Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 239787, 17 pages
doi:10.1155/2010/239787

Research Article

Periodic Solutions for a Numerical Discretization Neural Network

Chun Lu

School of Science, Qingdao Technological University, Qingdao 266520, China

Correspondence should be addressed to Chun Lu, mathlc@163.com

Received 21 January 2010; Accepted 4 March 2010

Academic Editor: Guang Zhang

The existence and global exponential stability of periodic solutions for a class of numerical discretization neural networks are considered. Using coincidence degree theory and Lyapunov method, sufficient conditions for the existence and global exponential stability of periodic solutions are obtained. Numerical simulations are given to illustrate the results.

1. Introduction

In this paper, we investigate the existence and stability of periodic solutions for a numerical discretization neural network, which results from the θ-method for neural networks with finite delays and distributed delays:

$$\begin{aligned}\dot{x}_i(t) &= -b_i(t)x_i(t)\\ &\quad + f_i\Bigg(t, x_1(t-\tau_{i1}(t)), \ldots, x_m(t-\tau_{im}(t));\\ &\qquad \int_0^{+\infty} k_{i1}(s)x_1(t-s)ds, \ldots, \int_0^{+\infty} k_{im}(s)x_m(t-s)ds\Bigg) + I_i(t),\\ &\qquad\qquad t \geq 0, \quad \tau_{ij}(t) \geq 0, \quad i, j = 1, \ldots, m.\end{aligned} \tag{1.1}$$

System (1.1) is a more general form of neural networks with delay. Many authors have discussed other neural networks with delays [1–8], and most of their systems can be deduced from (1.1).

For system (1.1), we make the following assumptions.

(H_1) For each $i \in \{1, \ldots, m\}$, $b_i(t)$ is bounded and continuous with $b_i(t) > 0$ and $b_i(t+\overline{\omega}) = b_i(t)$ for $t \in \mathbf{R}^+$, $I_i(t)$ is bounded and continuous, and $I_i(t+\overline{\omega}) = I_i(t)$ for $t \in \mathbf{R}^+$, where $\overline{\omega}$ is a positive constant.

(H_2) For each $i, j \in \{1, \ldots, m\}$, $k_{ij} : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies

$$\int_0^{+\infty} k_{ij}(s)ds = 1, \qquad \int_0^{+\infty} sk_{ij}(s)ds < +\infty. \tag{1.2}$$

(H_3) For each $i \in \{1, \ldots, m\}$, $f_i : \mathbf{R}^+ \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ is bounded and continuous and there exists nonnegative, bounded, and continuous functions $\alpha_{ij}(t)$, $\beta_{ij}(t)$ defined on $\mathbf{R}^+$ such that

$$\begin{aligned} &\left| f_i(t, u_1, \ldots, u_m; v_1, \ldots, v_m) - f_i(t, \overline{u}_1, \ldots, \overline{u}_m; \overline{v}_1, \ldots, \overline{v}_m) \right| \\ &\quad \le \sum_{j=1}^{m} \left[\alpha_{ij}(t) \left| u_j - \overline{u}_j \right| + \beta_{ij}(t) \left| v_j - \overline{v}_j \right| \right], \quad i = 1, \ldots, m. \end{aligned} \tag{1.3}$$

for any $(u_1, \ldots, u_m), (\overline{u}_1, \ldots, \overline{u}_m), (v_1, \ldots, v_m), (\overline{v}_1, \ldots, \overline{v}_m) \in \mathbf{R}^m$.

For system (1.1), we consider initial conditions of the form

$$x_i(t) = \varphi_i(t), \quad t \in (-\infty, 0], \; i = 1, \ldots, m, \tag{1.4}$$

where $\varphi_i(t)$ is bounded and continuous on $(-\infty, 0]$.

For convenience in our study, we adopt the following notations. Let $\mathbf{Z}$ denote the set of all integers, $\mathbf{Z}_0^+ = \{0, 1, 2, \ldots\}$, $[a, b]_{\mathbf{Z}} = \{a, a+1, \ldots, b-1, b\}$ where $a, b \in \mathbf{Z}$, $a \le b$, and $[a, \infty)_{\mathbf{Z}} = \{a, a+1, a+2, \ldots\}$ where $a \in \mathbf{Z}$. We begin approximating the continuous-time network (1.1) by replacing the integral terms with discrete sums of the form

$$\int_0^{+\infty} k_{ij}(s) x_j(t-s)ds \approx \sum_{[s/h]=1}^{+\infty} \omega_{ij}(h) k_{ij}\left(\left[\frac{s}{h}\right] h \right) x_j\left(\left[\frac{t}{h}\right] h - \left[\frac{s}{h}\right] h \right) \tag{1.5}$$

for $t \in [nh, (n+1)h)$, $s \in [ph, (p+1)h)$, $n \in \mathbf{Z}_0^+$, $p \in \mathbf{Z}^+$, where $[r]$ denotes the integer part of a real number r, $h > 0$ is a fixed number denoting a uniform discretization step size satisfying $h = \overline{\omega}/\omega$, $\omega \in N$, $\omega_{ij}(h) > 0$, for $h > 0$ and $\omega_{ij}(h) \approx h + O(h^2)$ for small $h > 0$. We note that $\omega_{ij}(h)$ are chosen so that the analogue kernels

$$\mathscr{K}_{ij}\left(\left[\frac{s}{h}\right] h \right) = \omega_{ij}(h) k_{ij}\left(\left[\frac{s}{h}\right] h \right) \tag{1.6}$$

satisfy the following properties:

(H_4) $\mathscr{K}_{ij} : \mathbf{Z}^+ \to [0, +\infty)$, $\sum_{p=1}^{+\infty} \mathscr{K}_{ij}(p) = 1$, $\sum_{p=1}^{+\infty} \mathscr{K}_{ij}(p)p < \infty$.

This analogue has been employed elsewhere (see, e.g., [9–11] in the formulation of discrete-time analogues of the distributed delay). With (1.5) and (1.6), we approximate (1.1) by differential equations with piecewise constant arguments of the form

$$\begin{aligned}\dot{x}_i(t) &= -b_i(t)x_i(t)\\ &\quad + f_i\Bigg(t, x_1(t-\tau_{i1}(t)), \dots, x_m(t-\tau_{im}(t)); \sum_{[s/h]=1}^{+\infty} \mathscr{K}_{i1}\left(\left[\frac{s}{h}\right]h\right)x_1\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right), \dots,\\ &\qquad \sum_{[s/h]=1}^{+\infty} \mathscr{K}_{im}\left(\left[\frac{s}{h}\right]h\right)x_m\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\Bigg) + I_i(t)\end{aligned} \tag{1.7}$$

for $t \in [nh, (n+1)h), s \in [ph, (p+1)h)$, and $n \in \mathbf{Z}_0^+, p \in \mathbf{Z}^+$. Noting that $[t/h] = n$, and $[s/h] = p$ and adopting the notation $u(n) = u(nh)$, we rewrite (1.7) as

$$\begin{aligned}\dot{x}_i(t) &= -\, b_i(t)x_i(t)\\ &\quad + f_i\Bigg(t, x_1(t-\tau_{i1}(t)), \dots, x_m(t-\tau_{im}(t)); \sum_{p=1}^{+\infty} \mathscr{K}_{i1}(p)x_1(n-p), \dots,\\ &\qquad \sum_{p=1}^{+\infty} \mathscr{K}_{im}(p)x_m(n-p)\Bigg) + I_i(t), \quad t \ge 0,\ i = 1, \dots, m.\end{aligned} \tag{1.8}$$

The initial values of (1.8) will be given below in (1.14).

The application of θ-method to the differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{1.9}$$

gives

$$x_{n+1} = x_n + h\theta f(t_n, x_n) + h(1-\theta)f(t_{n+1}, x_{n+1}), \quad 0 \le \theta \le 1, \tag{1.10}$$

where h is step size.

Applying θ-method (1.10) to (1.8) over the interval $[nh, t]$, where $t < (n+1)h$ and adopting the notation $u(n) = u(nh)$, we have

$$\begin{aligned}
x_i(t) &= x_i(n) + (t-nh)\theta \\
&\quad\times \Bigg(- b_i(n)x_i(n) + f_i\Bigg(n, x_1(n-\tau_{i1}(n)), \dots, x_m(n-\tau_{im}(n)); \sum_{p=1}^{+\infty} \mathscr{K}_{i1}(p)x_1(n-p), \dots, \\
&\qquad \sum_{p=1}^{+\infty} \mathscr{K}_{im}(p)x_m(n-p) \Bigg) + I_i(n) \Bigg) + (t-nh)(1-\theta) \\
&\quad\times \Bigg(- b_i(t)x_i(t) + f_i\Bigg(t, x_1(t-\tau_{i1}(t)), \dots, x_m(t-\tau_{im}(t)); \\
&\qquad \sum_{p=1}^{+\infty} \mathscr{K}_{i1}(p)x_1(n-p), \dots, \sum_{p=1}^{+\infty} \mathscr{K}_{im}(p)x_m(n-p) \Bigg) + I_i(t) \Bigg), \\
&\qquad i = 1, \dots, m, \quad n \in \mathbf{Z}_0^+, \quad \text{here } 0 \le \theta \le 1,
\end{aligned} \tag{1.11}$$

and by allowing $t \to (n+1)h$ in the above, we obtain

$$\begin{aligned}
&x_i(n+1) \\
&= x_i(n) + h\theta \\
&\quad\times \Bigg(- b_i(n)x_i(n) + f_i\Bigg(n, x_1(n-\tau_{i1}(n)), \dots, x_m(n-\tau_{im}(n)); \sum_{p=1}^{+\infty} \mathscr{K}_{i1}(p)x_1(n-p), \dots, \\
&\qquad \sum_{p=1}^{+\infty} \mathscr{K}_{im}(p)x_m(n-p) \Bigg) + I_i(n) \Bigg) + h(1-\theta) \\
&\quad\times \Bigg(- b_i(n+1)x_i(n+1) + f_i\Bigg(n+1, x_1(n+1-\tau_{i1}(n+1)), \dots, x_m(n+1-\tau_{im}(n+1)); \\
&\qquad \sum_{p=1}^{+\infty} \mathscr{K}_{i1}(p)x_1(n-p), \dots, \sum_{p=1}^{+\infty} \mathscr{K}_{im}(p)x_m(n-p) \Bigg) + I_i(n+1) \Bigg), \\
&\qquad i = 1, \dots, m, \quad n \in \mathbf{Z}_0^+, \quad \text{here } 0 \le \theta \le 1.
\end{aligned} \tag{1.12}$$

Then we have

$$
\begin{aligned}
&x_i(n+1) \\
&= \frac{1}{1+(1-\theta)hb_i(n+1)} \\
&\quad\times\Bigg(\big(1-\theta hb_i(n)\big)x_i(n)+\theta h\Bigg(f_i\Bigg(n,x_1(n-\tau_{i1}(n)),\dots,x_m(n-\tau_{im}(n)); \\
&\qquad\qquad \sum_{p=1}^{+\infty}\mathcal{K}_{i1}(p)x_1(n-p),\dots,\sum_{p=1}^{+\infty}\mathcal{K}_{im}(p)x_m(n-p)\Bigg)+I_i(n)\Bigg) \\
&\quad+(1-\theta)h\Bigg(f_i\Bigg(n+1,x_1(n+1-\tau_{i1}(n+1)),\dots,x_m(n+1-\tau_{im}(n+1)); \\
&\qquad\qquad \sum_{p=1}^{+\infty}\mathcal{K}_{i1}(p)x_1(n-p),\dots,\sum_{p=1}^{+\infty}\mathcal{K}_{im}(p)x_m(n-p)\Bigg)+I_i(n+1)\Bigg)\Bigg), \\
&\qquad\qquad i=1,\dots,m,\quad n\in\mathbf{Z}_0^+,\quad \text{here } 0\le\theta\le 1.
\end{aligned}
\tag{1.13}
$$

One can show that (1.13) converges towards (1.1) when $h \to 0^+$. In studying the discrete-time analogue (1.13), we assume that

(H_5) $h \in (0,\infty)$, $b_i : \mathbf{Z} \to (0,\infty)$, $\tau_{ij} : \mathbf{Z} \to \mathbf{Z}_0^+$, $i,j = 1,2,\dots,m$, and the function f_i satisfies (H_3). The system (1.13) is supplemented with initial values given by

$$x_i(s) = \varphi_i(s), \quad s \in \mathbf{Z}_0^- = \{0,-1,-2,\dots\}. \tag{1.14}$$

For convenience, we will use the following notations

$$I_\omega = \{0,1,\dots,\omega-1\}, \quad \overline{u} = \frac{1}{\omega}\sum_{k=0}^{\omega-1}u(k), \tag{1.15}$$

where $\{u(k)\}$ is ω-periodic sequence of real numbers defined for $k \in Z$ and the notations:

$$
\begin{gathered}
\alpha_{ij}^M = \max_{n\in\mathbf{Z}} \alpha_{ij}(n), \quad \beta_{ij}^M = \max_{n\in\mathbf{Z}} \beta_{ij}(n), \\
\underline{b_i} = \min_{n\in I_\omega}\{b_i(n)\}, \quad \overline{b_i} = \max_{n\in I_\omega}\{b_i(n)\}, \quad i=1,2,\dots,m, \\
M = \sup_{u\in\mathbf{R}^+\times\mathbf{R}^m\times\mathbf{R}^m}\{|f_i(u)|,\ i=1,2,\dots,m\}, \qquad J^M = \max_{n\in I_\omega}\{|I_i(n)|,\ i=1,2,\dots,m\}.
\end{gathered}
\tag{1.16}
$$

2. Existence of Periodic Solutions

In this section, based on Mawhin's continuation theorem (see [12–14]), we will study the existence of at least one periodic solution of (1.13).

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ a linear mapping, and $N : X \to Z$ a continuous mapping. This mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, it follows that $L \mid \operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible; we denote the inverse of that map by k_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$; if $QN(\overline{\Omega})$ is bounded and $k_P(I - Q)N : \overline{\Omega} \to X$ is compact, since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1. *Let L be a $Fredholm$ mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose that*

(a) *for each $\lambda \in (0, 1)$ every solution x of $Lx = \lambda N x$ is such that $x \notin \partial\Omega$,*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Theorem 2.2. *Assume that (H_1)–(H_5) and $\overline{b_i}h < 1$ hold, Then system (1.13) has at least one ω-periodic solution.*

Proof. Define

$$l_m = \{x = x(k) : x(k) \in \mathbf{R}^m,\ k \in \mathbf{Z}\},$$
$$|x| = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2}, \quad \forall x \in \mathbf{R}^m. \tag{2.1}$$

Let $l^\omega \subset l_m$ denote the subspace of all ω-periodic sequences equipped with the norm $\|\cdot\|$, that is, $\|x\| = \max_{k\in I_\omega}|x(k)|$, for any $x(k) = \{(x_1(k), x_2(k), \dots, x_m(k))^T,\ k \in \mathbf{Z}\} \in l^\omega$.

It is easy to prove that l^ω is a finite-dimensional Banach space.

Let

$$l_0^\omega = \left\{y = \{y(k)\} \in l^\omega : \frac{1}{\omega}\sum_{k=0}^{\omega-1} y(k) = 0\right\}, \qquad l_c^\omega = \{y = \{y(k)\} \in l^\omega : y(k) = c \in \mathbf{R}^m,\ k \in \mathbf{Z}\}, \tag{2.2}$$

then it follows that l_0^ω and l_c^ω are both closed linear subspaces of l^ω and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \qquad \dim l_c^\omega = m. \tag{2.3}$$

Now let us define $X = Y = l^{\omega}$, $(Lx)(k) = x(k+1) - x(k),\ x \in X,\ k \in \mathbf{Z}$, and

$$\begin{aligned}(Nx_i)(n) = &-\frac{h(\theta b_i(n) + (1-\theta)b_i(n+1))}{1+(1-\theta)hb_i(n+1)}x_i(n) + \frac{\theta h}{1+(1-\theta)hb_i(n+1)} \\ &\times \left(f_i\left(n, x_1(n-\tau_{i1}(n)), \ldots, x_m(n-\tau_{im}(n)); \sum_{p=1}^{+\infty}\mathscr{H}_{i1}(p)x_1(n-p), \ldots, \right.\right. \\ &\left.\left. \sum_{p=1}^{+\infty}\mathscr{H}_{im}(p)x_m(n-p) \right) + I_i(n) \right) + \frac{(1-\theta)h}{1+(1-\theta)hb_i(n+1)} \\ &\times \left(f_i\left(n+1, x_1(n+1-\tau_{i1}(n+1)), \ldots, x_m(n+1-\tau_{im}(n+1)); \right.\right. \\ &\left.\left. \sum_{p=1}^{+\infty}\mathscr{H}_{i1}(p)x_1(n-p), \ldots, \sum_{p=1}^{+\infty}\mathscr{H}_{im}(p)x_m(n-p) \right) + I_i(n+1) \right), \\ &i = 1, \ldots, m, \quad n \in \mathbf{Z}_0^+. \end{aligned} \tag{2.4}$$

It is easy to see that L is a bounded linear operator and

$$\operatorname{Ker} L = l_c^{\omega}, \qquad \operatorname{Im} L = l_0^{\omega}, \tag{2.5}$$

as well as

$$\dim \operatorname{Ker} L = m = \operatorname{codim} \operatorname{Im} L; \tag{2.6}$$

then it follows that L is a Fredholm mapping of index zero.

Define

$$Px = \frac{1}{\omega}\sum_{k=0}^{\omega-1}x(k), \quad x \in X, \qquad Qy = \frac{1}{\omega}\sum_{k=0}^{\omega-1}y(k), \quad y \in Y. \tag{2.7}$$

It is not difficult to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q). \tag{2.8}$$

Furthermore, the generalized inverse(to L) $k_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$(k_P y)(n) = \sum_{\nu=0}^{n-1}y(\nu) - \frac{1}{\omega}\sum_{\nu=1}^{\omega}\sum_{s=0}^{\nu-1}y(s), \quad n \in [0, \omega-1]_Z. \tag{2.9}$$

Clearly, QN and $k_P(I-Q)N$ are continuous, since X is a finite-dimensional Banach space. Using the Arzela-Ascoli theorem, it is not difficult to show that $\overline{k_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$. Since $\operatorname{Im} Q = \operatorname{Ker} L$, the isomorphic mapping J from $\operatorname{Im} Q$ to $\operatorname{Ker} L$ is I. We now are in position to search for an appropriate open, bounded subset $\Omega \subset X$ for the continuation theorem.

Corresponding to operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned} x_i(n+1) - x_i(n) = \lambda\Bigg(&-\frac{h(\theta b_i(n) + (1-\theta)b_i(n+1))}{1+(1-\theta)hb_i(n+1)}x_i(n) + \frac{\theta h}{1+(1-\theta)hb_i(n+1)} \\ &\times \Bigg(f_i\Bigg(n, x_1(n-\tau_{i1}(n)),\dots,x_m(n-\tau_{im}(n)); \sum_{p=1}^{+\infty}\mathcal{K}_{i1}(p)x_1(n-p),\dots, \\ &\sum_{p=1}^{+\infty}\mathcal{K}_{im}(p)x_m(n-p)\Bigg) + I_i(n)\Bigg) + \frac{(1-\theta)h}{1+(1-\theta)hb_i(n+1)} \\ &\times \Bigg(f_i\Bigg(n+1, x_1(n+1-\tau_{i1}(n+1)),\dots,x_m(n+1-\tau_{im}(n+1)); \\ &\sum_{p=1}^{+\infty}\mathcal{K}_{i1}(p)x_1(n-p),\dots,\sum_{p=1}^{+\infty}\mathcal{K}_{im}(p)x_m(n-p)\Bigg) + I_i(n+1)\Bigg)\Bigg), \\ &i = 1,\dots,m, \quad n \in \mathbf{Z}_0^+. \end{aligned} \tag{2.10}$$

Suppose that $\{(x_1(n), x_2(n),\dots,x_m(n))^T\} \in X$ is a solution of system (2.10) for a certain $\lambda \in (0,1)$. In view of (2.10) and condition $\overline{b_i}h < 1$ in Theorem 2.2, we have

$$\begin{aligned} \max_{n\in I_\omega}|x_i(n)| &= \max_{n\in I_\omega}|x_i(n+1)| \\ &\le \max_{n\in I_\omega}\Bigg(\Bigg(1 - \frac{\lambda h(\theta b_i(n) + (1-\theta)b_i(n+1))}{1+(1-\theta)hb_i(n+1)}\Bigg)|x_i(n)| + \frac{\lambda\theta h}{1+(1-\theta)hb_i(n+1)} \\ &\quad\times\Bigg|f_i\Bigg(n, x_1(n-\tau_{i1}(n)),\dots,x_m(n-\tau_{im}(n)); \sum_{p=1}^{+\infty}\mathcal{K}_{i1}(p)x_1(n-p),\dots, \\ &\quad\sum_{p=1}^{+\infty}\mathcal{K}_{im}(p)x_m(n-p)\Bigg) + I_i(n)\Bigg| + \frac{\lambda(1-\theta)h}{1+(1-\theta)hb_i(n+1)} \\ &\quad\times\Bigg|f_i\Bigg(n+1, x_1(n+1-\tau_{i1}(n+1)),\dots,x_m(n+1-\tau_{im}(n+1)); \\ &\quad\sum_{p=1}^{+\infty}\mathcal{K}_{i1}(p)x_1(n-p),\dots,\sum_{p=1}^{+\infty}\mathcal{K}_{im}(p)x_m(n-p)\Bigg) + I_i(n+1)\Bigg|\Bigg) \\ &\le \Bigg(1 - \frac{\lambda h\underline{b_i}}{1+h\overline{b_i}}\Bigg)\max_{n\in I_\omega}|x_i(n)| + \lambda h\Big(M + J^M\Big). \end{aligned} \tag{2.11}$$

Therefore,

$$\frac{\underline{b_i}}{1+h\overline{b_i}}\max_{n\in I_\omega}|x_i(n)| \le \left(M+J^M\right), \quad i=1,2,\ldots,m, \tag{2.12}$$

that is,

$$\max_{n\in I_\omega}|x_i(n)| \le \frac{\left(M+J^M\right)\left(1+h\overline{b_i}\right)}{\underline{b_i}} := A_i, \quad i=1,2,\ldots,m. \tag{2.13}$$

Denote $A = (\sum_{i=1}^{m} A_i^2)^{1/2} + B$, where $B > 0$ is a constant. Clearly, A is independent of λ. Now we take $\Omega = \{x \in X : \|x\| < A\}$. This Ω satisfies condition (i) in Lemma 2.1. When $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^m$, x is a constant vector in R^m with $\|x\| = A$. Then, if necessary, we can let A be greater such that

$$\begin{aligned}
x^T QNx &= \sum_{i=1}^{m}\Bigg(-x_i^2 h\sum_{k=0}^{\omega-1}\frac{\theta b_i(k)+(1-\theta)b_i(k+1)}{\omega(1+(1-\theta)hb_i(k+1))} \\
&\quad + x_i\sum_{k=0}^{\omega-1}\bigg(\frac{\theta h}{\omega(1+(1-\theta)hb_i(k+1))}\left(f_i(k,x_1\ldots,x_m)+I_i(k)\right) \\
&\quad + \frac{(1-\theta)h}{\omega(1+(1-\theta)hb_i(k+1))}\left(f_i(k+1,x_1\ldots,x_m)+I_i(k+1)\right)\bigg)\Bigg) \\
&\le \sum_{i=1}^{m}\left[-\frac{\underline{b_i}h}{1+h\overline{b_i}}|x_i|^2 + |x_i|h\left(M+J^M\right)\right] \\
&\le -\min_{1\le i\le m}\left\{\frac{\underline{b_i}h}{1+h\overline{b_i}}\right\}\|x\|^2 + \sqrt{m}h\left(M+J^M\right)\|x\| < 0.
\end{aligned} \tag{2.14}$$

Therefore, $QNx \ne 0$ for any $x \in \partial\Omega\cap\ker L$. Let $\psi(\gamma;x) = -\gamma x + (1-\gamma)QNx$, $\gamma \in [0,1]$, then for any $x \in \partial\Omega\cap\ker L$, $x^T\psi(\gamma;x) < 0$. From the homotopy invariance of Brower degree, it follows that

$$\deg\{JQN, \Omega\cap\operatorname{Ker} L, 0\} = \deg\{-x, \Omega\cap\operatorname{Ker} L, 0\} \ne 0. \tag{2.15}$$

Condition (b) of Lemma 2.1 is also satisfied. Thus, by Lemma 2.1 we conclude that $Lx = Nx$ has at least one solution in X, that is, (1.13) has at least one ω-periodic solution. The proof is complete. □

3. Stability of Periodic Solution

In this section, we shall construct appropriate Lyapunov functions to study the stability of periodic solutions of (1.13).

Theorem 3.1. *Assume that* (H_1)–(H_5) *holds. Furthermore, assume that* $\tau_{ij}(n) \equiv \tau_{ij} \geq 2$, $\tau_{ij} \in \mathbf{Z}^+$, $n \in \mathbf{Z}$, $i, j = 1, 2, \dots, m$ *are constants and*

$$\overline{b_i} h < 1, \qquad \theta \underline{b_i} > \sum_{j=1}^{m} \left(\alpha_{ji}^M + \beta_{ji}^M \right), \qquad \sum_{p=1}^{+\infty} p \lambda^p \mathscr{K}_{ji}(p) < +\infty, \tag{3.1}$$

then the ω*-periodic solution* $x^*(n) = \{(x_1^*(n), x_2^*(n), \dots, x_m^*(n))^T\}$ *of* (1.13) *is unique and global exponential stability in the sense that there exist constants* $\lambda > 1$ *and* $\delta \geq 1$ *such that*

$$\sum_{i=1}^{m} \left| x_i(n) - x_i^*(n) \right| \leq \delta \left(\frac{1}{\lambda} \right)^n \sum_{i=1}^{m} \left\{ \sup_{s \in \mathbf{Z}_0^-} \left| x_i(s) - x_i^*(s) \right| \right\}, \quad n \in \mathbf{Z}_0^+. \tag{3.2}$$

Proof. Let $x(n) = \{(x_1(n), x_2(n), \dots, x_m(n))^\top\}$ be an arbitrary solution of (1.12), and $x^*(n) = \{(x_1^*(n), x_2^*(n), \dots, x_m^*(n))^\top\}$ a ω-periodic solution of (1.12). Then

$$\begin{aligned} \left| x_i(n+1) - x_i^*(n+1) \right| \leq{} & \left| x_i(n) - x_i^*(n) \right| \left(1 - \theta \underline{b_i} h \right) \\ & + \theta h \sum_{j=1}^{m} \alpha_{ij}^M \left| x_j(n - \tau_{ij}) - x_j^*(n - \tau_{ij}) \right| \\ & + h \sum_{j=1}^{m} \beta_{ij}^M \sum_{p=1}^{+\infty} \mathscr{K}_{ij}(p) \left| x_j(n-p) - x_j^*(n-p) \right| \\ & + (1-\theta) h \sum_{j=1}^{m} \alpha_{ij}^M \left| x_j(n+1-\tau_{ij}) - x_j^*(n+1-\tau_{ij}) \right| \\ & \qquad i = 1, 2, \dots, m. \end{aligned} \tag{3.3}$$

Now we consider functions $\beta_i(\cdot, \cdot)$, $i = 1, 2, \dots, m$, defined by

$$\beta_i(v_i, n) = 1 - v_i(1 - \theta b_i(n) h) - h \sum_{j=1}^{m} \alpha_{ji}^M v_i^{\tau_{ji}+1} - h \sum_{j=1}^{m} \beta_{ji}^M \sum_{p=1}^{+\infty} v_i^{p+1} \mathscr{K}_{ji}(p), \tag{3.4}$$

where $v_i \in [1, \infty)$, $n \in I_\omega$, $i = 1, 2, \dots, m$, since

$$\begin{aligned} \beta_i(1, n) &= \theta b_i(n) h - h \sum_{j=1}^{m} \alpha_{ji}^M - h \sum_{j=1}^{m} \beta_{ji}^M \\ &= h \left(\theta b_i(n) - \sum_{j=1}^{m} \alpha_{ji}^M - \sum_{j=1}^{m} \beta_{ji}^M \right) \geq h\eta > 0, \quad i = 1, 2, \dots, m, \end{aligned} \tag{3.5}$$

where

$$\eta = \min_{1\leq i\leq m}\left\{ \underline{b_i}\theta - \sum_{j=1}^{m}\alpha_{ji}^{M} - \sum_{j=1}^{m}\beta_{ji}^{M} \right\}; \tag{3.6}$$

using the continuity of $\beta_i(v_i, n)$ on $[1,+\infty)$ with respect to v_i and the fact that $\beta_i(v_i, n) \to -\infty$ as $v_i \to \infty$ uniformly in $n \in I_\omega$, $i = 1,2,\ldots,m$, we see that there exists $v_i^*(n) \in (1,\infty)$ such that $\beta_i(v_i^*(n), n) = 0$ for $n \in I_\omega$, $i = 1,2,\ldots,m$. By choosing $\lambda = \min\{v_i^*(n),\ n \in I_\omega,\ i = 1,2,\ldots,m\}$ where $\lambda > 1$, we obtain $\beta_i(\lambda, n) \geq 0$ for all $n \in I_\omega$, $i = 1,2,\ldots,m$ with the implications

$$\lambda(1-\theta b_i(n)h) + h\sum_{j=1}^{m}\alpha_{ji}^{M}\lambda^{\tau_{ji}(n)+1} + h\sum_{j=1}^{m}\beta_{ji}^{M}\sum_{p=1}^{+\infty}\lambda^{p+1}\mathscr{K}_{ji}(p) \leq 1, \quad n\in I_\omega,\ i=1,2,\ldots,m. \tag{3.7}$$

Hence,

$$\lambda\left(1-\theta\underline{b_i}h\right) + h\sum_{j=1}^{m}\alpha_{ji}^{M}\lambda^{\tau_{ji}(n)+1} + h\sum_{j=1}^{m}\beta_{ji}^{M}\sum_{p=1}^{+\infty}\lambda^{p+1}\mathscr{K}_{ji}(p) \leq 1, \quad i=1,2,\ldots,m. \tag{3.8}$$

Now let us consider

$$u_i(n) = \lambda^n\frac{\left|x_i(n)-x_i^*(n)\right|}{h}, \quad n\in\mathbf{Z},\ i=1,2,\ldots,m. \tag{3.9}$$

Using (3.3) and (3.9), we derive that

$$\begin{aligned} u_i(n+1) \leq{}& \lambda\left(1-\theta\underline{b_i}h\right)u_i(n) + \sum_{j=1}^{m}\theta h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}u_j(n-\tau_{ij}) + \sum_{j=1}^{m}h\beta_{ij}^{M}\sum_{p=1}^{+\infty}\lambda^{p+1}\mathscr{K}_{ij}(p)u_j(n-p) \\ &+ \sum_{j=1}^{m}(1-\theta)h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}u_j(n+1-\tau_{ij}). \end{aligned} \tag{3.10}$$

We consider the Lyapunov function

$$\begin{aligned} V(n) = \sum_{i=1}^{m}\Bigg(& u_i(n) + \sum_{j=1}^{m}\theta h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}\sum_{s=n-\tau_{ij}}^{n-1}u_j(s) + \sum_{j=1}^{m}h\beta_{ij}^{M}\sum_{p=1}^{+\infty}\mathscr{K}_{ij}(p)\lambda^{p+1}\sum_{s=n-p}^{n-1}u_j(s) \\ &+\sum_{j=1}^{m}(1-\theta)h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}\sum_{s=n+1-\tau_{ij}}^{n-1}u_j(s)\Bigg). \end{aligned} \tag{3.11}$$

Consider the difference $\Delta V(n) = V(n+1) - V(n)$ along (3.11), we obtain

$$\begin{aligned}
\Delta V(n) = \sum_{i=1}^{m}\Bigg(& u_i(n+1) - u_i(n) + \sum_{j=1}^{m}\theta h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}\left(u_j(n) - u_j(n-\tau_{ij})\right) \\
& + \sum_{j=1}^{m} h\beta_{ij}^{M}\sum_{p=1}^{+\infty}\mathscr{K}_{ij}(p)\lambda^{p+1}\left(u_j(n) - u_j(n-p)\right) \\
& + \sum_{j=1}^{m}(1-\theta)h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}\left(u_j(n) - u_j(n+1-\tau_{ij})\right)\Bigg) \\
\le \sum_{i=1}^{m}\Bigg(& \left(\lambda\left(1-\theta\underline{b_i}h\right) - 1\right)u_i(n) + \sum_{j=1}^{m} h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}u_j(n) + \sum_{j=1}^{m} h\beta_{ij}^{M}\sum_{p=1}^{+\infty}\lambda^{p+1}\mathscr{K}_{ij}(p)u_j(n)\Bigg) \\
\le -\sum_{i=1}^{m}\Bigg(& 1 - \lambda\left(1-\theta\underline{b_i}h\right) - h\sum_{j=1}^{m}\alpha_{ji}^{M}\lambda^{\tau_{ji}+1} - h\sum_{j=1}^{m}\beta_{ji}^{M}\sum_{p=1}^{+\infty}\lambda^{p+1}\mathscr{K}_{ji}(p)\Bigg)u_i(n)
\end{aligned} \tag{3.12}$$

and by using (3.8) above, we deduce that $\Delta V(n) \le 0$ for $n \in \mathbf{Z}_0^+$. From this result and (3.9), we have

$$\sum_{i=1}^{m} u_i \le V(n) \le V(0) \quad \text{for } n \in \mathbf{Z}^+. \tag{3.13}$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{m} u_i &= \lambda^{n}\sum_{i=1}^{m}\frac{\left|x_i(n) - x_i^*(n)\right|}{h} \\
&\le \sum_{i=1}^{m}\Bigg(u_i(0) + \sum_{j=1}^{m}\theta h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}\sum_{s=-\tau_{ij}}^{-1}u_j(s) \\
&\qquad + \sum_{j=1}^{m} h\beta_{ij}^{M}\sum_{p=1}^{+\infty}\mathscr{K}_{ij}(p)\lambda^{p+1}\sum_{s=-p}^{-1}u_j(s) + \sum_{j=1}^{m}(1-\theta)h\alpha_{ij}^{M}\lambda^{\tau_{ij}+1}\sum_{s=1-\tau_{ij}}^{-1}u_j(s)\Bigg) \\
&\le \sum_{i=1}^{m}\frac{1}{h}\Bigg(1 + h\sum_{j=1}^{m}\alpha_{ji}^{M}\lambda^{\tau_{ji}+1}\tau_{ji} + h\sum_{j=1}^{m}\beta_{ji}^{M}\sum_{p=1}^{+\infty}\mathscr{K}_{ji}(p)\lambda^{p+1}p\Bigg)\sup_{s\in\mathbf{Z}_0^-}\left\{\left|x_i(s) - x_i^*(s)\right|\right\}.
\end{aligned} \tag{3.14}$$

Therefore, we obtain the assertion in Theorem 3.1:

$$\delta = \max_{1\le i\le m}\left\{1 + h\sum_{j=1}^{m}\alpha_{ji}^{M}\lambda^{\tau_{ji}+1}\tau_{ji} + h\sum_{j=1}^{m}\beta_{ji}^{M}\sum_{p=1}^{\infty}\mathscr{K}_{ji}(p)\lambda^{p+1}p\right\} \ge 1. \tag{3.15}$$

We conclude from (3.2) that the unique periodic solution of (1.13) is global exponential stability. This completes the proof. □

4. Numerical Simulations

Remark 4.1. If $\theta = 1/2$ or $\theta = 1$, then (1.13) reduces to a discrete neural network by Trapezoidal method or Euler method (see Examples 4.2 or 4.3).

Example 4.2. One has

$$\begin{aligned}
x(n+1) &= \frac{2}{2+h(\cos(7(n+1)\pi)+5)} \\
&\quad \times \Bigg((1-2h(\cos(7n\pi)+5))x(n) + \frac{y(n-3)h}{6+(y(n-3))^2} + \frac{x(n-1)h}{1+(x(n-1))^2} \\
&\quad + \frac{z(n-1)h}{4+(z(n-1))^2} + \frac{\sum_{p=1}^{100}(e^h-1)e^{-ph}x(n-p)h}{1+\left(\sum_{p=1}^{100}(e^h-1)e^{-ph}x(n-p)\right)^2} + \\
&\quad 5\sin(n\pi)+5\sin((n+1)\pi) \Bigg), \\
y(n+1) &= \frac{2}{3+h(4\sin(5(n+1)\pi)+2)} \\
&\quad \times \Bigg((1-2h(\cos(5n\pi)+5))y(n) + \frac{x(n-4)h}{4+(x(n-4))^2} + \frac{y(n-1)h}{1+(y(n-1))^2} \\
&\quad + \frac{z(n-6)h}{1+(z(n-6))^2} + \frac{\sum_{p=1}^{100}(e^h-1)e^{-ph}x(n-p)h}{2+\left(\sum_{p=1}^{100}(e^h-1)e^{-ph}x(n-p)\right)^2} \\
&\quad +5\cos(n\pi)+4\cos((n+1)\pi) \Bigg), \\
z(n+1) &= \frac{2}{2+h(4\cos(3(n+1)\pi)+1)} \\
&\quad \times \Bigg((1-2h(\cos(3n\pi)+5))z(n) + \frac{x(n-3)h}{6+(x(n-3))^2} + \frac{y(n-1)h}{1+(y(n-1))^2} \\
&\quad + \frac{z(n-7)h}{1+(z(n-7))^2} + \frac{\sum_{p=1}^{100}(e^h-1)e^{-ph}x(n-p)h}{4+\left(\sum_{p=1}^{100}(e^h-1)e^{-ph}x(n-p)\right)^2} \\
&\quad +7\cos(n\pi)+ 6\cos((n+1)\pi) \Bigg),
\end{aligned} \tag{4.1}$$

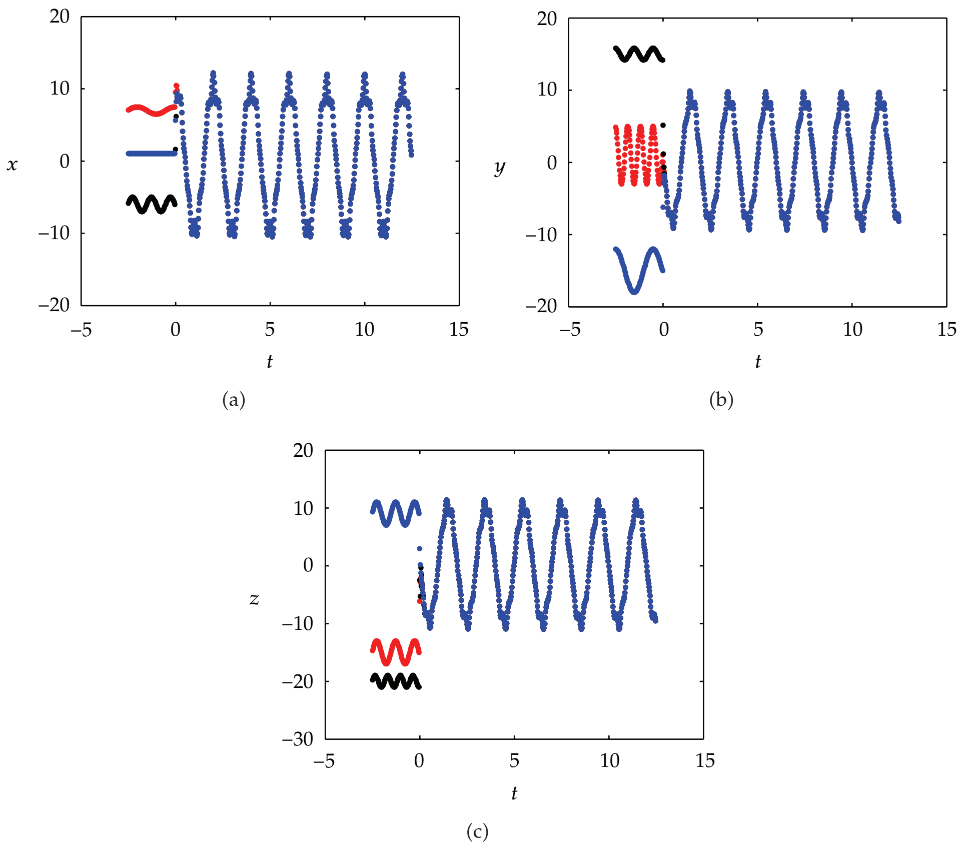

Figure 1: $\theta = 1/2$, $h = 0.025$.

with initial conditions

(1) $\varphi_1(n) = \sin(\pi nh) + 7,\ \varphi_2(n) = 4\cos(3\pi nh) + 1,\ \varphi_3(n) = 2\sin(2\pi nh) - 15,$

(2) $\varphi_1(n) = 0.5\sin(2\pi nh) - 6,\ \varphi_2(n) = 0.8\cos(2\pi nh) + 15,\ \varphi_3(n) = \sin(3\pi nh) - 6,$

(3) $\varphi_1(n) = 1,\ \varphi_2(n) = 3\cos(\pi nh) - 15,\ \varphi_3(n) = 2\sin(2\pi nh) + 9,$

respectively.

It is easy to verify that

$$\overline{b}_i h < 1, \qquad \frac{\underline{b}_i}{2} > \sum_{j=1}^{m}\left(\alpha_{ji}^{M} + \beta_{ji}^{M}\right), \tag{4.2}$$

and the function $\mathscr{K}_{ij}(p)$ satisfies

$$\sum_{p=1}^{100}\left(e^{h} - 1\right)e^{-ph} \approx 1, \qquad \sum_{p=1}^{100} p\left(e^{h} - 1\right)e^{-ph} < \sum_{p=1}^{+\infty} p\left(e^{h} - 1\right)e^{-ph} < +\infty. \tag{4.3}$$

So condition (H_4) is satisfied. Furthermore, using the same method as Theorem 3.1, we can choose a constant λ such that

$$\sum_{p=1}^{100} p\lambda^p \mathscr{K}_{ji}(p) = \sum_{p=1}^{100} p\lambda^p \left(e^h - 1\right)e^{-ph} < \sum_{p=1}^{+\infty} p\lambda^p \left(e^h - 1\right)e^{-ph} = \frac{\lambda(e^h - 1)e^{-h}}{(1 - \lambda e^{-h})^2} < +\infty. \tag{4.4}$$

Therefore, by using Theorems 2.2 and 3.1, the system (4.1) has a unique positive 2-periodic solution which is global exponential stability (see Figure 1).

Example 4.3. One has

$$\begin{aligned}
x(n+1) &= x(n) - 4h(\sin(7n\pi) + 5)x(n) + \frac{x(n)h}{1 + [x(n)]^2} + \frac{y(n)h}{2 + [y(n)]^2} \\
&\quad + \frac{z(n)h}{3 + [z(n)]^2} + \frac{x(n-1)h}{1 + [x(n-1)]^2} + \frac{\sum_{p=1}^{200}(e^h - 1)e^{-ph}x(n-p)h}{1 + \left[\sum_{p=1}^{200}(e^h - 1)e^{-ph}x(n-p)\right]^2} + 5\sin(n\pi), \\
y(n+1) &= y(n) - 4h[\sin(5n\pi) + 7]y(n) + \frac{x(n)h}{1 + [x(n)]^2} + \frac{y(n)h}{2 + [y(n)]^2} \\
&\quad + \frac{z(n)h}{3 + [z(n)]^2} + \frac{x(n-1)h}{1 + [x(n-1)]^2} + \frac{\sum_{p=1}^{200}(e^h - 1)e^{-ph}x(n-p)h}{2 + \left[\sum_{p=1}^{200}(e^h - 1)e^{-ph}x(n-p)\right]^2} + 6\cos(n\pi), \\
z(n+1) &= z(n) - 4h[\sin(3n\pi) + 7]z(n) + \frac{x(n)h}{1 + [x(n)]^2} + \frac{y(n)h}{2 + [y(n)]^2} \\
&\quad + \frac{z(n)h}{3 + [z(n)]^2} + \frac{x(n-1)h}{1 + [x(n-1)]^2} + \frac{\sum_{p=1}^{200}(e^h - 1)e^{-ph}x(n-p)h}{3 + \left[\sum_{p=1}^{200}(e^h - 1)e^{-ph}x(n-p)\right]^2} + 7\sin(n\pi),
\end{aligned} \tag{4.5}$$

with initial conditions

(1) $\varphi_1(n) = 40,\ \varphi_2(n) = 1,\ \varphi_3(n) = -20,$

(2) $\varphi_1(n) = 50,\ \varphi_2(n) = 20,\ \varphi_3(n) = -8,$

(3) $\varphi_1(n) = -10,\ \varphi_2(n) = -15,\ \varphi_3(n) = -25,$

respectively.

Similarly, it is straight forward to check that all the conditions needed in Theorems 2.2 and 3.1 are satisfied. Therefore, system (4.5) has exactly one 2-periodic solution which is global exponentially stability (see Figure 2).

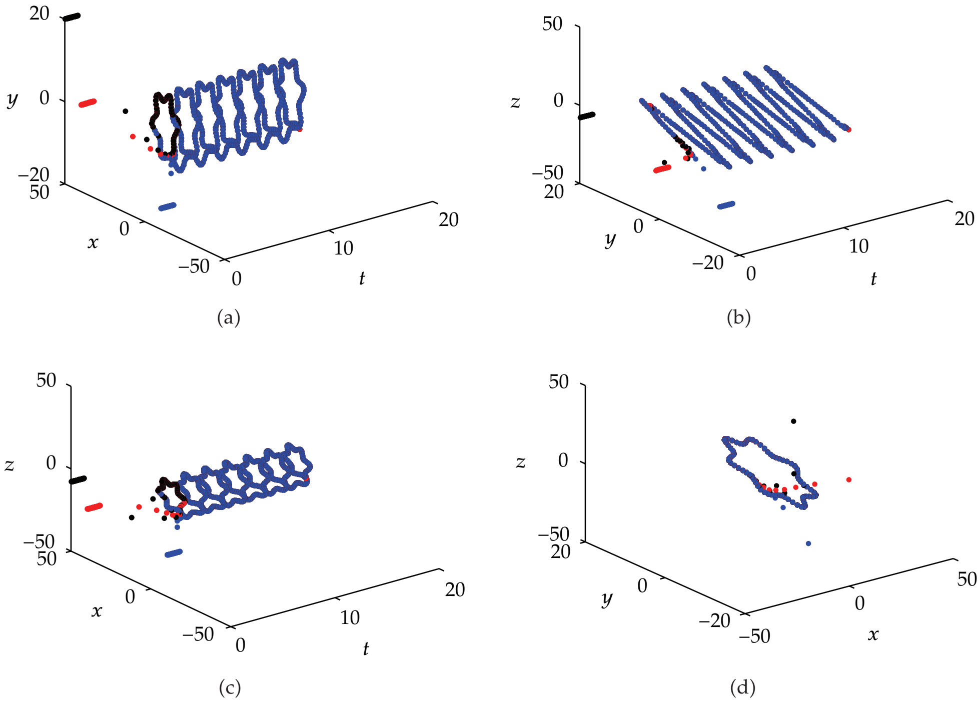

Figure 2: $\theta = 1$, $h = 0.025$.

Acknowledgment

This work is supported by Scientific Research Fund of Heilongjiang Provincial Education Department of PR China (no. 11531428).

References

[1] J. H. Park, "A new stability analysis of delayed cellular neural networks," *Applied Mathematics and Computation*, vol. 181, no. 1, pp. 200–205, 2006.

[2] M. Fan and X. Zou, "Global asymptotic stability of a class of nonautonomous integro-differential systems and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 57, no. 1, pp. 111–135, 2004.

[3] L. Zhang and L. Si, "Existence and global attractivity of almost periodic solution for DCNNs with time-varying coefficients," *Computers & Mathematics with Applications*, vol. 55, no. 8, pp. 1887–1894, 2008.

[4] S. Guo and L. Huang, "Periodic oscillation for a class of neural networks with variable coefficients," *Nonlinear Analysis: Real World Applications*, vol. 6, no. 3, pp. 545–561, 2005.

[5] H. Zhao and J. Cao, "New conditions for global exponential stability of cellular neural networks with delays," *Neural Networks*, vol. 18, no. 10, pp. 1332–1340, 2005.

[6] D. Zhou and J. Cao, "Globally exponential stability conditions for cellular neural networks with time-varying delays," *Applied Mathematics and Computation*, vol. 131, no. 2-3, pp. 487–496, 2002.

[7] L. Zhang and L. Si, "Existence and exponential stability of almost periodic solution for BAM neural networks with variable coefficients and delays," *Applied Mathematics and Computation*, vol. 194, no. 1, pp. 215–223, 2007.

[8] S. Guo and L. Huang, "Stability of nonlinear waves in a ring of neurons with delays," *Journal of Differential Equations*, vol. 236, no. 2, pp. 343–374, 2007.

[9] S. Mohamad and A. G. Naim, “Discrete-time analogues of integrodifferential equations modelling bidirectional neural networks,” *Journal of Computational and Applied Mathematics*, vol. 138, no. 1, pp. 1–20, 2002.
[10] S. Mohamad and K. Gopalsamy, “Neuronal dynamics in time varying environments: continuous and discrete time models,” *Discrete and Continuous Dynamical Systems*, vol. 6, no. 4, pp. 841–860, 2000.
[11] P. Liu and X. Cui, “A discrete model of competition,” *Mathematics and Computers in Simulation*, vol. 49, no. 1-2, pp. 1–12, 1999.
[12] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, vol. 568 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1977.
[13] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, vol. 1 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1975.
[14] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Publications of the Mathematical Society of Japan, no. 9, The Mathematical Society of Japan, Tokyo, Japan, 1966.