Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2010, Article ID 246783, 11 pages
doi:10.1155/2010/246783

Research Article

Permanence of a Discrete Periodic Volterra Model with Mutual Interference and Beddington-DeAngelis Functional Response

Runxin Wu

Department of Mathematics and Physics, Fujian University of Technology, Fuzhou, Fujian 350014, China

Correspondence should be addressed to Runxin Wu, runxinwu@163.com

Received 4 March 2010; Accepted 12 May 2010

Academic Editor: Leonid Berezansky

This paper discuss a discrete periodic Volterra model with mutual interference and Beddington-DeAngelis functional response. By using the comparison theorem of difference equation, sufficient conditions are obtained for the permanence of the system. After that,we give an example to show the feasibility of our main result.

1. Introduction

In 1971, Hassell introduced the concept of mutual interference m $(0 < m \leq 1)$ and established a Volterra model with mutual interference as follows: (see [1])

$$\begin{aligned} \dot{x} &= xg(x) - \varphi(x)y^m, \\ \dot{y} &= y\left(-d + k\varphi(x)y^{m-1} - q(y)\right). \end{aligned} \tag{1.1}$$

Recently, Wang and Zhu [2] proposed following system:

$$\begin{aligned} \dot{x}(t) &= x(t)(r_1(t) - b_1(t)x(t)) - \frac{c_1(t)x(t)}{k + x(t)}y^m(t), \\ \dot{y}(t) &= y(t)\left(-r_2(t) - b_2(t)y(t)\right) + \frac{c_2(t)x(t)}{k + x(t)}y^m(t). \end{aligned} \tag{1.2}$$

Motivated by the works of Wang and Zhu [2], Lin and Chen [3] considered an almost periodic Volterra model with mutual interference and Beddington-DeAngelis function response as follows, which is the generalization of the model (1.2):

$$\begin{aligned}\dot{x}(t) &= x(t)(r_1(t) - b_1(t)x(t)) - \frac{k_1(t)x(t)}{a(t) + d(t)x(t) + c(t)y(t)}y^m(t),\\ \dot{y}(t) &= \left(-r_2(t) - b_2(t)y(t)\right) + \frac{k_2(t)x(t)}{a(t) + d(t)x(t) + c(t)y(t)}y^m(t).\end{aligned} \tag{1.3}$$

Sufficient conditions which guarantee the permanence and existence of a unique globally attractive positive almost periodic solution of the system are obtained by applying the comparison theorem of the differential equation and constructing a suitable Lyapunov functional.

On the other hand, it has been found that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations (see [4, 5]). This motivated us to propose and study the discrete analogous of predator-prey system (1.4):

$$\begin{aligned}x(n+1) &= x(n)\exp\left\{r_1(n) - b_1(n)x(n) - \frac{k_1(n)}{a(n) + d(n)x(n) + c(n)y(n)}y^m(n)\right\},\\ y(n+1) &= y(n)\exp\left\{-r_2(n) - b_2(n)y(n) + \frac{k_2(n)x(n)}{a(n) + d(n)x(n) + c(n)y(n)}y^{m-1}(n)\right\},\end{aligned} \tag{1.4}$$

where $x(n)$ is the density of prey species at nth generation, $y(n)$ is the density of predator species at nth generation. Also, $r_1(n)$, $b_1(n)$ denote the intrinsic growth rate and density-dependent coefficient of the prey, respectively, $r_2(n)$, $b_2(n)$ denote the death rate and density-dependant coefficient of the predator, respectively, $k_1(n)$ is the capturing rate of the predator, $k_2(n)/k_1(n)$ is the rate of conversion of nutrients into the reproduction of the predator. Further, m is mutual interference constant. In this paper, we assume that all the coefficients $a(n)$, $d(n)$, $c(n)$, $r_i(n)$, $k_i(n)$, $b_i(n)$, $i = 1, 2$, are all positive ω-periodic sequences and $0 < m < 1$. Here, for convenience, we denote $\overline{f} = (1/\omega)\sum_{n=0}^{\omega-1} f(n)$, $f^u = \sup_{n\in I_\omega}\{f(n)\}$, and $f^l = \inf_{n\in I_\omega}\{f(n)\}$ where $I_\omega = \{0, 1, 2, \ldots, \omega - 1\}$.

The remaining part of this paper is organized as follows: in Section 2 we will introduce some definitions and establish several useful lemmas. The permanence of system (1.4) is then studied in Section 3. In Section 4, we give an example to show the feasibility of our main result.

By the biological meaning, we will focus our discussion on the positive solution of system (1.4). So it is assumed that the initial conditions of (1.4) are of the form

$$x(0) > 0, \quad y(0) > 0. \tag{1.5}$$

One can easily show that the solution of (1.4) with the initial condition (1.5) are defined and remain positive for all $n \in N$ where $N = \{0, 1, 2, \ldots\}$.

2. Preliminaries

In this section, we will introduce the definition of permanence and several useful lemmas.

Definition 2.1. System (1.4) is said to be permanent if there exist positive constants x^*, y^*, x_*, y_*, which are independent of the solution of system (1.4), such that for any position solution $(x(n), y(n))$ of system (1.4) satisfies

$$\begin{aligned} x_* &\le \liminf_{n\to+\infty} x(n) \le \limsup_{n\to+\infty} x(n) \le x^*, \\ y_* &\le \liminf_{n\to+\infty} y(n) \le \limsup_{n\to+\infty} x(n) \le y^*. \end{aligned} \tag{2.1}$$

Lemma 2.2 ([6]). *Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and*

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\} \tag{2.2}$$

for $n \in N$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n) \le \frac{1}{b^l}\exp(a^u - 1). \tag{2.3}$$

Lemma 2.3 ([6]). *Assume that $\{x(n)\}$ satisfies*

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \ge N_0. \tag{2.4}$$

$\limsup_{n\to+\infty} x(n) \le x^*$ *and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants and $N_0 \in N$. Then*

$$\liminf_{n\to+\infty} x(n) \ge \frac{a^l \exp\{a^l - b^u x^*\}}{b^u}. \tag{2.5}$$

Lemma 2.4 ([7]). *The problem*

$$x(n+1) = x(n)\exp\{a(n) - b(n)x(n)\} \tag{2.6}$$

with $x(0) = x_0 > 0$ has at least one periodic positive solution $x^*(n)$ if both $b : Z \to R^+$ and $a : Z \to R^+$ are ω-periodic sequences with $\overline{a} > 0$. Moreover, if $b(n) = b$ is a constant and $a^u < 1$, then $bx(n) \le 1$ for n sufficiently large, where $x(n)$ is any solution of (2.6).

Lemma 2.5 ([8]). *Suppose that $f : Z^+ \times [0, +\infty)$ and $g : Z^+ \times [0, +\infty)$ with $f(n, x) \le g(n, x)$ for $n \in Z^+$ and $x \in [0, +\infty)$. Assume that $g(n, x)$ is nondecreasing with respect to the argument x. If $x(n)$ and $u(n)$ are solutions of*

$$x(n+1) = f(n, x(n)), \qquad u(n+1) = g(n, u(n)), \tag{2.7}$$

respectively, and $x(0) \leq u(0)$ $(x(0)u(0))$, *then*

$$x(n) \leq u(n) \quad (x(n) \geq u(n)) \tag{2.8}$$

for all $n > 0$.

3. Permanence

In this section, we establish a permanent result for system (1.4).

Proposition 3.1. *If* (H_1): $(1-m)r_2^u < 1$ *holds, then for any positive solution* $(x(n), y(n))$ *of system* (1.4), *there exist positive constants* x^* *and* y^*, *which are independent of the solution of the system, such that*

$$\limsup_{n\to+\infty} x(n) \leq x^*, \qquad \limsup_{n\to+\infty} y(n) \leq y^*. \tag{3.1}$$

Proof. Let $(x(n), y(n))$ be any positive solution of system (1.4), from the first equation of (1.4), it follows that

$$x(n+1) \leq x(n)\exp\{r_1(n) - b_1(n)x(n)\}. \tag{3.2}$$

By applying Lemma 2.2, we obtain

$$\limsup_{n\to+\infty} x(n) \leq x^*, \tag{3.3}$$

where

$$x^* = \frac{1}{b_1^l}\exp\left(r_1^u - 1\right). \tag{3.4}$$

Denote $P(n) = (1/y(n))^{1-m}$. Then from the second equation of (1.4), it follows that

$$P(n+1) = P(n)\exp\left\{(1-m)r_2(n) + \frac{(1-m)b_2(n)}{\sqrt[1-m]{P(n)}} - \frac{(1-m)k_2(n)x(n)}{a(n)+d(n)x(n)+c(n)\sqrt[m-1]{P(n)}}P(n)\right\}, \tag{3.5}$$

which leads to

$$P(n+1) \geq P(n)\exp\left\{(1-m)r_2(n) - (1-m)k_2^u P(n)\right\}. \tag{3.6}$$

Consider the following auxiliary equation:

$$J(n+1) = J(n)\exp\left\{(1-m)r_2(n) - (1-m)k_2^u J(n)\right\}. \tag{3.7}$$

By Lemma 2.4, (3.7) has at least one positive ω-periodic solution and we denote one of them as $J*(n)$. Now (H_1) and Lemma 2.4 imply $(1-m)K_2^u J(n) \le 1$ for n sufficiently large, where $J(n)$ is any solution of (3.7). Consider the following function:

$$g_1(n,J) = J\exp\{(1-m)r_2(n) - (1-m)K_2^u J\}. \tag{3.8}$$

It is not difficult to see that $g_1(n,J)$ is nondecreasing with respect to the argument J. Then applying Lemma 2.5 to (3.6) and (3.7), we easily obtain that $P(n) \ge J^*(n)$. So $\liminf_{n\to+\infty} P(n) \ge (J^*(n))^l$, which together with that transformation $P(n) = (1/y(n))^{1-m}$, produces

$$\limsup_{n\to+\infty} y(n) \le \frac{1}{\sqrt[1-m]{(J^*(n))^l}} \triangleq y^*. \tag{3.9}$$

This ends the proof of Proposition 3.1. □

Proposition 3.2. *Assume that*

$$(H_2)\text{:}\ \left(r_1(n) - \frac{k_1(n)(y^*)^m}{a^l}\right)^l > 0 \tag{3.10}$$

hold, then for any positive solution $(x(n), y(n))$ of system (1.4), there exist positive constants x_* and y_*, which are independent of the solution of the system, such that

$$\liminf_{n\to+\infty} x(n) \ge x_*, \qquad \liminf_{n\to+\infty} y(n) \ge y_*, \tag{3.11}$$

where y^* can be seen in Proposition 3.1.

Proof. Let $(x(n), y(n))$ be any positive solution of system (1.4). From (H_2), there exists a small enough positive constant ε such that

$$\left((r_1(n) - \frac{k_1(n)(y^*+\varepsilon)^m}{a(n)}\right)^l > 0. \tag{3.12}$$

Also, according to Proposition 3.1, for above ε, there exists $N_1 > 0$ such that for $n > N_1$,

$$y(n) < y^* + \varepsilon. \tag{3.13}$$

Then from the first equation of (1.4), for $n > N_1$, we have

$$x(n+1) \ge x(n)\exp\left\{r_1(n) - \frac{k_1(n)(y^*+\varepsilon)^m}{a(n)} - b_1(n)x(n)\right\}. \tag{3.14}$$

Let $e(n,\varepsilon) = r_1(n) - (k_1(n)/a(n))(y^* + \varepsilon)^m$, so the above inequality follows that

$$x(n+1) \geq x(n)\exp\{e(n,\varepsilon) - b_1(n)x(n)\}. \tag{3.15}$$

From (3.12) and (3.15), by Lemma 2.3, we have

$$\liminf_{n\to+\infty} x(n) \geq \frac{(e(n,\varepsilon))^l}{b_1^u}\exp\left\{(e(n,\varepsilon))^l - b_u^l x^*\right\}. \tag{3.16}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\liminf_{n\to+\infty} x(n) \geq \frac{e^l}{b_1^u}\exp\left\{e^l - b_1^u x^*\right\} \triangleq x_*, \tag{3.17}$$

where

$$e(n) = r_1(n) - \frac{k_1(n)}{a(n)}(y^*)^m. \tag{3.18}$$

From above ε, there exists $N_2 > N_1$ such that $n \geq N_2$, $x(n) \geq x_* - \varepsilon$. So from (3.5), we obtain that

$$P(n+1) \leq P(n)\exp\left\{(1-m)\left(r_2(n) + b_2(n)\left(y^* + \varepsilon\right)\right) - \frac{(1-m)k_2(n)(x_* - \varepsilon)}{a^u + d^u(x_* - \varepsilon) + c^u\left(y^* + \varepsilon\right)}P(n)\right\}. \tag{3.19}$$

Consider the following auxiliary equation:

$$L(n+1) = L(n)\exp\left\{(1-m)\left(r_2(n) + b_2(n)\left(y^* + \varepsilon\right)\right) - \frac{(1-m)k_2(n)(x_* - \varepsilon)}{a^u + d^u(x_* - \varepsilon) + c^u\left(y^* + \varepsilon\right)}L(n)\right\}. \tag{3.20}$$

By Lemma 2.4, (3.20) has at least one positive ω-periodic solution and we denote one of them as $L^*(n)$.

Let

$$R(n) = \ln P(n), \qquad W(n) = \ln L^*(n). \tag{3.21}$$

Then,

$$R(n+1)-R(n)\leq(1-m)\big(r_2(n)+b_2(n)\big(y^*+\varepsilon\big)\big)-\frac{(1-m)k_2(n)(x_*-\varepsilon)}{a^u+d^u(x_*-\varepsilon)+c^u\big(y^*+\varepsilon\big)}\exp\{R(n)\},$$
$$W(n+1)-W(n)=(1-m)\big(r_2(n)+b_2(n)\big(y^*+\varepsilon\big)\big)-\frac{(1-m)k_2(n)(x_*-\varepsilon)}{a^u+d^u(x_*-\varepsilon)+c^u\big(y^*+\varepsilon\big)}\exp\{W(n)\}. \quad (3.22)$$

Set

$$U(n)=R(n)-W(n). \quad (3.23)$$

Then

$$U(n+1)-U(n)\leq-\frac{(1-m)k_2(n)(x_*-\varepsilon)}{a^u+d^u(x_*-\varepsilon)+c^u\big(y^*+\varepsilon\big)}\exp\{W(n)\}\big[\exp\{U(n)\}-1\big]. \quad (3.24)$$

In the following we distinguish three cases.

Case 1. $\{U(n)\}$ is eventually positive. Then, from (3.24), we see that $U(n+1)<U(n)$ for any sufficient large n. Hence, $\lim_{n\to+\infty}U(n)=0$, which implies that

$$\limsup_{n\to+\infty}P(n)\leq(L^*(n))^u. \quad (3.25)$$

Case 2. $\{U(n)\}$ is eventually negative. Then, from (3.23), we can also obtain (3.25).

Case 3. $\{U(n)\}$ oscillates about zero. In this case, we let $\{U(n_{st})\}\ \ (s,t\in N)$ be the positive semicycle of $\{U(n)\}$, where $U(n_{s1})$ denotes the first element of the sth positive semicycle of $\{U(n)\}$. From (3.24), we know that $U(n+1)<U(n)$ if $U(n)>0$. Hence, $\limsup_{n\to+\infty}U(n)=\limsup_{n\to+\infty}U(n_{s1})$. From (3.24), and $U(n_{s1-1})<0$, we can obtain

$$\begin{aligned}U(n_{s1})\ &\leq\frac{(1-m)k_2(n_{s1-1})(x_*-\varepsilon)}{a^u+d^u(x_*-\varepsilon)+c^u\big(y^*+\varepsilon\big)}\exp\{W(n_{s1})\}\big[1-\exp\{U(n_{s1})\}\big],\\&\leq\frac{(1-m)k_2^u(x_*-\varepsilon)}{a^u+d^u(x_*-\varepsilon)+c^u\big(y^*+\varepsilon\big)}(L^*(n))^u.\end{aligned} \quad (3.26)$$

From (3.21) and (3.23), we easily obtain

$$\limsup_{n\to+\infty}P(n)\leq(L^*(n))^u\exp\left\{\frac{(1-m)k_2^u(x_*-\varepsilon)}{a^u+d^u(x_*-\varepsilon)+c^u\big(y^*+\varepsilon\big)}(L^*(n))^u\right\}. \quad (3.27)$$

Setting $\varepsilon\to0$ in the above inequality leads to

$$\limsup_{n\to+\infty}P(n)\leq(L^*(n))^u\exp\left\{\frac{(1-m)k_2^ux_*}{a^u+d^ux_*+c^uy^*}(L^*(n))^u\right\}\triangleq P^*. \quad (3.28)$$

which together with that transformation $P(n) = (1/y(n))^{1-m}$, we have

$$\liminf_{n\to+\infty} y(n) \geq \frac{1}{\sqrt[1-m]{P^*}} \triangleq y_*. \tag{3.29}$$

Thus, we complete the proof of Proposition 3.2. □

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *hold, then system* (1.4) *is permanent.*

It should be noticed that, from the proof of Propositions 3.1 and 3.2, one knows that under the conditions of Theorem 3.3, the set $\Omega = \{(x,y) \mid x_* \leq x \leq x^*, y_* \leq y \leq y^*\}$ is an invariant set of system (1.4).

4. Example

In this section,we give an example to show the feasibility of our main result.

Example 4.1. Consider the following system:

$$\begin{aligned} x(n+1) &= x(n)\exp\left\{0.7 + 0.1\cos(n) - 0.7x(n) - \frac{0.4y^{0.6}(n)}{2+x(n)+y(n)}\right\}, \\ y(n+1) &= y(n)\exp\left\{-0.8 - 0.1\sin(n) - (1.1 + 0.1\cos(n))y(n) + \frac{0.6x(n)y^{-0.4}(n)}{2+x(n)+y(n)}\right\}, \end{aligned} \tag{4.1}$$

where $m = 0.6$, $r_1(n) = 0.7 + 0.1\cos(n)$, $b_1(n) = 0.7$, $k_1(n) = 0.4$, $a(n) = 2$, $d(n) = 1$, $c(n) = 1$, $r_2(n) = 0.8 + 0.1\sin(n)$, $b_2(n) = 1.1 + 0.1\cos(n)$, $k_2(n) = 0.6$.

By simple computation, we have $y^* \approx 1.1405$.Thus, one could easily see that

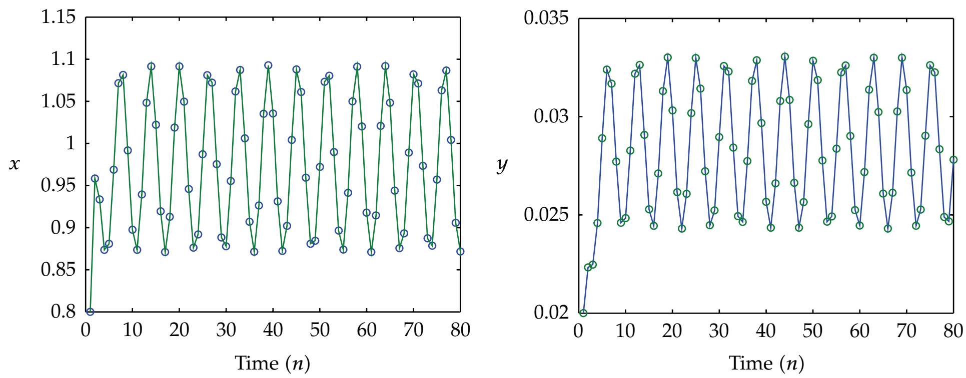
$$(1-m)(r_2(n))^u \approx 0.36 < 1, \left(r_1(n) - \frac{k_1(n)}{a^l}(y^*)^m\right)^l \approx 0.3836 > 0. \tag{4.2}$$

Clearly, conditions (H_1) and (H_2) are satisfied,then system (4.1) is permanent.

Figure 1 shows the dynamics behavior of system (4.1).

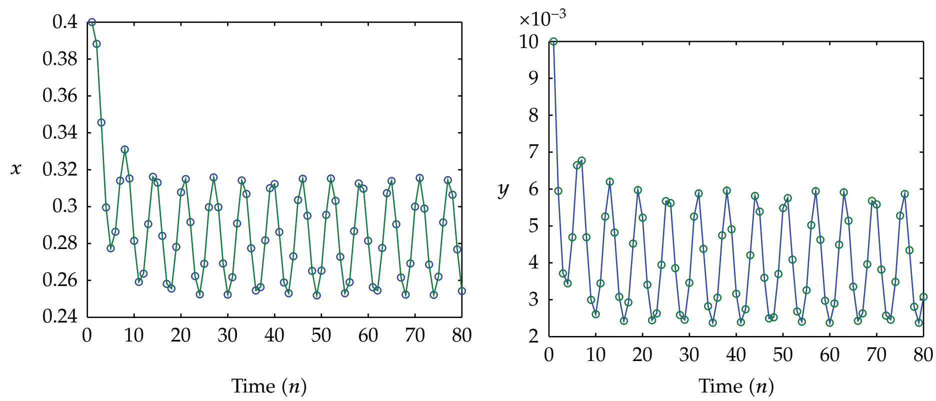
Example 4.2. Consider the following system:

$$\begin{aligned} x(n+1) &= x(n)\exp\left\{0.2 + 0.1\cos(n) - 0.7x(n) - \frac{0.4y^{0.8}(n)}{2+x(n)+y(n)}\right\}, \\ y(n+1) &= y(n)\exp\left\{-0.8 - 0.4\sin(n) - (0.8 + 0.1\cos(n))y(n) + \frac{0.6x(n)y^{-0.2}(n)}{2+x(n)+y(n)}\right\}, \end{aligned} \tag{4.3}$$



(a) (b)

Figure 1: Dynamics behavior of system (4.1) with initial condition $(x(0), y(0)) = (0.8, 0.02)$.



(a) (b)

Figure 2: Dynamics behavior of system (4.3) with initial condition $(x(0), y(0)) = (0.4, 0.01)$.

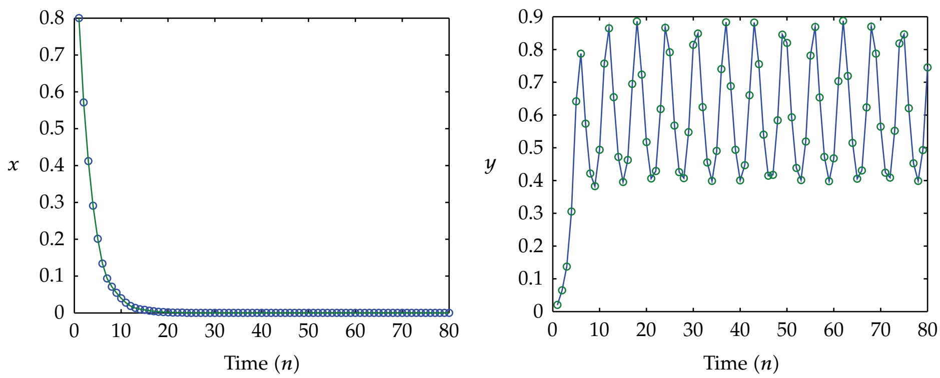
where $m = 0.8$, $r_1(n) = 0.2 + 0.1\cos(n)$, $b_1(n) = 0.7$, $k_1(n) = 0.4$, $a(n) = 2$, $d(n) = 1$, $c(n) = 1$, $r_2(n) = 0.8 + 0.4\sin(n)$, $b_2(n) = 0.8 + 0.1\cos(n)$, $k_2(n) = 0.6$.

By simple computation, we have $y^* \approx 37.3346$. Thus, one could easily see that

$$(1-m)(r_2(n))^u \approx 0.24 < 1, \left(r_1(n) - \frac{k_1(n)}{a^l}(y^*)^m\right)^l \approx -3.5201 < 0. \tag{4.4}$$

Clearly, condition (H_1) is satisfied and condition (H_2) is not satisfied, but the system (4.3) is permanent. It shows that conditions (H_1) and (H_2) are sufficient for the system (1.4) but not necessary.

Figure 2 shows the dynamics behavior of system (4.3).



(a) (b)

Figure 3: Dynamics behavior of system (4.5) with initial condition $(x(0), y(0)) = (0.8, 0.02)$.

Example 4.3. Consider the following system:

$$x(n+1) = x(n)\exp\left\{0.2 + 0.1\cos(n) - 0.7x(n) - \frac{2y^{0.8}(n)}{2 + x(n) + y(n)}\right\},$$
$$y(n+1) = y(n)\exp\left\{-0.8 - 0.4\sin(n) - (0.8 + 0.1\cos(n))y(n) + \frac{3x(n)y^{-0.2}(n)}{2 + x(n) + y(n)}\right\}, \tag{4.5}$$

where $m = 0.8$, $r_1(n) = 0.2 + 0.1\cos(n)$, $b_1(n) = 0.7$, $k_1(n) = 2$, $a(n) = 2$, $d(n) = 1$, $c(n) = 1$, $r_2(n) = 0.8 + 0.4\sin(n)$, $b_2(n) = 0.8 + 0.1\cos(n)$, $k_2(n) = 3$.

By simple computation, we have $y^* \approx 116670$. Thus, one could easily see that

$$(1-m)(r_2(n))^u \approx 0.24 < 1, \left(r_1(n) - \frac{k_1(n)}{a^l}(y^*)^m\right)^l \approx -11313 < 0. \tag{4.6}$$

Clearly,condition (H_1) is satisfied and (H_2) is not satisfied, then the species x is extinct and the species y is permanent.

Figure 3 shows the dynamics behavior of system (4.5).

Acknowledgment

The work is supported by the Technology Innovation Platform Project of Fujian Province (2009J1007).

References

[1] M. P. Hassel, "Density dependence in single-species population," *Journal of Animal Ecology*, vol. 44, pp. 283–295, 1975.

[2] K. Wang and Y. L. Zhu, "Global attractivity of positive periodic solution for a Volterra model," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 493–501, 2008.

[3] X. Lin and F. Chen, “Almost periodic solution for a Volterra model with mutual interference and Beddington-DeAngelis functional response,” *Applied Mathematics and Computation*, vol. 214, no. 2, pp. 548–556, 2009.

[4] L. J. Chen and L. J. Chen, “Permanence of a discrete periodic Volterra model with mutual interference,” *Discrete Dynamics in Nature and Society*, vol. 2009, Article ID 205481, 9 pages, 2009.

[5] B. Apanasov and X. Xie, “Discrete actions on nilpotent Lie groups and negatively curved spaces,” *Differential Geometry and Its Applications*, vol. 20, no. 1, pp. 11–29, 2004.

[6] F. D. Chen, “Permanence and global stability of nonautonomous Lotka-Volterra system with predator-prey and deviating arguments,” *Applied Mathematics and Computation*, vol. 173, no. 2, pp. 1082–1100, 2006.

[7] Y.-H. Fan and W.-T. Li, “Permanence for a delayed discrete ratio-dependent predator-prey system with Holling type functional response,” *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 357–374, 2004.

[8] L. Wang and M. Q. Wang, *Ordinary Difference Equation*, Xinjiang University Press, Xinjiang, China, 1991.