leq -2$. This a contradiction. □

From the discussion above, we know that $\deg(F)$ and $\deg(\Phi)$ are interrelated. This is too severe for (1.4). So as in [8], by Remark 2.8, it is also natural to suppose that $\text{Dom}(\Phi) = (\mathbb{T}^1 \setminus \mathbf{1})^n \cup \{\mathbf{1}^n\}$.

2.3.2. The Case $\text{Dom}(\Phi) = (\mathbb{T}^1 \setminus \mathbf{1})^n \cup \{\mathbf{1}^n\}$

The following assumptions (H1) and (A) are cited from [8].

(H1) $\Phi : (\mathbb{T}^1 \setminus \{\mathbf{1}\})^n \cup \{\mathbf{1}^n\} \rightarrow \mathbb{T}^1$ is continuous, $\Phi((\mathbb{T}^1 \setminus \{\mathbf{1}\})^n) = \mathbb{T}^1 \setminus \{\mathbf{1}\}$.

(A) There exists a constant $\delta > 0$ such that for $0 < t_k < \delta$, $k = 1, \dots, n$, we have $\Phi(h_*(t_1), \dots, h_*(t_n)) \in \overrightarrow{(\mathbf{1}, i)}$, and for $1 - \delta < t_k < 1$, $k = 1, \dots, n$, we have $\Phi(h_*(t_1), \dots, h_*(t_n)) \in \overrightarrow{(-i, \mathbf{1})}$.

Under assumptions (H1) and (A), as in [8], define

$$\begin{aligned}\Psi(t_1, \dots, t_n) &:= h_*^{-1}(\Phi(h_*(t_1), \dots, h_*(t_n))), \quad t_j \in (0, 1), \quad j = 1, \dots, n, \\ \Psi(0, \dots, 0) &= 0, \quad \Psi(1, \dots, 1) = 1.\end{aligned}\tag{2.27}$$

The function Ψ defined on $(0, 1)^n \cup \{(0, \dots, 0), (1, \dots, 1)\}$ is said to be the induced map of Φ . Let us note that

$$\lim_{t_j \rightarrow 0, j=1, \dots, n} \Psi(t_1, \dots, t_n) = 0, \quad \lim_{t_j \rightarrow 1, j=1, \dots, n} \Psi(t_1, \dots, t_n) = 1.\tag{2.28}$$

Lemma 2.15 (see [8]). *Under assumptions (H1) and (A), the induced map Ψ of Φ is continuous.*

Lemma 2.16. *Let \tilde{F} be the lift of F with $\tilde{F}(1) = 0$, and let Ψ be the induced map of Φ . Let $f \in H_1^0(\mathbb{T}^1, -1)$ be a solution of (1.4), and let \tilde{f} be a lift of f with $\tilde{f}(1) = 0$. Then (1.4) is equivalent to*

$$\Psi(\tilde{f}(t), \tilde{f}^3(t), \dots, \tilde{f}^{2n-1}(t)) = \tilde{F}(t), \quad t \in I.\tag{2.29}$$

Lemma 2.16 can be proved by means of the method which is used to prove Lemma 2.11.

As in [8], we also need the following assumption.

(H2) [8] There are nonnegative real constants $\alpha_j, \beta_j, j = 1, \dots, n$ with $\beta_1 \geq \alpha_1 > 0, \beta_j \geq \alpha_j \geq 0, j = 2, \dots, n$, such that

$$\sum_{j=1}^n \alpha_j (t_j - s_j) \leq \Psi(t_1, \dots, t_n) - \Psi(s_1, \dots, s_n) \leq \sum_{j=1}^n \beta_j (t_j - s_j)\tag{2.30}$$

for all $t_j \geq s_j$ in $(0, 1), j = 1, \dots, n$.

Lemma 2.17 (see [8]). *If Ψ is differentiable in $(0, 1)^n$ with respect to each variable and for every k there exist α_k, β_k such that $\alpha_1 > 0, 0 \leq \alpha_k \leq \partial\Psi/\partial t_k \leq \beta_k$, then Ψ satisfies (H2).*

3. Existence of Strictly Decreasing Solutions

Let $C^0(I)$ denote the set of all continuous maps on $I = [0, 1]$. It is well known that $C^0(I)$ equipped with the supremum norm $\|g\| = \sup\{|g(t)| : t \in I\}$ is a Banach space. Let $C^0(I, I)$ denote the set of all continuous self-mappings on I ; $C^0(I, I)$ is a compact closed subset of $C^0(I)$.

The following lemma is useful in the proof of Theorem 3.2.

Lemma 3.1 (see [12]). *Let $i = 1, 2$, and suppose that g_i is a self-homeomorphism of I such that $|g_i(x) - g_i(y)| \leq N|x - y|$ for all $x, y \in I$, where $N > 0$ is a constant. Then*

$$(i) \|g_1^n - g_2^n\| \leq \left(\sum_{i=0}^{n-1} N^i\right) \|g_1 - g_2\| \text{ for all } n \in \mathbb{Z};$$

$$(ii) \|g_1 - g_2\| \leq N \|g_1^{-1} - g_2^{-1}\|.$$

For $0 \leq l \leq L$, define

$$\begin{aligned} \mathfrak{F}(I, l, L) &= \left\{ g \in C^0(I, I) \mid g(0) = 0, g(1) = 1, l(t-s) \leq g(t) - g(s) \leq L(t-s), \forall s \leq t \in I \right\}, \\ \mathfrak{F}'(I, l, L) &= \left\{ g \in C^0(I, I) \mid g(0) = 1, g(1) = 0, l(t-s) \leq g(s) - g(t) \leq L(t-s), \forall s \leq t \in I \right\}. \end{aligned} \quad (3.1)$$

Both $\mathfrak{F}(I, l, L)$ and $\mathfrak{F}'(I, l, L)$ are compact convex subsets of $C^0(I)$ [1, 13].

Theorem 3.2. *Suppose (H1), (H2) hold and $F \in H_1^0(\mathbb{T}^1, -1)$ with a Lipschitz constant $M > \alpha_1$. Equation (1.4) has a solution $f \in H_1^0(\mathbb{T}^1, -1)$ with a Lipschitz constant M/α_1 .*

Proof. Let G and Ψ be the induced maps of F and Φ defined in Section 2. By Lemma 2.16 (1.4) is equivalent to the following equation:

$$\Psi\left(g(t), g^3(t), \dots, g^{2^{n-1}}(t)\right) = G(t), \quad t \in I. \quad (3.2)$$

By Lemma 2.3, the map $G \in C^0(I)$ is strictly decreasing $G(0) = 1$, and $G(1) = 0$. The map G can be extended to a lift of F , and

$$|G(t_1) - G(t_2)| \leq M|t_1 - t_2|, \quad \forall t_1, t_2 \in I. \quad (3.3)$$

Let $M_0 = \sum_{j=1}^n \beta_j (\alpha_1^{-1} M)^{2^j - 2}$. Note that $\mathfrak{F}'(I, 0, \alpha_1^{-1} M) \neq \emptyset$, since $M > \alpha_1$. Define an operator

$$\mathcal{L} : \mathfrak{F}'(I, 0, \alpha_1^{-1} M) \longrightarrow C^0(I) \quad (3.4)$$

by $g \mapsto \mathfrak{L}_g$, where

$$\mathfrak{L}_g(t) = \Psi\left(t, g^2(t), \dots, g^{2^{n-2}}(t)\right), \quad t \in I. \quad (3.5)$$

By (H2), we have for any $t \geq s \in I$,

$$\begin{aligned} \mathfrak{L}_g(t) - \mathfrak{L}_g(s) &= \Psi\left(t, g^2(t), \dots, g^{2^{n-2}}(t)\right) - \Psi\left(s, g^2(s), \dots, g^{2^{n-2}}(s)\right) \\ &\geq \alpha_1(t-s) + \sum_{j=2}^n \alpha_j \left(g^{2^{j-2}}(t) - g^{2^{j-2}}(s)\right) \\ &\geq \alpha_1(t-s), \end{aligned} \quad (3.6)$$

$$\begin{aligned} \mathfrak{L}_g(t) - \mathfrak{L}_g(s) &= \Psi\left(t, g^2(t), \dots, g^{2^{n-2}}(t)\right) - \Psi\left(s, g^2(s), \dots, g^{2^{n-2}}(s)\right) \\ &\leq \beta_1(t-s) + \sum_{j=2}^n \beta_j \left(g^{2^{j-2}}(t) - g^{2^{j-2}}(s)\right) \\ &\leq M_0(t-s). \end{aligned}$$

Hence, \mathfrak{L}_g is increasing, $\mathfrak{L}_g(0) = 0$, $\mathfrak{L}_g(1) = 1$, and $\mathfrak{L}_g \in \mathfrak{F}(I, \alpha_1, M_0)$. Thus $\mathfrak{L}_g^{-1} \in \mathfrak{F}(I, M_0^{-1}, \alpha_1^{-1})$.

Define $\mathfrak{T} : \mathfrak{F}'(I, 0, \alpha_1^{-1}M) \rightarrow C^0(I)$ by

$$\mathfrak{T}(g) = \mathfrak{L}_g^{-1} \circ G, \quad (3.7)$$

where \mathfrak{L}_g is defined in (3.5) and G is the induced map of F . For convenience, denote $\mathfrak{T}(g) = \mathfrak{T}_g$. Then, we have $\mathfrak{T}_g(0) = 1, \mathfrak{T}_g(1) = 0$. Since \mathfrak{L}_g^{-1} is increasing, we have that for all $s, t \in I$ with $t \geq s$,

$$\begin{aligned} 0 \leq \mathfrak{T}_g(s) - \mathfrak{T}_g(t) &= \mathfrak{L}_g^{-1} \circ G(s) - \mathfrak{L}_g^{-1} \circ G(t) \\ &\leq \alpha_1^{-1}(G(s) - G(t)) \leq \alpha_1^{-1}M(t-s). \end{aligned} \quad (3.8)$$

Hence, $\mathfrak{T}(g) \in \mathfrak{F}'(I, 0, \alpha_1^{-1}M)$.

Furthermore, for any $g_1, g_2 \in \mathfrak{F}'(I, 0, \alpha_1^{-1}M)$, we have

$$\begin{aligned}
\|\mathfrak{T}_{g_1} - \mathfrak{T}_{g_2}\| &= \|\mathfrak{L}_{g_1}^{-1} \circ G - \mathfrak{L}_{g_2}^{-1} \circ G\| \\
&= \|\mathfrak{L}_{g_1}^{-1} - \mathfrak{L}_{g_2}^{-1}\| \\
&\leq \alpha_1^{-1} \|\mathfrak{L}_{g_1} - \mathfrak{L}_{g_2}\|, \\
\|\mathfrak{L}_{g_1} - \mathfrak{L}_{g_2}\| &\leq \sup_{t \in I} \left| \Psi(t, g_1^2(t), \dots, g_1^{2n-2}(t)) - \Psi(t, g_2^2(t), \dots, g_2^{2n-2}(t)) \right| \quad (3.9) \\
&\leq \sum_{j=2}^n \beta_j \|g_1^{2j-2} - g_2^{2j-2}\| \\
&\leq \sum_{j=2}^n \beta_j \sum_{i=1}^{2j-2} (\alpha_1^{-1}M)^{i-1} \|g_1 - g_2\|,
\end{aligned}$$

where Lemma 3.1 and (H2) are applied. Finally we get that

$$\|\mathfrak{T}_{g_1} - \mathfrak{T}_{g_2}\| \leq \sum_{j=2}^n \beta_j \sum_{i=1}^{2j-2} \alpha_1^{-i} M^{i-1} \|g_1 - g_2\|. \quad (3.10)$$

Hence \mathfrak{T} maps $\mathfrak{F}'(I, 0, \alpha_1^{-1}M)$ continuously into itself. By Schauder's fixed point theorem, \mathfrak{T} has a fixed $g \in \mathfrak{F}'(I, 0, \alpha_1^{-1}M)$. By Lemma 2.2, the map g can be extended to a map \tilde{g} such that $\tilde{g}(t+1) - \tilde{g}(t) = -1$, for all $t \in \mathbb{R}$, and there exists a unique $f \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ such that \tilde{g} is a lift of f . Note that for any $t, s \in \mathbb{R}$ with $t < s$, there is an integer k and a nonnegative integer m such that $t \in [k, k+1)$ and $s \in [k+m, k+m+1)$. Thus

$$\begin{aligned}
|\tilde{g}(t) - \tilde{g}(s)| &\leq |\tilde{g}(t) - \tilde{g}(k+1)| + \sum_{j=1}^m |\tilde{g}(k+j) - \tilde{g}(k+j+1)| + |\tilde{g}(s) - \tilde{g}(k+m)| \\
&\leq |\tilde{g}(t-k) - \tilde{g}(1)| + (m-1)|\tilde{g}(0) - \tilde{g}(1)| + |\tilde{g}(s-k-m) - \tilde{g}(0)| \quad (3.11) \\
&\leq \frac{M}{\alpha_1} [1 - (t-k) + m - 1 + s - k - m] = \frac{M}{\alpha_1} (s-t).
\end{aligned}$$

Obviously $f \in H_1^0(\mathbb{T}^1, -1)$ and f is a solution of (1.4) with a Lipschitz constant M/α_1 . \square

4. Uniqueness and Stability

Let $F_1, F_2 \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ and \tilde{F}_1, \tilde{F}_2 be their lifts, respectively. By (iv) of Lemma 2.1, we can define

$$\|F_1 - F_2\|_{\mathbb{R}} := \sup_{t \in \mathbb{R}} |\tilde{F}_1(t_1) - \tilde{F}_2(t_2)|. \quad (4.1)$$

Definition 4.1. For a given small constant $\epsilon > 0$, one says that F_1 is $\epsilon - C^0$ —close to F_2 if

$$\|F_1 - F_2\|_{\mathbb{R}} < \epsilon. \quad (4.2)$$

Now as in [8], we give the definition that (1.4) is stable.

Definition 4.2. Let $F_1, F_2 \in C^0(\mathbb{T}^1, \mathbb{T}^1)$ and f_1, f_2 be corresponding solutions of (1.4) on F_1, F_2 . One says that (1.4) is stable if for all $\epsilon > 0$ there is a $\delta > 0$ such that if F_1 is $\delta - C^0$ —close to F_2 , then f_1 is $\epsilon - C^0$ —close to f_2 .

Theorem 4.3. *Suppose that the conditions in Theorem 3.2 still hold, and*

$$r = \sum_{j=2}^n \beta_j \sum_{i=1}^{2j-2} \alpha_1^{-i} M^{i-1} < 1. \quad (4.3)$$

Then (1.4) has a unique solution $f \in H_1^0(\mathbb{T}^1, -1)$ with a Lipschitz constant M/α_1 . Moreover, (1.4) is stable.

Proof. By Banach contraction theorem and (3.10), the solution f is unique.

Suppose that $F_1, F_2 \in H_1^0(\mathbb{T}^1, -1)$ both satisfy the conditions in Theorem 3.2 and f_1, f_2 are the unique solutions of (1.4) corresponding to the given $F_j, j = 1, 2$, respectively. Assume that \tilde{F}_i, \tilde{f}_i be the lifts of $F_i, f_i, i = 1, 2$, respectively. Let G_i, g_i be the restrictions on $[0, 1]$ of $\tilde{F}_i, \tilde{f}_i, i = 1, 2$. Let Ψ be the induced map of Φ defined in (2.27). By the proof of Theorem 3.2, we have

$$g_i = \mathfrak{L}_{g_i}^{-1} \circ G_i, \quad i = 1, 2, \quad (4.4)$$

where \mathfrak{L}_{g_i} is defined as (3.5) and $\mathfrak{L}_{g_i}^{-1} \in \mathfrak{F}(I, M_0^{-1}, \alpha_1^{-1}), i = 1, 2$. By (3.10), we have

$$\begin{aligned} \|g_1 - g_2\| &= \left\| \mathfrak{L}_{g_1}^{-1} \circ G_1 - \mathfrak{L}_{g_2}^{-1} \circ G_2 \right\| \\ &\leq \left\| \mathfrak{L}_{g_1}^{-1} \circ G_1 - \mathfrak{L}_{g_1}^{-1} \circ G_2 \right\| + \left\| \mathfrak{L}_{g_1}^{-1} \circ G_2 - \mathfrak{L}_{g_2}^{-1} \circ G_2 \right\| \\ &\leq \alpha_1^{-1} \|G_1 - G_2\| + r \|g_1 - g_2\|. \end{aligned} \quad (4.5)$$

Thus

$$\|g_1 - g_2\| \leq [(1 - r)\alpha_1]^{-1} \|G_1 - G_2\|. \quad (4.6)$$

For any $t \in \mathbb{R}$, there is an integer k with $t \in [k, k + 1)$ such that

$$\left| \tilde{f}_1(t) - \tilde{f}_2(t) \right| = \left| \tilde{f}_1(t - k) + k - \tilde{f}_2(t - k) - k \right| = \left| g_1(t - k) - g_2(t - k) \right|. \quad (4.7)$$

This means that

$$\left\| \tilde{f}_1 - \tilde{f}_2 \right\|_{\mathbb{R}} = \|g_1 - g_2\|. \quad (4.8)$$

By the same argument, we have

$$\left\| \tilde{F}_1 - \tilde{F}_2 \right\|_{\mathbb{R}} = \|G_1 - G_2\|. \quad (4.9)$$

Thus

$$\left\| \tilde{f}_1 - \tilde{f}_2 \right\|_{\mathbb{R}} \leq [(1 - r)\alpha_1]^{-1} \left\| \tilde{F}_1 - \tilde{F}_2 \right\|_{\mathbb{R}}. \quad (4.10)$$

By the discussion above, we know that (1.4) is stable. \square

5. Example

Example 5.1. Consider the equation

$$(f(z))^{359/360} (f^3(z))^{(1/720\pi i) \ln(f^3(z))} = \exp\left(\frac{2\pi i(e - z^{1/2\pi i})}{e - 1}\right). \quad (5.1)$$

Obviously

$$F(z) = \exp\left(\frac{2\pi i(e - z^{1/2\pi i})}{e - 1}\right), \quad (5.2)$$

$$\Phi(z_1, z_2) = z_1^{359/360} z_2^{(1/720\pi i) \ln(z_2)}.$$

The map F has a lift \tilde{F} and $\tilde{F}|_{[0,1]} = (e - e^t)/(e - 1)$. Moreover, F has a Lipschitz constant 2. The induced map of Φ is

$$\begin{aligned} \Psi(t_1, t_2) &= \frac{359}{360}t_1 + \frac{1}{360}t_2^2, \quad 0 < t_j < 1, \quad j = 1, 2, \\ \Psi(0, 0) &= 0, \quad \Psi(1, 1) = 1. \end{aligned} \quad (5.3)$$

It is easy to verify that both (H1) and (H2) hold. The constants are $\alpha_1 = \beta_1 = 359/360$, $\alpha_2 = 0$, $\beta_2 = 1/180$. By Theorem 3.2, (5.1) has a solution $f \in H_1^0(\mathbb{T}^1, -1)$ with a Lipschitz constant 720/359.

Note that $\alpha_1^{-1} = 359/360 < 1$. We have

$$r = \beta_2 \sum_{i=1}^2 \alpha_1^{-i} M^{i-1} < \frac{1}{180} \sum_{i=1}^2 2^{i-1} = \frac{1}{60}. \quad (5.4)$$

Thus by Theorem 4.3, the solution is unique and the equation is stable.

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