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Research Article

Convergence Properties for Asymptotically almost Negatively Associated Sequence

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We get the strong law of large numbers, strong growth rate, and the integrability of supremum for the partial sums of asymptotically almost negatively associated sequence. In addition, the complete convergence for weighted sums of asymptotically almost negatively associated sequences is also studied.

1. Introduction

Definition 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively associated (NA) if, for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \leq 0, \tag{1.1}$$

whenever f and g are coordinate-wise nondecreasing such that this covariance exists. An infinite sequence $\{X_n,\ n \geq 1\}$ is NA if every finite subcollection is NA.

The concept of negative association was introduced by Joag-Dev and Proschan [1] and Block et al. [2]. By inspecting the proof of maximal inequality for the NA random variables in Matuła [3], one also can allow negative correlations provided they are small. Primarily motivated by this, Chandra and Ghosal [4, 5] introduced the following dependence.

Definition 1.2. A sequence $\{X_n,\ n \geq 1\}$ of random variables is called asymptotically almost negatively associated (AANA) if there exists a nonnegative sequence $q(n) \to 0$ as $n \to \infty$ such that

$$\operatorname{Cov}(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})) \leq q(n)\left[\operatorname{Var}(f(X_n))\operatorname{Var}(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k}))\right]^{1/2} \tag{1.2}$$

for all $n, k \geq 1$ and for all coordinate-wise nondecreasing continuous functions f and g whenever the variances exist.

The family of AANA sequence contains NA (in particular, independent) sequences (with $q(n) = 0$, $n \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated. An example of an AANA sequence which is not NA was constructed by Chandra and Ghosal [4].

Since the concept of AANA sequence was introduced by Chandra and Ghosal [4], many applications have been found. For example, Chandra and Ghosal [4] derived the Kolmogorov-type inequality and the strong law of large numbers of Marcinkiewicz-Zygmund, Chandra and Ghosal [5] obtained the almost sure convergence of weighted averages, Ko et al. [6] studied the Hájek-Rényi-type inequality, and Wang et al. [7] established the law of the iterated logarithm for product sums. Recently, Yuan and An [8] established some Rosenthal-type inequalities for maximum partial sums of AANA sequence. As applications of these inequalities, they derived some results on L_p convergence, where $1 < p < 2$, and complete convergence. In addition, they estimated the rate of convergence in Marcinkiewicz-Zygmund strong law for partial sums of identically distributed random variables.

The main purpose of the paper is to study the strong law of large numbers, strong growth rate, and the integrability of supremum for AANA sequence. In addition, the complete convergence for weighted sums of AANA sequence is also studied.

Throughout the paper, we let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Denote $S_n \doteq \sum_{i=1}^{n} X_i$. Let $X^{(a)} = -aI(X < -a) + XI(|X| \leq a) + aI(X > a)$ for some $a > 0$, and let $I(A)$ be the indicator function of the set A. For $p > 1$, let $q \doteq p/(p-1)$ be the dual number of p. We assume that $\phi(x)$ is a positive increasing function on $(0, \infty)$ satisfying $\phi(x) \uparrow \infty$ as $x \to \infty$ and $\psi(x)$ is the inverse function of $\phi(x)$. Since $\phi(x) \uparrow \infty$, it follows that $\psi(x) \uparrow \infty$. For easy notation, we let $\phi(0) = 0$ and $\psi(0) = 0$. The $a_n = O(b_n)$ denotes that there exists a positive constant C such that $|a_n/b_n| \leq C$. C denotes a positive constant which may be different in various places. The main results of this paper are dependent on the following lemmas.

Lemma 1.3 (cf. Yuan and An [8, Lemma 2.1]). *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$, and let $f_1, f_2, \ldots$ be all nondecreasing (or nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$.*

Lemma 1.4. *Let $1 < p \leq 2$, and let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficients $\{q(n), n \geq 1\}$ and $EX_n = 0$ for each $n \geq 1$. If $\sum_{n=1}^{\infty} q^2(n) < \infty$, then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1\leq i\leq n}|S_i|^p\right) \leq C_p \sum_{i=1}^{n} E|X_i|^p \tag{1.3}$$

for all $n \geq 1$, where $C_p = 2^p[2^{2-p}p + (6p)^p(\sum_{n=1}^{\infty} q^2(n))^{p/q}]$.

Proof. We use the same notations as that in the study by Yuan and An [8]. They proved that

$$E\left|\max_{1\le i\le n} S_i\right|^p \le 2^{2-p}p\sum_{i=1}^{n}\|X_i\|_p^p + (6p)^p\left(\sum_{i=1}^{n-1}q^{2/q}(i)\|X_i\|_p\right)^p,$$
$$E\left|\max_{1\le i\le n}(-S_i)\right|^p \le 2^{2-p}p\sum_{i=1}^{n}\|X_i\|_p^p + (6p)^p\left(\sum_{i=1}^{n-1}q^{2/q}(i)\|X_i\|_p\right)^p, \tag{1.4}$$
$$\max_{1\le i\le n}|S_i|^p \le 2^{p-1}\left|\max_{1\le i\le n} S_i\right|^p + 2^{p-1}\left|\max_{1\le i\le n}(-S_i)\right|^p.$$

By (1.4) and Hölder's inequality, we have

$$\begin{aligned} E\left(\max_{1\le i\le n}|S_i|^p\right) &\le 2^{p-1}E\left|\max_{1\le i\le n} S_i\right|^p + 2^{p-1}E\left|\max_{1\le i\le n}(-S_i)\right|^p \\ &\le 2^p\left[2^{2-p}p\sum_{i=1}^{n}E|X_i|^p + (6p)^p\left(\sum_{i=1}^{n}q^{2/q}(i)\|X_i\|_p\right)^p\right] \\ &\le 2^p\left[2^{2-p}p\sum_{i=1}^{n}E|X_i|^p + (6p)^p\left(\sum_{i=1}^{n}q^2(i)\right)^{p/q}\sum_{i=1}^{n}E|X_i|^p\right] \\ &\le 2^p\left[2^{2-p}p + (6p)^p\left(\sum_{n=1}^{\infty}q^2(n)\right)^{p/q}\right]\sum_{i=1}^{n}E|X_i|^p = C_p\sum_{i=1}^{n}E|X_i|^p. \end{aligned} \tag{1.5}$$

This completes the proof of the lemma. □

We point out that Lemma 1.4 has been studied by Yuan and An [8]. But here we give the accurate coefficient C_p. And Lemma 1.4 generalizes and improves the result of Lemma 2.2 in the study by Ko et al. [6].

Lemma 1.5 (cf. Fazekas and Klesov [9, Theorem 2.1] and Hu et al. [10, Lemma 1.5]). *Let $\{X_n, n \ge 1\}$ be a sequence of random variables. Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers, and let $\alpha_1, \alpha_2, \ldots$ be nonnegative numbers. Let r and C be fixed positive numbers. Assume that, for each $n \ge 1$,*

$$E\left(\max_{1\le l\le n}|S_l|^r\right) \le C\sum_{l=1}^{n}\alpha_l, \tag{1.6}$$

$$\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty, \tag{1.7}$$

then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \quad a.s., \tag{1.8}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad a.s., \tag{1.9}$$

where

$$\beta_n = \max_{1\le k\le n} b_k v_k^{\delta/r}, \quad \forall 0<\delta<1, \qquad v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{b_k^r}, \qquad \lim_{n\to\infty}\frac{\beta_n}{b_n} = 0,$$

$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{b_l}\right|^r\right) \le 4C\sum_{l=1}^{n}\frac{\alpha_l}{b_l^r} < \infty, \tag{1.10}$$

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{b_l}\right|^r\right) \le 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty.$$

If further one assumes that $\alpha_n > 0$ for infinitely many n, then

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{\beta_l}\right|^r\right) \le 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r} < \infty. \tag{1.11}$$

Lemma 1.6 (cf. Fazekas and Klesov [9, Corollary 2.1] and Hu [11, Corollary 2.1.1]). *Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers, and let $\alpha_1, \alpha_2, \ldots$ be nonnegative numbers. Denote $\Lambda_k = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ for $k \ge 1$. Let r be a fixed positive number satisfying* (1.6). *If*

$$\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^r} - \frac{1}{b_{l+1}^r}\right) < \infty, \tag{1.12}$$

$$\frac{\Lambda_n}{b_n^r} \text{ is bounded}, \tag{1.13}$$

then (1.8)–(1.11) *hold.*

Lemma 1.7 (cf. Yuan and An [8, Theorem 2.1]). *Let $\{X_n, n \ge 1\}$ be a sequence of AANA random variables with $EX_i = 0$ for all $i \ge 1$ and $p \in (3\cdot 2^{k-1}, 4\cdot 2^{k-1}]$, where integer number $k \ge 1$. If $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$, then there exists a positive constant D_p depending only on p such that, for all $n \ge 1$,*

$$E\left(\max_{1\le i\le n}|S_i|^p\right) \le D_p\left\{\sum_{i=1}^{n}E|X_i|^p + \left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\}. \tag{1.14}$$

Lemma 1.8. *Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n)\sum_{i=1}^{n}\frac{1}{\psi(i)} = O(n). \tag{1.15}$$

If $E[\phi(|X|)] < \infty$, then $\sum_{n=1}^{\infty}(1/\psi(n))E|X|I(|X| > \psi(n)) < \infty$.

Proof. Since $\psi(x)$ is an increasing function of x, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{\psi(n)}E|X|I(|X| > \psi(n)) &= \sum_{n=1}^{\infty}\frac{1}{\psi(n)}\sum_{i=n}^{\infty}E|X|I\left(\psi(i) < |X| \le \psi(\mathrm{i}+1)\right) \\
&= \sum_{i=1}^{\infty}E|X|I\left(\psi(i) < |X| \le \psi(i+1)\right)\sum_{n=1}^{i}\frac{1}{\psi(n)} \\
&\le \sum_{i=1}^{\infty}P\left(\psi(i) < |X| \le \psi(i+1)\right)\psi(i+1)\sum_{n=1}^{i}\frac{1}{\psi(n)} \\
&\le C\sum_{i=1}^{\infty}P\left(\psi(i) < |X| \le \psi(i+1)\right)i \\
&\le CE\left[\phi(|X|)\right] < \infty.
\end{aligned} \tag{1.16}$$

The proof is complete. □

2. Strong Law of Large Numbers and Growth Rate for AANA Sequence

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$, and let $\{b_n, n \ge 1\}$ be a nondecreasing unbounded sequence of positive numbers; $1 < p \le 2$. Assume that*

$$\sum_{n=1}^{\infty}\frac{E|X_n|^p}{b_n^p} < \infty, \tag{2.1}$$

then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \quad a.s., \tag{2.2}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad a.s., \tag{2.3}$$

where

$$\beta_n = \max_{1\le k\le n} b_k v_k^{\delta/2}, \quad \forall 0<\delta<1, \qquad v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{b_k^p}, \qquad \lim_{n\to\infty}\frac{\beta_n}{b_n}=0,$$
$$\alpha_k = C_p E|X_k|^p, \quad k\ge 1,\ C_p \text{ is defined in Lemma 1.4},$$
$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{b_l}\right|^p\right) \le 4\sum_{l=1}^{n}\frac{\alpha_l}{b_l^p}<\infty,$$
$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{b_l}\right|^p\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^p}<\infty. \tag{2.4}$$

If further one assumes that $\alpha_n>0$ for infinitely many n, then

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{\beta_l}\right|^p\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^p}<\infty. \tag{2.5}$$

Proof. By Lemma 1.4, we have

$$E\left(\max_{1\le k\le n}|S_k|^p\right) \le C_p\sum_{k=1}^{n}E|X_k|^p = \sum_{k=1}^{n}\alpha_k. \tag{2.6}$$

It follows from (2.1) that

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n^p} = C_p\sum_{n=1}^{\infty}\frac{E|X_n|^p}{b_n^p}<\infty. \tag{2.7}$$

Thus, (2.2)–(2.5) follow from (2.6), (2.7), and Lemma 1.5 immediately. We complete the proof of the theorem. □

Theorem 2.2. *Let $\{X_n, n\ge 1\}$ be a sequence of AANA random variables with $\sum_{n=1}^{\infty} q^2(n)<\infty$, $1\le p<2$. Denote $Q_n=\max_{1\le k\le n}EX_k^2$ for $n\ge 1$ and $Q_0=0$. Assume that*

$$\sum_{n=1}^{\infty}\frac{Q_n}{n^{2/p}}<\infty, \tag{2.8}$$

then

$$\lim_{n\to\infty}\frac{1}{n^{1/p}}\sum_{i=1}^{n}(X_i-EX_i)=0 \quad a.s., \tag{2.9}$$

and with the growth rate

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n}(X_i - EX_i) = O\left(\frac{\beta_n}{n^{1/p}}\right) \quad a.s., \tag{2.10}$$

where

$$\beta_n = \max_{1\le k\le n} k^{1/p} v_k^{\delta/2}, \quad \forall 0<\delta<1, \qquad v_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{k^{2/p}}, \qquad \lim_{n\to\infty}\frac{\beta_n}{n^{1/p}} = 0,$$
$$\alpha_k = C_2(kQ_k - (k-1)Q_{k-1}), \quad k\ge 1, \ C_2 \text{ is defined in Lemma 1.4}, \tag{2.11}$$

$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{l^{1/p}}\right|^2\right) \le 4\sum_{l=1}^{n}\frac{\alpha_l}{l^{2/p}} < \infty, \tag{2.12}$$

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{l^{1/p}}\right|^2\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}} < \infty. \tag{2.13}$$

If further one assumes that $\alpha_n > 0$ *for infinitely many* n, *then*

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{\beta_l}\right|^2\right) \le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^2} < \infty. \tag{2.14}$$

In addition, for any $r \in (0,2)$,

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{l^{1/p}}\right|^r\right) \le 1 + \frac{4r}{2-r}\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}} < \infty. \tag{2.15}$$

Proof. Assume that $EX_n = 0$, $b_n = n^{1/p}$, and $\Lambda_n = \sum_{l=1}^{n}\alpha_l$, $n\ge 1$. By Lemma 1.4, we can see that

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^2\right) \le C_2\sum_{i=1}^{n}EX_i^2 \le C_2 nQ_n = \sum_{k=1}^{n}\alpha_k. \tag{2.16}$$

It is a simple fact that $\alpha_k \ge 0$ for all $k\ge 1$. It follows from (2.8) that

$$\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^2} - \frac{1}{b_{l+1}^2}\right) = C_2\sum_{l=1}^{\infty} lQ_l\left(\frac{1}{l^{2/p}} - \frac{1}{(l+1)^{2/p}}\right) \le \frac{2C_2}{p}\sum_{l=1}^{\infty}\frac{Q_l}{l^{2/p}} < \infty. \tag{2.17}$$

That is to say that (1.12) holds. By Remark 2.1 in Fazekas and Klesov [9], (1.12) implies (1.13). By Lemma 1.6, we can obtain (2.9)–(2.14) immediately. By (2.13), it follows that

$$E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r\right) = \int_0^\infty P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r > t\right)dt \leq 1 + \int_1^\infty P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right| > t^{1/r}\right)dt$$
$$\leq 1 + E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^2\right)\int_1^\infty t^{-2/r}dt \leq 1 + \frac{4r}{2-r}\sum_{l=1}^\infty \frac{\alpha_l}{l^{2/p}} < \infty. \tag{2.18}$$

The proof is complete. □

Theorem 2.3. *Let* $p \in (3\cdot 2^{k-1}, 4\cdot 2^{k-1}]$, *where integer number* $k \geq 1$, *and let* $\{X_n, n \geq 1\}$ *be a sequence of AANA random variables with* $EX_i = 0$ *for all* $i \geq 1$ *and* $\sum_{n=1}^\infty q^{q/p}(n) < \infty$. *Let* $\{b_n, n \geq 1\}$ *be a nondecreasing unbounded sequence of positive numbers. Assume that*

$$\sum_{n=1}^\infty \frac{n^{p/2-1}}{b_n^p}E|X_n|^p < \infty,$$
$$\sum_{k=1}^\infty E|X_k|^p \sum_{n=k+1}^\infty \frac{n^{p/2-2}}{b_n^p} < \infty, \tag{2.19}$$

then (1.8)–(1.11) *hold (for* $C = 1$*), where*

$$\alpha_1 = 2D_pE|X_1|^p, \quad \alpha_k = 2D_p\left(k^{p/2-1}\sum_{j=1}^k E\left|X_j\right|^p - (k-1)^{p/2-1}\sum_{j=1}^{k-1}E\left|X_j\right|^p\right), \quad k \geq 2, \tag{2.20}$$

and D_p *is defined in Lemma 1.7.*

Proof. Since $p > 2$, $0 < 2/p < 1$. By C_r's inequality,

$$\left(\sum_{i=1}^n |X_i|^p\right)^{2/p} \leq \sum_{i=1}^n X_i^2, \tag{2.21}$$

which implies that

$$\sum_{i=1}^n E|X_i|^p \leq E\left(\sum_{i=1}^n X_i^2\right)^{p/2}. \tag{2.22}$$

By Jensen's inequality, we have

$$\left(\sum_{i=1}^n EX_i^2\right)^{p/2} \leq E\left(\sum_{i=1}^n X_i^2\right)^{p/2}. \tag{2.23}$$

By (2.22)-(2.23) and C_r's inequality,

$$\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} \leq 2E\left(\sum_{i=1}^{n} X_i^2\right)^{p/2} \leq 2n^{p/2-1}\sum_{i=1}^{n} E|X_i|^p. \tag{2.24}$$

It follows from Lemma 1.7 and (2.24) that

$$E\left(\max_{1\leq i\leq n}|S_i|^p\right) \leq 2D_p n^{p/2-1}\sum_{i=1}^{n} E|X_i|^p = \sum_{l=1}^{n}\alpha_l. \tag{2.25}$$

It is a simple fact that

$$0 \leq \alpha_k \leq C_1(p)\left(k^{p/2-1}E|X_k|^p + k^{p/2-2}\sum_{j=1}^{k-1} E\left|X_j\right|^p\right), \tag{2.26}$$

where $C_1(p)$ is a positive number depending only on p and D_p. By (2.19),

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{b_n^p} \leq C_1(p)\left(\sum_{n=1}^{\infty}\frac{n^{p/2-1}}{b_n^p}E|X_n|^p + \sum_{k=1}^{\infty}E|X_k|^p\sum_{n=k+1}^{\infty}\frac{n^{p/2-2}}{b_n^p}\right) < \infty. \tag{2.27}$$

The desired results follow from (2.25)–(2.27) and Lemma 1.5 immediately. □

3. Complete Convergence for Weighted Sums of AANA Random Variables

Theorem 3.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed AANA random variables with $\sum_{n=1}^{\infty} q^2(n) < \infty$, $EX = 0$, $EX^2 < \infty$, and $E[\phi(|X|)] < \infty$. Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies* (1.15)*. Let $\{a_{ni}, n \geq 1, i \geq 1\}$ be a triangular array of positive constants such that*

(i) $\max_{1\leq i\leq n} a_{ni} = O(1/\psi(n))$,

(ii) $\sum_{i=1}^{n} a_{ni}^2 = O(\log^{-1-\alpha} n)$ *for some* $\alpha > 0$.

Then, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon\right) < \infty. \tag{3.1}$$

Proof. For each $n \geq 1$, denote

$$X_j^{(n)} = -\psi(n)I(X_j < -\psi(n)) + X_j I(|X_j| \leq \psi(n)) + \psi(n)I(X_j > \psi(n)), \quad 1 \leq j \leq n,$$
$$T_j^{(n)} = \sum_{i=1}^{j}\left(a_{ni}X_i^{(n)} - Ea_{ni}X_i^{(n)}\right), \quad 1 \leq j \leq n,$$
$$A = \bigcap_{i=1}^{n}\left(X_i = X_i^{(n)}\right) = \bigcap_{i=1}^{n}(|X_i| \leq \psi(n)), \qquad B = \overline{A} = \bigcup_{i=1}^{n}\left(X_i \neq X_i^{(n)}\right) = \bigcup_{i=1}^{n}(|X_i| > \psi(n)),$$
$$E_n = \left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > \varepsilon\right). \tag{3.2}$$

It is easy to check that

$$\sum_{i=1}^{j}a_{ni}X_i = T_j^{(n)} + \sum_{i=1}^{j}Ea_{ni}X_i^{(n)} + \sum_{i=1}^{j}a_{ni}X_i I(|X_i| > \psi(n))$$
$$+ \sum_{i=1}^{j}a_{ni}\psi(n)\left[I(X_j < -\psi(n)) - I(X_j > \psi(n))\right],$$
$$E_n = E_nA + E_nB = \left(\max_{1\leq j\leq n}\left|T_j^{(n)} + \sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right| > \varepsilon\right) + E_nB$$
$$\subset \left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right|\right) + B. \tag{3.3}$$

Therefore,

$$P(E_n) \leq P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right|\right) + P(B)$$
$$\leq P\left(\max_{1\leq j\leq n}\left|T_j^{(n)}\right| > \varepsilon - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right|\right) + \sum_{i=1}^{n}P(|X_i| > \psi(n)). \tag{3.4}$$

Firstly, we will show that

$$\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.5}$$

It follows from Lemma 1.8 and Kronecker's lemma that

$$\frac{1}{\psi(n)}\sum_{i=1}^{n}E|X|I\left(|X|>\psi(i)\right)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty. \tag{3.6}$$

By $EX=0$, condition (i), (3.6), and $\psi(n)\uparrow\infty$, we can see that

$$\begin{aligned}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right| &\le \sum_{i=1}^{n}\left|Ea_{ni}X_iI\left(|X_i|\le\psi(n)\right)\right|+\sum_{i=1}^{n}a_{ni}\psi(n)EI\left(|X_i|>\psi(n)\right)\\ &\le \sum_{i=1}^{n}a_{ni}E|X_i|I\left(|X_i|>\psi(n)\right)+\sum_{i=1}^{n}a_{ni}E|X_i|I\left(|X_i|>\psi(n)\right)\\ &\le \frac{C}{\psi(n)}\sum_{i=1}^{n}E|X|I\left(|X|>\psi(i)\right)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty,\end{aligned} \tag{3.7}$$

which implies (3.5). By (3.4) and (3.5), we can see that, for sufficiently large n,

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon\right)\le P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}\right)+\sum_{i=1}^{n}P\left(|X_i|>\psi(n)\right). \tag{3.8}$$

To prove (3.1), it suffices to show that

$$\begin{gathered}\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}\right)<\infty,\\ \sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{\mathrm{n}}P\left(|X_i|>\psi(n)\right)<\infty.\end{gathered} \tag{3.9}$$

By Markov's inequality, Lemma 1.4, C_r inequality, $EX^2<\infty$, and condition (ii), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}\right)\\ &\quad\le C\sum_{n=1}^{\infty}n^{-1}E\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|^2\right)\le C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}E\left|a_{ni}X_i^{(n)}\right|^2\\ &\quad\le C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}a_{ni}^2EX^2I\left(|X|\le\psi(n)\right)+C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}a_{ni}^2\psi^2(n)EI\left(|X|>\psi(n)\right)\\ &\quad\le C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}a_{ni}^2EX^2I\left(|X|\le\psi(n)\right)+C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}a_{ni}^2EX^2I\left(|X|>\psi(n)\right)\\ &\quad\le C\sum_{n=1}^{\infty}n^{-1}\sum_{i=1}^{n}a_{ni}^2\le C\sum_{n=1}^{\infty}n^{-1}\log^{-1-\alpha}n<\infty.\end{aligned} \tag{3.10}$$

It follows from $E[\phi(|X|)] < \infty$ that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P(|X_i| > \psi(n)) &= \sum_{n=1}^{\infty} P(|X| > \psi(n)) = \sum_{n=1}^{\infty} P(\phi(|X|) > n) \\ &\le CE[\phi(|X|)] < \infty.\end{aligned} \tag{3.11}$$

□

Theorem 3.2. *Let $\{X_n, n \ge 1\}$ be a sequence of AANA random variables, and let $\{a_{ni}, n \ge 1, i \ge 1\}$ be an array of positive numbers. Let $\{b_n, n \ge 1\}$ be an increasing sequence of positive integers, and let $\{c_n, n \ge 1\}$ be a sequence of positive numbers. If, for some $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, where integer number $k \ge 1$, $0 < t < 2$, and for any $\varepsilon > 0$, the following conditions are satisfied:*

$$\begin{gathered}\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni} X_i| \ge \varepsilon b_n^{1/t}\right) < \infty, \\ \sum_{n=1}^{\infty} c_n b_n^{-p/t} \sum_{i=1}^{b_n} |a_{ni}|^p E|X_i|^p I\left(|a_{ni} X_i| < \varepsilon b_n^{1/t}\right) < \infty, \\ \sum_{n=1}^{\infty} c_n b_n^{-p/t} \left[\sum_{i=1}^{b_n} a_{ni}^2 E X_i^2 I\left(|a_{ni} X_i| < \varepsilon b_n^{1/t}\right)\right]^{p/2} < \infty,\end{gathered} \tag{3.12}$$

and $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$, then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1 \le i \le b_n} \left|\sum_{j=1}^{i} \left[a_{nj} X_j - a_{nj} E X_j I\left(\left|a_{nj} X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} < \infty. \tag{3.13}$$

Proof. Note that if the series $\sum_{n=1}^{\infty} c_n$ is convergent, then (3.13) holds. Therefore, we will consider only such sequences $\{c_n, n \ge 1\}$ for which the series $\sum_{n=1}^{\infty} c_n$ is divergent.

Let

$$\begin{gathered}Y_i^{(n)} = -\varepsilon b_n^{1/t} I\left(a_{ni} X_i < -\varepsilon b_n^{1/t}\right) + a_{ni} X_i I\left(|a_{ni} X_i| < \varepsilon b_n^{1/t}\right) + \varepsilon b_n^{1/t} I\left(a_{ni} X_i > \varepsilon b_n^{1/t}\right), \\ S'_{ni} = \sum_{j=1}^{i} Y_j^{(n)}, \quad n \ge 1, i \ge 1.\end{gathered} \tag{3.14}$$

Note that

$$\begin{aligned}P&\left\{\max_{1 \le i \le b_n} \left|\sum_{j=1}^{i} \left[a_{nj} X_{nj} - a_{nj} E X_j I\left(\left|a_{nj} X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} \\ &\le C \sum_{i=1}^{b_n} P\left(|a_{ni} X_i| \ge \varepsilon b_n^{1/t}\right) + 2^p \varepsilon^{-p} b_n^{-p/t} E\left(\max_{1 \le i \le b_n} |S'_{ni} - ES'_{ni}|\right)^p.\end{aligned} \tag{3.15}$$

Using the C_r inequality and Jensen's inequality, we can estimate $E|Y_i^{(n)} - EY_i^{(n)}|^p$ in the following way:

$$E\left|Y_i^{(n)} - EY_i^{(n)}\right|^p \le C|a_{ni}|^p E|X_i|^p I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) + Cb_n^{p/t} P\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right). \tag{3.16}$$

By (3.15), (3.16), and Lemma 1.7, we can get

$$\begin{aligned} &P\left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left[a_{nj}X_j - a_{nj}EX_jI\left(\left|a_{nj}X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \ge \varepsilon b_n^{1/t}\right\} \\ &\quad \le C\sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \ge \varepsilon b_n^{1/t}\right) + Cb_n^{-p/t}\sum_{i=1}^{b_n}|a_{ni}|^p E|X_i|^p I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) \\ &\quad + Cb_n^{-p/t}\left[\sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right)\right]^{p/2}. \end{aligned} \tag{3.17}$$

Therefore, we can conclude that (3.13) holds by (3.12) and (3.17). □

Theorem 3.3. *Let* $1 \le r \le 2$ *and let* $\{X_n, n \ge 1\}$ *be a sequence of AANA random variables with* $EX_n = 0$ *and* $E|X_n|^r < \infty$ *for* $n \ge 1$. *Let* $\{a_{ni}, n \ge 1, i \ge 1\}$ *be an array of real numbers satisfying the condition*

$$\sum_{i=1}^{n}|a_{ni}|^r E|X_i|^r = O\left(n^{\delta}\right) \quad \text{as } n \longrightarrow \infty \tag{3.18}$$

and $\sum_{n=1}^{\infty} q^{q/p}(n) < \infty$ *for some* $0 < \delta \le 2/p$ *and* $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, *where integer number* $k \ge 1$. *Then, for any* $\varepsilon > 0$ *and* $\alpha r \ge 1$,

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left(\max_{1\le i\le n}\left|\sum_{j=1}^{i} a_{nj}X_j\right| \ge \varepsilon n^{\alpha}\right) < \infty. \tag{3.19}$$

Proof. Take $c_n = n^{\alpha r-2}$, $b_n = n$, and $1/t = \alpha$ in Theorem 3.2. By (3.18), we have

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \geq \varepsilon b_n^{1/t}\right) \leq C\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{i=1}^{n} \frac{|a_{ni}|^r E|X_i|^r}{n^{\alpha r}} \leq C\sum_{n=1}^{\infty} n^{-2+\delta} < \infty,$$

$$\sum_{n=1}^{\infty} c_n b_n^{-p/t} \sum_{i=1}^{b_n} |a_{ni}|^p E|X_i|^p I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) \leq \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}|^r E|X_i|^r \leq C\sum_{n=1}^{\infty} n^{-2+\delta} < \infty,$$

$$\begin{aligned}\sum_{n=1}^{\infty} c_n b_n^{-p/t} \left[\sum_{i=1}^{b_n} a_{ni}^2 EX_i^2 I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right)\right]^{p/2} &\leq C\sum_{n=1}^{\infty} n^{\alpha r-2-(\alpha rp/2)} \left(\sum_{i=1}^{n} |a_{ni}|^r E|X_i|^r\right)^{p/2} \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha r-2-(\alpha rp/2)+(\delta p/2)} \leq C\sum_{n=1}^{\infty} n^{\alpha r(1-p/2)-1} < \infty\end{aligned} \tag{3.20}$$

following from $\delta p/2 \leq 1$. By the assumption $EX_n = 0$ for $n \geq 1$ and (3.18), we get

$$\begin{aligned}&\frac{1}{n^{\alpha}} \max_{1\leq i\leq n} \left|\sum_{j=1}^{i} a_{nj} EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon n^{\alpha}\right)\right| \\ &\quad\leq \frac{1}{n^{\alpha}} \sum_{j=1}^{n} \left|a_{nj} EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon n^{\alpha}\right)\right| = \frac{1}{n^{\alpha}} \sum_{j=1}^{n} \left|a_{nj} EX_j I\left(\left|a_{nj}X_j\right| \geq \varepsilon n^{\alpha}\right)\right| \\ &\quad\leq \frac{1}{n^{\alpha r}} \sum_{j=1}^{n} \left|a_{nj}\right|^r E\left|X_j\right|^r \leq Cn^{\delta-\alpha r} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty\end{aligned} \tag{3.21}$$

following from $\delta < 1$ and $\alpha r \geq 1$. We get the desired result from Theorem 3.2 immediately. The proof is complete. □

Theorem 3.4. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables satisfying $\sum_{n=1}^{\infty} q^2(n) < \infty$, and let $\{a_{ni}, n \geq 1, i \geq 1\}$ be an array of positive numbers. Let $\{b_n, n \geq 1\}$ be an increasing sequence of positive integers, and let $\{c_n, n \geq 1\}$ be a sequence of positive numbers. If, for some $1 < p \leq 2$, $0 < t < 2$, and for any $\varepsilon > 0$, the following conditions are satisfied:*

$$\begin{gathered}\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_i| \geq \varepsilon b_n^{1/t}\right) < \infty, \\ \sum_{n=1}^{\infty} c_n b_n^{-p/t} \sum_{i=1}^{b_n} |a_{ni}|^p E|X_i|^p I\left(|a_{ni}X_i| < \varepsilon b_n^{1/t}\right) < \infty,\end{gathered} \tag{3.22}$$

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq i\leq b_n} \left|\sum_{j=1}^{i} \left[a_{nj}X_j - a_{nj} EX_j I\left(\left|a_{nj}X_j\right| < \varepsilon b_n^{1/t}\right)\right]\right| \geq \varepsilon b_n^{1/t}\right\} < \infty. \tag{3.23}$$

Proof. The proof is similar to that of Theorem 3.2, so we omit it. □

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