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Research Article

Existence and Multiplicity of Solutions to Discrete Conjugate Boundary Value Problems

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We consider the existence and multiplicity of solutions to discrete conjugate boundary value problems. A generalized asymptotically linear condition on the nonlinearity is proposed, which includes the asymptotically linear as a special case. By classifying the linear systems, we define index functions and obtain some properties and the concrete computation formulae of index functions. Then, some new conditions on the existence and multiplicity of solutions are obtained by combining some nonlinear analysis methods, such as Leray-Schauder principle and Morse theory. Our results are new even for the case of asymptotically linear.

1. Introduction

Let $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$ be the sets of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$. Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, and $\Delta^2 u(n) = \Delta(\Delta u(n))$. Let A be an $m \times m$ matrix. A^τ or x^τ denotes the transpose of matrix A or vector x. The set of eigenvalues of matrix A will be denoted by $\sigma(A)$, and the determinant of matrix A will be denoted by $\det A$.

Discrete boundary value problems (BVPs for short) arise in the study of solid state physics, combinatorial analysis, chemical reactions, population dynamics, and so forth. Besides, they are also natural consequences of the discretization of continuous BVPs. Thus, these problems have been studied by many scholars.

Discrete two-point BVPs

$$\begin{gathered} \Delta^2 u(n-1) + f(n, u(n)) = 0, \quad n \in \mathbb{Z}(1, T), \\ u(0) = A, \quad u(T+1) = B \end{gathered} \tag{1.1}$$

often appear in electrical circuit analysis, mathematical physics, finite elasticity, and so forth as the mathematical models, where $f : \mathbb{Z}(1,T) \times \mathbb{R}^d \to \mathbb{R}^d$ with $d \in \mathbb{N}$, $T > 0$ is a given integer, and A, B are given constants.

We may think of (1.1) as being a discrete analogue of the continuous BVPs:

$$\begin{gathered} u'' + f(t,u) = 0, \quad a \le t \le b, \\ u(a) = A, \quad u(b) = B, \end{gathered} \tag{1.2}$$

which have been studied by many scholars because of its numerous applications in science and technology. In particular, Hale, Walter, Mawhin, and so forth have obtained some significant results on the existence, uniqueness, and multiplicity of solutions of (1.2). We refer the readers to [1–3] and references therein for further details.

Let

$$y(n) = \frac{Bn - A(n-T-1)}{T+1}, \qquad z(n) = u(n) - y(n). \tag{1.3}$$

Then (1.1) reduces to

$$\begin{gathered} \Delta^2 z(n-1) + f\big(n, z(n) + y(n)\big) = 0, \quad n \in \mathbb{Z}(1,T), \\ z(0) = 0 = z(T+1). \end{gathered} \tag{1.4}$$

Hence, in the following, we can only consider the discrete conjugate BVPs, that is,

$$\begin{gathered} \Delta^2 u(n-1) + f(n, u(n)) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1). \end{gathered} \tag{1.5}$$

As being remarked in [4], the nature of the solution of a continuous problem is not identical with that of the solution of its discrete analogue. And since discrete analogs of continuous problems yield interesting dynamical systems in their own right, many scholars have investigated BVPs (1.5) independently. There are fundamental questions that arise for BVPs (1.5). Does a solution exist, is it unique, and how many solutions can be found if BVPs (1.5) have multiple solutions? How to find the lower bound or the upper bound of the number of solutions of BVPs (1.5)? Furthermore, how to obtain the precise number of solutions of BVPs (1.5)?

In recent years, the existence, uniqueness, and multiplicity of solutions of discrete BVPs have been studied by many authors. In fact, early in 1968, Lasota [5] studied the discretizations of (1.2) with $f(t,u)$ replaced by $f(t,u,u')$ and proved that the discrete problem had one and only one solution with f satisfying a Lipschitz condition. Note that under certain conditions the solution of a nonhomogeneous BVPs can be expressed in terms of Green's functions. For example, suppose that $u(n)$ is a solution of (1.1). Then

$$u(n) = \sum_{m=1}^{T} G(n,m) f(m, u(m)) + w(n), \tag{1.6}$$

where $G(n,m)$ is Green's function for

$$\begin{gathered}\Delta^2 u(n-1)=0, \quad n\in\mathbb{Z}(1,T),\\ u(0)=0=u(T+1),\\ w(n)=A+\frac{B-A}{T+1}n.\end{gathered}\tag{1.7}$$

Let $\mathcal{B}=\{u \mid u \text{ is a real-valued function defined on } \mathbb{Z}(0,T+1), u(0)=u(T+1)=0\}$, and define $\mathcal{T}:\mathcal{B}\to\mathcal{B}$ by

$$\mathcal{T}u(n)=\sum_{m=1}^{T}G(n,m)f(m,u(m))+w(n)\tag{1.8}$$

for n in $\mathbb{Z}(0,T+1)$. Then there is a one-to-one correspondence between the fixed points of $\mathcal{T}$ and the solutions of BVPs (1.1). When the nonlinearity f satisfies growth conditions known as Lipschitz conditions, a unique solution of BVPs (1.1) can be obtained by using Contraction Mapping Theorem see [6, 7] for more details.

Note that discrete BVPs model numerous physical phenomena in nature hence it is of fundamental importance to know the criteria that ensure the existence of at least one meaningful solution. And since discrete BVPs often have multiple solutions, it is useful to have a collection of results that yield existence of solutions without the implication that the solutions must be unique. To this end, many scholars have obtained some significant results on the existence and multiplicity of solutions of discrete BVPs by using various analytic techniques and various fixed-point theorems, for example, the upper and lower solution method [8–10], the conical shell fixed point theorems [11, 12], the Brouwer and Schauder fixed point theorems [9, 13, 14], and topological degree theory [15, 16]. As we know, critical-point theory (which includes the minimax method and Morse theory, etc.) has played an important role in dealing with the existence and multiplicity of solutions to continuous systems [2, 17]. It is natural for us to think that critical-point theory may be applied to study the existence and multiplicity of solutions to discrete systems. In fact, in recent papers [18–25], the authors have applied critical-point theory to study the existence and multiplicity of periodic solutions to discrete systems. We also refer to [26–31] for the discrete BVPs. In [26], Agarwal et al. employed the Mountain Pass Lemma to study (1.5) and obtained the existence of multiple solutions. Very recently, B. Zheng and Q. Zhang [32] studied discrete BVPs (1.5) with $f(n,u(n))=V'(u(n))$ and obtained the existence of exactly three solutions by using both Morse theory and degree theory, and so forth. To the best of our knowledge, [32] is among a few works dealing with discrete BVPs by using Morse theory. Hence, further studies on application of Morse theory to discrete BVPs are still perspective.

Here, we consider the case $f(n,u(n))=\nabla V(n,u(n))$ that is, we consider the following discrete conjugate BVPs:

$$\begin{gathered}\Delta^2 u(n-1)+\nabla V(n,u(n))=0, \quad n\in\mathbb{Z}(1,T),\\ u(0)=0=u(T+1),\end{gathered}\tag{1.9}$$

where $V(n,\cdot) \in C^1(\mathbb{R}^d,\mathbb{R})$ for every $n \in \mathbb{Z}(1,T)$, $\nabla V(n,z)$ denotes the gradient of V with respect to z, and $d \geq 2$, $T > 0$ are given integers.

Assume

$$\nabla V(n,z) = A(n,z)z + o(|z|) \tag{1.10}$$

as $|z| \to \infty$, where $A : \mathbb{Z}(1,T) \times \mathbb{R}^d \to \mathrm{GL}_s(\mathbb{R}^d)$, and

$$A_1(n) \leq A(n,z) \leq A_2(n) \tag{1.11}$$

for every $n \in \mathbb{Z}(1,T)$, and $z \in \mathbb{R}^d$, $A_1, A_2 : \mathbb{Z}(1,T) \to \mathrm{GL}_s(\mathbb{R}^d)$, $\mathrm{GL}_s(\mathbb{R}^d)$ denotes the group of $d \times d$ real nonsingular symmetric matrices, and $|z|$ denotes the Euclidean norm of z in $\mathbb{R}^d$. Throughout this paper, for any $A_1, A_2 \in \mathrm{GL}_s(\mathbb{R}^d)$, we denote $A_1 \leq A_2$ if $A_2 - A_1$ is semi-positive definite, and we denote $A_1 < A_2$ if $A_2 - A_1$ is positive definite. For any $A_1, A_2 : \mathbb{Z}(1,T) \to \mathrm{GL}_s(\mathbb{R}^d)$, we denote $A_1 \leq A_2$ if $A_1(n) \leq A_2(n)$ for every $n \in \mathbb{Z}(1,T)$, and we denote $A_1 < A_2$ if $A_1(n) \leq A_2(n)$ for every $n \in \mathbb{Z}(1,T)$ and $\{n \mid A_1(n) < A_2(n)\} \neq \emptyset$.

If $A(n,z) \equiv A(n)$ in (1.10), then (1.10) is usually called an asymptotically linear condition. So here we call (1.10) and (1.11) generalized asymptotically linear conditions. Our results are new even for the case of asymptotically linear case.

The rest of this paper is organized as follows. In Section 2, firstly, we classify the linear systems

$$\begin{gathered}\Delta^2 u(n-1) + A(n)u(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1)\end{gathered} \tag{1.12}$$

for every $A : \mathbb{Z}(1,T) \to \mathrm{GL}_s(\mathbb{R}^d)$. This classification gives a pair of integers $(i(A), \nu(A)) \in \mathbb{Z}(0,dT) \times \mathbb{Z}(0,d)$. We call $i(A)$ and $\nu(A)$ the index and nullity of A, respectively. Secondly, we give some properties of the index and nullity together with the concrete computation formulae. And finally, we introduce the definition of relative Morse index and give its precise description. By using both results in Section 2 and Leray-Schauder principle, we obtain some solvable conditions of (1.9) in Section 3. However, we cannot exclude the possibility that the solution we found is trivial. To this end, we make use of Morse theory to obtain the existence and multiplicity of nontrivial solutions to (1.9). Examples are also included to illustrate the results obtained.

2. Index Theory for Linear Systems

To establish the index theory for (1.12), we introduce the following finite dimensional sequence space:

$$E = \left\{u \mid u = (u(0), u(1), \dots, u(T), u(T+1))^{\tau}, u(0) = 0 = u(T+1)\right\}, \tag{2.1}$$

where $u(n) = (u_1(n), u_2(n), \ldots, u_d(n))^\tau \in \mathbb{R}^d$ for every $n \in \mathbb{Z}(0, T+1)$. Define the inner product on E as follows:

$$\langle u, v \rangle = \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)), \quad \forall u, v \in E, \tag{2.2}$$

by which the norm $\|\cdot\|$ on E can be induced by

$$\|u\| = \left(\sum_{n=0}^{T} |\Delta u(n)|^2 \right)^{1/2}, \quad \forall u \in E, \tag{2.3}$$

where $(\cdot, \cdot)$ is the usual inner product on $\mathbb{R}^d$, and $|\cdot|$ is the usual norm on $\mathbb{R}^d$.

Define a linear map $\Gamma : E \to \mathbb{R}^{dT}$ by

$$\Gamma u = (u_1(1), u_1(2), \ldots, u_1(T), u_2(1), u_2(2), \ldots, u_2(T), \ldots, u_d(1), u_d(2), \ldots, u_d(T))^\tau. \tag{2.4}$$

It is easy to see that the map Γ defined in (2.4) is a linear homeomorphism, and $(E, \langle \cdot, \cdot \rangle)$ is a Hilbert space, which can be identified with $\mathbb{R}^{dT}$.

Define

$$q_A(u, v) = \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)) - \sum_{n=1}^{T} (A(n)u(n), v(n)), \quad \forall u, v \in E. \tag{2.5}$$

For any $u, v \in E$, if $q_A(u, v) = 0$, we say that u and v are q_A orthogonal. For any two subspaces E_1 and E_2 of E, if $q_A(u, v) = 0$ for any $u \in E_1$ and $v \in E_2$, we say that E_1 and E_2 are q_A orthogonal.

For any subspace E_1 of E, we say that q_A is positive definite (or negative definite) on E_1 if $q_A(u, u) > 0$ (or $q_A(u, u) < 0$) for all $u \in E_1 \setminus \{0\}$. And if $q_A(u, u) = 0$ for all $u \in E_1$, then E_1 is called a null subspace of E.

Proposition 2.1. *For any $A : \mathbb{Z}(1, T) \to GL_s(\mathbb{R}^d)$, the following results hold.*

(1) *There are $\{\lambda_i(A)\}_{i=1}^m \subset \mathbb{R}$ with $\lambda_1(A) < \lambda_2(A) < \cdots < \lambda_m(A)$ such that*

$$\begin{gathered} \Delta^2 u(n-1) + (A(n) + \lambda_i(A) I_d) u(n) = 0, \quad n \in \mathbb{Z}(1, T), \\ u(0) = 0 = u(T+1) \end{gathered} \tag{2.6}$$

has a nontrivial solution. If $E_i(A)$ denotes the subspace of solutions with respect to $\lambda_i(A)$, then $\dim E_i(A) := n_i \leq d$ *and* $E = \bigoplus_{i=1}^m E_i(A)$.

(2) *The space E has a q_A orthogonal decomposition*

$$E = E^+(A) \bigoplus E^0(A) \bigoplus E^-(A) \tag{2.7}$$

such that q_A is positive definite, negative definite, and null on $E^+(A)$, $E^-(A)$, and $E^0(A)$, respectively.

To prove Proposition 2.1, we need the following lemma.

Lemma 2.2. *For any* $u = \{u(n)\}_{n=0}^{T+1} \in E$*, the following inequalities hold.*

$$4\sin^2\frac{\pi}{2(T+1)}\sum_{n=1}^{T}|u(n)|^2 \le \sum_{n=0}^{T}|\Delta u(n)|^2 \le 4\cos^2\frac{\pi}{2(T+1)}\sum_{n=1}^{T}|u(n)|^2. \tag{2.8}$$

Proof. Note that

$$\sum_{n=0}^{T}(\Delta u(n), \Delta u(n)) = \sum_{n=1}^{T}2(u(n), u(n)) - \sum_{n=1}^{T-1}2(u(n), u(n+1)) = (A\Gamma u, \Gamma u), \tag{2.9}$$

where

$$A = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B \end{pmatrix}_{dT\times dT}, \qquad B = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T\times T}. \tag{2.10}$$

Assume that λ is an eigenvalue of B and that $\xi = (\xi_1, \xi_2, \ldots, \xi_T)^\tau$ is an eigenvector associated to λ. Define the sequence $\{v(n)\}_{n=0}^{T+1}$ as

$$v(i) = \xi_i, \quad i = 1, 2, \ldots, T, \qquad v(0) = 0 = v(T+1). \tag{2.11}$$

Then $\{v(n)\}_{n=0}^{T+1}$ satisfies

$$\begin{gathered} \Delta^2 v(n-1) + \lambda v(n) = 0, \quad n \in \mathbb{Z}(1, T), \\ v(0) = 0 = v(T+1). \end{gathered} \tag{2.12}$$

Equation (2.12) has a nontrivial solution if and only if

$$\lambda = \lambda_k = 4\sin^2\frac{k\pi}{2(T+1)}, \quad k = 1, 2, \ldots, T; \tag{2.13}$$

see [33]. So $\sigma(A) = \sigma(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_T\}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_T$ and

$$\begin{aligned} \lambda_{\min} &= \min\{\lambda_1, \lambda_2, \ldots, \lambda_T\} = 4\sin^2\frac{\pi}{2(T+1)} = \lambda_1, \\ \lambda_{\max} &= \max\{\lambda_1, \lambda_2, \ldots, \lambda_T\} = 4\cos^2\frac{\pi}{2(T+1)} = \lambda_T. \end{aligned} \tag{2.14}$$

Noticing that for any real symmetric $dT \times dT$ matrix A, we have

$$\lambda_1(\Gamma u, \Gamma u) \le (A\Gamma u, \Gamma u) \le \lambda_T(\Gamma u, \Gamma u), \quad \forall u \in \mathbb{R}^{dT}. \tag{2.15}$$

Since $(\Gamma u, \Gamma u) = \sum_{n=1}^{T} |u(n)|^2$, the inequalities (2.8) now follow from (2.9) and (2.15). □

Remark 2.3. In the following, we rewrite (2.8) as

$$\lambda_1 |u|^2 \le \|u\|^2 \le \lambda_T |u|^2 \tag{2.16}$$

for simplicity.

Proof of Proposition 2.1. (1) We claim that the norm $\|\cdot\|_{\lambda_0}$ induced by the inner product

$$(u, v)_{\lambda_0} := \sum_{n=0}^{T}(\Delta u(n), \Delta v(n)) + \sum_{n=1}^{T}((\lambda_0 I_d - A(n))u(n), v(n)), \quad \forall u, v \in E \tag{2.17}$$

is equivalent to $\|\cdot\|$, where λ_0 is a positive number satisfying $\lambda_0 I_d > A$. In fact, it is easy to see that there exists $c \in (0, +\infty)$ such that

$$0 \le \sum_{n=1}^{T}((\lambda_0 I_d - A(n))u(n), u(n)) \le c\sum_{n=1}^{T}|u(n)|^2 \le \frac{c}{\lambda_1}\sum_{n=0}^{T}|\Delta u(n)|^2 = \frac{c}{\lambda_1}\|u\|^2. \tag{2.18}$$

Hence

$$\|u\|^2 \le \|u\|_{\lambda_0}^2 \le \left(1 + \frac{c}{\lambda_1}\right)\|u\|^2. \tag{2.19}$$

Define a bilinear function

$$a(u, v) = \sum_{n=1}^{T}(u(n), v(n)), \quad \forall u, v \in E, \tag{2.20}$$

and then

$$|a(u, v)| \le \left(\sum_{n=1}^{T}|u(n)|^2\right)^{1/2}\left(\sum_{n=1}^{T}|v(n)|^2\right)^{1/2} \le \frac{1}{\lambda_1}\|u\|\|v\| \le \frac{1}{\lambda_1}\|u\|_{\lambda_0}\|v\|_{\lambda_0}. \tag{2.21}$$

Hence, there exists a unique continuous linear operator $K_{\lambda_0} : E \to E$ satisfying

$$\sum_{n=1}^{T}(u(n), v(n)) = (u, K_{\lambda_0} v)_{\lambda_0}. \tag{2.22}$$

It is easy to see that K_{λ_0} is self-adjoint, and hence all the eigenvalues of K_{λ_0} are real. Therefore, there exist μ_i, $i = 1, 2, \ldots, m$ and e_{ij}, $j = 1, 2, \ldots, n_i$ such that

$$\left(e_{ij}, e_{lk}\right)_{\lambda_0} = \begin{cases} 1, & i = l \text{ and } j = k, \\ 0, & i \neq l \text{ or } j \neq k, \end{cases} \qquad K_{\lambda_0} e_{ij} = \mu_i e_{ij}, \tag{2.23}$$

where n_i is the multiplicity of μ_i with $\sum_i n_i = dT$. By (2.22) and (2.23) we have

$$\mu_i \left(e_{ij}, u\right)_{\lambda_0} = \sum_{n=1}^{T} \left(e_{ij}(n), u(n)\right), \quad \forall u \in E. \tag{2.24}$$

In particular, $\mu_i = \sum_{n=1}^{T} |e_{ij}(n)|^2 > 0$. Without loss of generality we assume that μ_i is strictly monotonously decreasing, that is, $\mu_1 > \mu_2 > \cdots > \mu_m$. Denote $\lambda_i(A) = 1/\mu_i - \lambda_0$ and $E_i(A) = \operatorname{span}\{e_{ij}\}_{j=1}^{n_i}$, then $E = \bigoplus_{i=1}^{m} E_i(A)$. We claim that for every $\lambda_i(A)$, $e_{ij} = \{e_{ij}(n)\}_{n=0}^{T+1} \in E$ is a nontrivial solution of (2.6). In fact, by (2.24), for any $u \in E$, we have

$$\mu_i \sum_{n=0}^{T} \left(\Delta e_{ij}(n), \Delta u(n)\right) + \mu_i \sum_{n=1}^{T} \left((\lambda_0 I_d - A(n)) e_{ij}(n), u(n)\right) = \sum_{n=1}^{T} \left(e_{ij}(n), u(n)\right), \tag{2.25}$$

and since $\mu_i > 0$, the above equality means

$$\sum_{n=1}^{T} \left(\Delta^2 e_{ij}(n-1) + \left(A(n) + \frac{1}{\mu_i} I_d - \lambda_0 I_d \right) e_{ij}(n), u(n) \right) = 0, \quad \forall u \in E. \tag{2.26}$$

Therefore e_{ij} satisfies (2.6). Now, we have proved the first result of Proposition 2.1 except $\dim E_i(A) = n_i \le d$.

Set $u(n) = y_1(n)$, $\Delta u(n-1) = -y_2(n)$; then (2.6) is equivalent to

$$\begin{gathered} \Delta y_1(n) = -y_2(n+1), \quad n \in \mathbb{Z}(0, T), \\ \Delta y_2(n) = (A(n) + \lambda_i(A) I_d) y_1(n), \quad n \in \mathbb{Z}(1, T), \\ y_1(0) = 0 = y_1(T+1), \end{gathered} \tag{2.27}$$

which is also equivalent to

$$\begin{gathered} y(n+1) = B(n) y(n), \quad n \in \mathbb{Z}(1, T), \\ y_1(0) = 0 = y_1(T+1), \end{gathered} \tag{2.28}$$

where

$$y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix}, \qquad B(n) = \begin{pmatrix} I_d - \lambda_i(A) I_d - A(n) & -I_d \\ \lambda_i(A) I_d + A(n) & I_d \end{pmatrix}. \tag{2.29}$$

Since $\det(B(n)) \equiv 1$, $B(n)$ is a nonsingular $2d \times 2d$ matrix for every n. So, we can assume that $\Phi(n)$ is the fundamental matrix of equation $y(n+1) = B(n)y(n)$ satisfying $\Phi(0) = I_{2d}$. The general solution of $y(n+1) = B(n)y(n)$ can be given by $y(n) = \Phi(n)c$, where $c = \binom{c_1}{c_2} \in \mathbb{R}^{2d}$ and $c_i \in \mathbb{R}^d, i = 1, 2$. Set

$$\Phi(n) = \begin{pmatrix} \Phi_{11}(n) & \Phi_{12}(n) \\ \Phi_{21}(n) & \Phi_{22}(n) \end{pmatrix}, \quad \text{where } \Phi_{ij}(n) \text{ is } d \times d \text{ matrix}, \quad i, j = 1, 2, \tag{2.30}$$

then $y_1(n) = \Phi_{11}(n)c_1 + \Phi_{12}(n)c_2$. By $y_1(0) = 0 = y_1(T+1)$ and $\Phi(0) = I_{2d}$, we have $c_1 = 0$ and

$$E_i(A) \cong \left\{ c_2 \in \mathbb{R}^d \mid \Phi_{12}(T+1)c_2 = 0 \right\} \subseteq \mathbb{R}^d. \tag{2.31}$$

Hence, $\dim E_i(A) = n_i \leq d$.

(2) For any $u \in E$ with $u = \sum_{i,j} c_{ij}e_{ij}$, by (2.23) and (2.24), we have

$$q_A(u,u) = (u,u)_{\lambda_0} - \lambda_0(u, K_{\lambda_0}u)_{\lambda_0} = \sum_{i,j}\left(\left|c_{ij}\right|^2 - \lambda_0\mu_i\left|c_{ij}\right|^2 \right) = \sum_{i,j}\lambda_i(A)\mu_i\left|c_{ij}\right|^2. \tag{2.32}$$

Hence, if we denote

$$\begin{aligned} E^+(A) &= \left\{ u = \sum c_{ij}e_{ij} \mid c_{ij} = 0, \text{if } \lambda_i(A) \leq 0 \right\}, \\ E^0(A) &= \left\{ u = \sum c_{ij}e_{ij} \mid c_{ij} = 0, \text{if } \lambda_i(A) \neq 0 \right\}, \\ E^-(A) &= \left\{ u = \sum c_{ij}e_{ij} \mid c_{ij} = 0, \text{if } \lambda_i(A) \geq 0 \right\}, \end{aligned} \tag{2.33}$$

then the results hold. □

Definition 2.4. For any $A : \mathbb{Z}(1,T) \to \mathrm{GL}_s(\mathbb{R}^d)$, define the index of A as $i(A) := \dim E^-(A)$, and define the nullity of A as $\nu(A) := \dim E^0(A)$.

In the following we shall discuss the properties of $(i(A), v(A))$.

Proposition 2.5. *For any $A : \mathbb{Z}(1,T) \to GL_s(\mathbb{R}^d)$, one has the following.*

(1) *$\nu(A)$ is the dimension of the solution subspace of (1.12), and $\nu(A) \in \mathbb{Z}(0,d)$.*

(2) *$i(A) = \sum_{\lambda_i(A)<0} n_i$, where λ_i and n_i are defined in the proof of Proposition 2.1.*

Proof. (1) By Proposition 2.5, if $\lambda_i(A) \neq 0$ for any i, then (1.12) has only a trivial solution, and hence $E^0(A) = \{0\}$, $\nu(A) = 0$. If $\lambda_i(A) = 0$ for some i with multiplicity n_i, then by the proof of Proposition 2.1, $E_i(A)$ is the solution subspace of (1.12) and $\nu(A) := \dim E_i(A) \in \mathbb{Z}(1,d)$.

(2) By the proof of Proposition 2.1, $E^-(A) = \bigoplus_{\lambda_i(A)<0} E_i(A)$, and $E_i(A)$ and $E_j(A)$ are q_A orthogonal if $i \neq j$. Hence the result holds. □

Remark 2.6. By (1) of Proposition 2.5, $\nu(A) \geq 0$, and $\nu(A) = 0$ if and only if $\lambda_i(A) \neq 0$ for any $i \in \mathbb{Z}(1, dT)$ which holds if and only if (1.12) has only a trivial solution.

Proposition 2.7. *For any* $A_1, A_2 : \mathbb{Z}(1,T) \to GL_s(\mathbb{R}^d)$*, the following results hold.*

(1) *If* $A_1 \le A_2$*, then* $i(A_1) \le i(A_2)$.

(2) *If* $A_1 \le A_2$*, then* $i(A_1) + \nu(A_1) \le i(A_2) + \nu(A_2)$.

(3) *If* $A_1 < A_2$*, then* $i(A_1) + \nu(A_1) \le i(A_2)$.

To prove Proposition 2.7, we firstly prove the following lemma.

Lemma 2.8. *Let* E_1 *be a subspace of* E *satisfying*

$$q_A(u,u) \le 0, \quad \forall u \in E_1, \tag{2.34}$$

and then

$$\dim E_1 \le i(A) + \nu(A). \tag{2.35}$$

Moreover, if

$$q_A(u,u) < 0, \quad \forall u \in E_1 \setminus \{0\}, \tag{2.36}$$

then

$$\dim E_1 \le i(A). \tag{2.37}$$

Proof. Without loss of generality we can assume that $\dim E_1 = k \ge 1$ and $E_1 = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$. Let $e_i = e_i^+ + e_i^*$, where $e_i^* \in E^-(A) \oplus E^0(A)$, $e_i^+ \in E^+(A)$. To prove $i(A) + \nu(A) \ge k$, we only need to prove that $e_1^*, e_2^*, \ldots, e_k^*$ is linear independent. If not, there exist not all zero constants $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\sum_{i=1}^k \alpha_i e_i^* = 0$. So $e := \sum_{i=1}^k \alpha_i e_i = \sum_{i=1}^k \alpha_i e_i^+ \in E^+(A)$, and hence $q_A(e,e) > 0$. This is a contradiction to (2.34). This implies that $e_1^*, e_2^*, \ldots, e_k^*$ is linear independent and $i(A) + \nu(A) := \dim E^-(A) + \dim E^0(A) = \dim(E^-(A) \oplus E^0(A)) \ge k$. The first part is proved.

To prove the second part, let $e_i = e_i^+ + e_i^- + e_i^0$, where $e_i^* \in E^*(A)$, $* = +, 0, -$. To prove $i(A) \ge k$, we only need to prove that $e_1^-, e_2^-, \ldots, e_k^-$ is linear independent. If not, there exist not all zero constants $c_1, c_2, \ldots, c_k$ such that $\sum_{i=1}^k c_i e_i^- = 0$. So, $e := \sum_{i=1}^k c_i e_i = \sum_{i=1}^k (c_i e_i^+ + c_i e_i^0) \in E^+(A) \oplus E^0(A)$ and $q_A(e,e) \ge 0$. This is a contradiction to (2.36). □

Proof of Proposition 2.7. For any $u \in E$, denote $u = u^+ + u^0 + u^-$, where $u^* \in E^*(A_1)$, $* = +, -, 0$.

(1) From Lemma 2.8, we only need to show that

$$q_{A_2}(u,u) < 0, \quad \forall u \in E^-(A_1) \setminus \{0\}. \tag{2.38}$$

In fact, for every $u = u^- \in E^-(A_1)$, if $u^- \ne 0$, then

$$q_{A_2}(u,u) \le q_{A_1}(u,u) = q_{A_1}(u^-, u^-) < 0. \tag{2.39}$$

(2) From Lemma 2.8, we only need to show that

$$q_{A_2}(u,u) \leq 0, \quad \forall u \in E^-(A_1) \oplus E^0(A_1). \tag{2.40}$$

In fact, for every $u = u^- + u^0 \in E^-(A_1) \oplus E^0(A_1)$, one has

$$q_{A_2}(u,u) \leq q_{A_1}(u,u) = q_{A_1}\left(u^0,u^0\right) + q_{A_1}(u^-,u^-) \leq 0. \tag{2.41}$$

(3) From Lemma 2.8, we only need to show that

$$q_{A_2}(u,u) < 0, \quad \forall u \in E^-(A_1) \oplus E^0(A_1) \setminus \{0\}. \tag{2.42}$$

In fact, for every $u = u^- + u^0$ with $u^- \in E^-(A_1)$, $u^0 \in E^0(A_1)$, if $u^- \neq 0$, then

$$q_{A_2}(u,u) \leq q_{A_1}(u,u) = q_{A_1}\left(u^0,u^0\right) + q_{A_1}(u^-,u^-) < 0. \tag{2.43}$$

If $u^- = 0$, $u^0 \neq 0$, then $\{u^0(n)\}_{n=0}^{T+1}$ is a nontrivial solution of

$$\begin{gathered} \Delta^2 u(n-1) + A_1(n)u(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1). \end{gathered} \tag{2.44}$$

From $A_1 < A_2$ we have

$$q_{A_2}\left(u^0,u^0\right) < q_{A_1}\left(u^0,u^0\right) = 0. \tag{2.45}$$

Hence (2.42) holds. □

Proposition 2.9. *If $U \in O(d)$, that is, $U \in GL_s(\mathbb{R}^d)$ and $U^\tau U = I_d$, then for any $A : \mathbb{Z}(1,T) \to GL_s(\mathbb{R}^d)$, $i(U^\tau AU) = i(A), \nu(U^\tau AU) = \nu(A)$. In particular, for any $A \in GL_s(\mathbb{R}^d)$, we have*

$$i(A) = \sum_{i=1}^{d} \sharp\{k \in \mathbb{Z}(1,T) \mid \lambda_k < a_i\}, \qquad \nu(A) = \sum_{i=1}^{d} \sharp\{k \in \mathbb{Z}(1,T) \mid \lambda_k = a_i\}, \tag{2.46}$$

where λ_k is given by (2.13), and $\{a_i\}_{i=1}^d = \sigma(A)$ is the set of eigenvalues of A.

Proof. Firstly, we claim that

$$\lambda_i(U^\tau AU) = \lambda_i(A). \tag{2.47}$$

In fact, since $\lambda_0 I_d > A$ if and only if $\lambda_0 I_d > U^\tau AU$, we can choose $\lambda_0(U^\tau AU) = \lambda_0(A)$. By (2.22) and (2.23), it is easy to see that $\mu_i(U^\tau AU) = \mu_i(A)$, and hence (2.47) holds. Therefore, by Definition 2.4,

$$i(U^\tau AU) = i(A), \qquad \nu(U^\tau AU) = \nu(A). \tag{2.48}$$

Since

$$E^*\left(\mathrm{diag}\{A_1, A_2\}\right) \cong E^*(A_1) \oplus E^*(A_2), \quad * = +, 0, -, \tag{2.49}$$

we have

$$i\left(\mathrm{diag}\{A_1, A_2\}\right) = i(A_1) + i(A_2), \qquad \nu\left(\mathrm{diag}\{A_1, A_2\}\right) = \nu(A_1) + \nu(A_2). \tag{2.50}$$

Note that the scalar eigenvalue problem

$$\begin{gathered} \Delta^2 y(n-1) + \lambda y(n) = 0, \quad n \in \mathbb{Z}(1, T), \\ y(0) = 0 = y(T+1) \end{gathered} \tag{2.51}$$

has a nontrivial solution if and only if $\lambda = \lambda_k = 4\sin^2(k\pi/2(T+1))$, $k \in \mathbb{Z}(1, \mathrm{T})$. By Proposition 2.1 and Definition 2.4, we see that for any $\alpha \in \mathbb{R}$,

$$i(\alpha) = \sharp\{k \in \mathbb{Z}(1, T) \mid \lambda_k < \alpha\}, \qquad \nu(\alpha) = \sharp\{k \in \mathbb{Z}(1, T) \mid \lambda_k = \alpha\}. \tag{2.52}$$

Since $\{a_i\}_{i=1}^d$ is the set of eigenvalues of A, there exists an orthogonal matrix U such that

$$U^\tau AU = \mathrm{diag}\{a_1, a_2, \ldots, a_d\}. \tag{2.53}$$

From (2.48), (2.50), and (2.52) we have

$$i(A) = \sum_{i=1}^{d} \sharp\{k \in \mathbb{Z}(1, T) \mid \lambda_k < a_i\}, \qquad \nu(A) = \sum_{i=1}^{d} \sharp\{k \in \mathbb{Z}(1, T) \mid \lambda_k = a_i\}. \tag{2.54}$$

This completes the proof. □

Proposition 2.10. *For any $A : \mathbb{Z}(1, T) \to GL_s(\mathbb{R}^d)$ with $i(A) = 0$, one has*

$$\sum_{n=0}^{T} |\Delta u(n)|^2 \geq \sum_{n=1}^{T} (A(n)u(n), u(n)), \quad \forall u \in E. \tag{2.55}$$

And the equality holds if and only if $u \in E^0(A)$.

Proof. For any $u \in E$ with $u = \sum_{i,j} c_{ij} e_{ij}$, we have

$$\sum_{n=0}^{T} |\Delta u(n)|^2 = \sum_{n=1}^{T} (A(n)u(n), u(n)) + \sum_{i,j} \lambda_i(A)\mu_i |c_{ij}|^2. \tag{2.56}$$

Because $i(A) = 0$, by definition, $\lambda_i(A) \geq 0$ for any i. So the inequality holds. And the equality holds if and only if $c_{ij} = 0$ as $\lambda_i(A) \neq 0$, that is, $u \in E^0(A)$. □

By now, we have proved the monotonicity and have offered the computation formulae of the indices. These will play an important role in discussing nonlinear Hamiltonian systems in the next section. In the end of this section, we shall introduce the relative Morse index, which is a precise expression of the number $i(A_2) - i(A_1)$ as $A_2 > A_1$.

Definition 2.11. For any $A_1, A_2 : \mathbb{Z}(1,T) \to \mathrm{GL}_s(\mathbb{R}^d)$ with $A_1 < A_2$, define

$$I(A_1, A_2) = \sum_{\lambda \in [0,1)} \nu(A_1 + \lambda(A_2 - A_1)). \tag{2.57}$$

If $A_1 = \alpha_1 I_d$, $A_2 = \alpha_2 I_d$, where $\alpha_1 < \alpha_2$ are two real numbers, then by Proposition 2.9, we have

$$\begin{aligned} I(\alpha_1 I_d, \alpha_2 I_d) = \sum_{\lambda \in [0,1)} \nu(\alpha_1 I_d + \lambda(\alpha_2 - \alpha_1) I_d) &= d\sharp\{k \in \mathbb{Z}(1,T) \mid \lambda_k \in [\alpha_1, \alpha_2)\}, \\ i(\alpha_1 I_d) &= d\sharp\{k \in \mathbb{Z}(1,T) \mid \lambda_k < \alpha_1\}, \\ i(\alpha_2 I_d) &= d\sharp\{k \in \mathbb{Z}(1,T) \mid \lambda_k < \alpha_2\}. \end{aligned} \tag{2.58}$$

So

$$I(\alpha_1 I_d, \alpha_2 I_d) = i(\alpha_2 I_d) - i(\alpha_1 I_d). \tag{2.59}$$

This gives us a steer toward the following result.

Proposition 2.12. *For any $A_1, A_2 : \mathbb{Z}(1,T) \to GL_s(\mathbb{R}^d)$ with $A_1 < A_2$, one has*

$$I(A_1, A_2) = i(A_2) - i(A_1). \tag{2.60}$$

Proof. Denote $i(\lambda) := i(A_1 + \lambda(A_2 - A_1))$ for $\lambda \in [0,1]$, $\nu(\lambda) := \nu(A_1 + \lambda(A_2 - A_1))$; then to prove (2.60), we only need to prove that

$$i(\lambda + 0) = i(\lambda) + \nu(\lambda), \quad \forall \lambda \in [0,1), \tag{2.61}$$

$$i(\lambda - 0) = i(\lambda), \quad \forall \lambda \in (0,1] \tag{2.62}$$

hold. In fact, if (2.61) and (2.62) hold, then the function $\lambda \to i(\lambda)$ is integer-valued, left continuous, and nondecreasing. So, for any $\lambda_1 \in [0,1]$, $i(\lambda_1) - i(0)$ must equal to the sum of the jumps $i(\lambda)$ incurred in $[0,\lambda_1)$. By (2.61) and (2.62), this is precisely the sum of $\nu(\lambda)$, $0 \le \lambda < \lambda_1$, that is,

$$i(\lambda_1) - i(0) = \sum_{\lambda\in[0,\lambda_1)} \nu(\lambda). \tag{2.63}$$

Hence, if we choose $\lambda_1 = 1$, then (2.60) holds.

From (3) of Proposition 2.7, to prove (2.61), we only need to prove $i(\lambda+0) \le i(\lambda)+\nu(\lambda)$ which is also equivalent to $dT - i(\lambda) - \nu(\lambda) \le dT - i(\lambda+0)$. For any $s \in [0,1]$, set $m^+(s) = dT - i(s) - \nu(s)$, we only need to prove

$$m^+(\lambda) \le m^+(\lambda+0) + \nu(\lambda+0). \tag{2.64}$$

Similar to the proof of Lemma 2.8, it is easy to know that for $\varepsilon > 0$ sufficiently small, if

$$q_{B_{\lambda+\varepsilon}}(u,u) \ge 0, \quad \forall u \in E^+(\lambda), \tag{2.65}$$

then (2.64) holds, where $E^+(\lambda) = E^+(A_1+\lambda(A_2-A_1))$, $B_{\lambda+\varepsilon}(n) = A_1(n)+(\lambda+\varepsilon)(A_2(n)-A_1(n))$. While as $\varepsilon > 0$ is sufficiently small and $u \in E^+(\lambda)$, we have

$$\begin{aligned} q_{B_{\lambda+\varepsilon}}(u,u) &= q_{B_\lambda}(u,u) - \varepsilon\sum_{n=1}^{T}((A_2(n)-A_1(n))u(n),u(n)) \\ &= q_{B_\lambda}(u^+,u^+) - \varepsilon\sum_{n=1}^{T}((A_2(n)-A_1(n))u(n),u(n)) \ge 0, \end{aligned} \tag{2.66}$$

where $B_\lambda(n) = A_1(n) + \lambda(A_2(n) - A_1(n))$. Hence $i(\lambda+0) \le i(\lambda) + \nu(\lambda)$.

On the other hand, from (1) of Proposition 2.7, to prove (2.62), we only need to prove $i(\lambda) \le i(\lambda-0)$. By Lemma 2.8, to prove $i(\lambda) \le i(\lambda-0)$, we only need to prove

$$q_{B_{\lambda-\varepsilon}}(u,u) < 0, \quad \forall u \in E^-(\lambda) \setminus \{0\}, \tag{2.67}$$

where $\varepsilon > 0$ is sufficiently small, $B_{\lambda-\varepsilon}(n) = A_1(n) + (\lambda-\varepsilon)(A_2(n) - A_1(n))$. And as $\varepsilon > 0$ is sufficiently small, we have

$$\begin{aligned} q_{B_{\lambda-\varepsilon}}(u,u) &= q_{B_\lambda}(u,u) + \varepsilon\sum_{n=1}^{T}((A_2(n)-A_1(n))u(n),u(n)) \\ &= q_{B_\lambda}(u^-,u^-) + \varepsilon\sum_{n=1}^{T}((A_2(n)-A_1(n))u(n),u(n)) < 0, \end{aligned} \tag{2.68}$$

where $B_\lambda(n) = A_1(n) + \lambda(A_2(n) - A_1(n))$. This completes the proof. □

Proposition 2.13. *For any* $A : \mathbb{Z}(1,T) \to GL_s(\mathbb{R}^d)$, *there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$, *one has*

$$\nu(A + \varepsilon I_d) = 0, \tag{2.69}$$

$$\nu(A - \varepsilon I_d) = 0, \tag{2.70}$$

$$i(A - \varepsilon I_d) = i(A), \tag{2.71}$$

$$i(A + \varepsilon I_d) = i(A) + \nu(A). \tag{2.72}$$

Proof. From Proposition 2.12 we have $i(A+I_d) = i(A) + I(A, A+I_d)$. From Definition 2.11 and Proposition 2.12, we know that $I(A, A+I_d) = \sum_{\lambda\in[0,1)} \nu(A+\lambda I_d)$ is finite, and then there must exist some $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $\nu(A+\varepsilon I_d) = 0$ and

$$i(A + \varepsilon I_d) = i(A) + \sum_{\lambda\in[0,1)} \nu(A + \lambda\varepsilon I_d) = i(A) + \nu(A). \tag{2.73}$$

This proves (2.69) and (2.72).

To prove (2.70) and (2.71), note that $I(A - I_d, A) = i(A) - i(A - I_d)$ and

$$I(A - I_d, A) = \sum_{\lambda\in[0,1)} \nu(A - (1-\lambda)I_d) = \sum_{\lambda\in(0,1]} \nu(A - \lambda I_d). \tag{2.74}$$

Since $I(A-I_d, A)$ is finite, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $\nu(A-\varepsilon I_d) = 0$. Hence

$$i(A - \varepsilon I_d) = i(A) - I(A - \varepsilon I_d, A) = i(A) - \sum_{\lambda\in(0,1]} \nu(A - \lambda\varepsilon I_d) = i(A). \tag{2.75}$$

This proves (2.70) and (2.71). □

3. Main Results

In this section, firstly, we shall obtain the existence of solutions to (1.9) by using both the index theory in Section 2 and Leray-Schauder principle. Then, we obtain the multiplicity of solutions to (1.9) by using Morse theory.

Theorem 3.1. *Assume that*

(1) *there exist* $A : \mathbb{Z}(1,T) \times \mathbb{R}^d \to GL_s(\mathbb{R}^d)$ *and* $h : \mathbb{Z}(1,T) \times \mathbb{R}^d \to \mathbb{R}^d$ *which are both continuous with respect to the second variable, where* $h(n,z) = o(|z|)$ *as* $|z| \to \infty$ *for every* $n \in \mathbb{Z}(1,T)$ *and*

$$\nabla V(n,z) = A(n,z)z + h(n,z); \tag{3.1}$$

(2) *there exist* $A_1, A_2 : \mathbb{Z}(1,T) \to GL_s(\mathbb{R}^d)$ *satisfying* $A_1 \le A_2$, $i(A_1) = i(A_2)$, $\nu(A_2) = 0$ *such that*

$$A_1(n) \le A(n,z) \le A_2(n), \quad \forall z \in \mathbb{R}^d \tag{3.2}$$

for every $n \in \mathbb{Z}(1,T)$. *Then* (1.9) *has at least one solution.*

To prove Theorem 3.1, we need the following Leray-Schauder principle; see [34] for detailed proof.

Lemma 3.2. *Assume that* $(X, \|\cdot\|_X)$ *is a Banach space and that* $\Phi : X \to X$ *is completely continuous. If the set* $\{\|x\|_X \mid x \in X, x = \lambda\Phi x, 0 < \lambda < 1\}$ *is bounded, then* Φ *must have a fixed point in a closed ball* B_R *in* X, *where*

$$\begin{gathered} B_R = \{x \mid x \in X, \|x\|_X \le R\}, \\ R = \sup\{\|x\|_X \mid x = \lambda\Phi x, 0 < \lambda < 1\}. \end{gathered} \tag{3.3}$$

Proof of Theorem 3.1. Assume that (3.2) holds. Since $\nu(A_2) = 0$, from (1) of Proposition 2.5, we know that the system

$$\begin{gathered} \Delta^2 u(n-1) + A_2(n)u(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1) \end{gathered} \tag{3.4}$$

has only a trivial solution. Define $\Gamma_1 : E \to E$ as

$$(\Gamma_1 u)(n) = \Delta^2 u(n-1) + A_2(n)u(n); \tag{3.5}$$

then Γ_1 is an invertible operator. Define $\Gamma_2 : E \to E$ as

$$(\Gamma_2 u)(n) = A_2(n)u(n) - \nabla V(n, u(n)); \tag{3.6}$$

then finding the solutions to (1.9) is equivalent to finding solutions to

$$\Gamma_1 u - \Gamma_2 u = 0 \tag{3.7}$$

in E, which is also equivalent to finding the fixed points of $\Gamma_1^{-1}\Gamma_2$ in E since Γ_1 is invertible. By Lemma 3.2, we only need to prove that the possible solutions to

$$\begin{gathered} \Delta^2 u(n-1) + (1-\lambda)A_2(n)u(n) + \lambda\nabla V(n, u(n)) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1) \end{gathered} \tag{3.8}$$

are priori bounded with respect to the norm $\|\cdot\|$ in E, where $\lambda \in (0,1)$. If not, there exist $\{u_k\} \subset E$, $\{\lambda_k\} \subset (0,1)$ with $\|u_k\| \to \infty$ such that

$$\begin{aligned} &\Delta^2 u_k(n-1) + (1-\lambda_k)A_2(n)u_k(n) + \lambda_k \nabla V(n,u_k(n)) = 0, \quad n \in \mathbb{Z}(1,T), \\ &u_k(0) = 0 = u_k(T+1). \end{aligned} \tag{3.9}$$

Denote $v_k(n) = u_k(n)/\|u_k\|$, $B_k(n) = (1-\lambda_k)A_2(n) + \lambda_k A(n,u_k(n))$, $e_k(n) = \lambda_k(\nabla V(n,u_k(n)) - A(n,u_k(n))u_k(n))/\|u_k\|$; then (3.9) is equivalent to

$$\begin{aligned} &\Delta^2 v_k(n-1) + B_k(n)v_k(n) + e_k(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ &v_k(0) = 0 = v_k(T+1). \end{aligned} \tag{3.10}$$

From (3.1), $e_k \to 0$ as $\|u_k\| \to \infty$. We may assume that $v_k \to v_0$, $\lambda_k \to \lambda_0$, and $b_{ij}^k \to b_{ij}$, where $B_k(n) = (b_{ij}^{(k)})_{d\times d}(n)$. Denote $B_0(n) = (b_{ij}(n))_{d\times d}$; let $k \to \infty$ in (3.10); we have

$$\begin{aligned} &\Delta^2 v_0(n-1) + B_0(n)v_0(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ &v_0(0) = 0 = v_0(T+1). \end{aligned} \tag{3.11}$$

On the other hand, (3.2) implies that $A_1 \le B_k \le A_2$, and hence $A_1 \le B_0 \le A_2$. By $i(A_2) = i(A_1)$, $\nu(A_2) = 0$, and Proposition 2.7, we have $\nu(A_1) = \nu(A_2) = \nu(B_0) = 0$. This contradicts the fact that (3.11) has a nontrivial solution. □

Example 3.3. Let

$$V(n,z) = \sum_{i=1}^{d} F_i(z_i) + \frac{1}{2}\sum_{i=1}^{d} a_{ii}(n)z_i^2 + \sum_{1\le i<j\le d} a_{ij}(n)z_i z_j + \left(\sum_{i=1}^{d} z_i^2 + 1\right)^{3/4} \sin n, \tag{3.12}$$

where $z = (z_1, z_2, \ldots, z_d)^\tau$, $F_i(t) = \int_0^t s f_i(s)\,ds$, $f_i : \mathbb{R} \to [0,\alpha]$ is continuous and $\alpha > 0$, $a_{ij}(n) = a_{ji}(n)$, $i = 1,2,\ldots,d$, $j = 1,2,\ldots,d$. Set

$$A(n) = \left(a_{ij}(n)\right)_{d\times d} = U^\tau \operatorname{diag}\{\alpha_1(n), \alpha_2(n), \ldots, \alpha_d(n)\}U, \tag{3.13}$$

where $U \in O(d)$. Since

$$\begin{aligned} \nabla V(n,z) &= \left(z_1 f_1(z_1), z_2 f_2(z_2), \ldots, z_d f_d(z_d)\right)^\tau + A(n)z + \frac{3}{2}\left(\sum_{i=1}^{d} z_i^2 + 1\right)^{-1/4} z \sin n \\ &= \operatorname{diag}\{f_1(z_1), f_2(z_2), \ldots, f_d(z_d)\}z + A(n)z + \frac{3}{2}\left(\sum_{i=1}^{d} z_i^2 + 1\right)^{-1/4} z \sin n, \end{aligned} \tag{3.14}$$

$V(n,z)$ satisfies (3.1) with

$$A(n,z) = A(n) + \operatorname{diag}\{f_1(z_1), f_2(z_2), \ldots, f_d(z_d)\},$$
$$A_1(n) = A(n), \qquad A_2(n) = A(n) + \alpha I_d, \tag{3.15}$$
$$h(n,z) = \frac{3}{2}\left(\sum_{i=1}^{d} z_i^2 + 1\right)^{-1/4} z \sin n.$$

If $\nu(\alpha_i) = 0$ for every $i \in \mathbb{Z}(1,d)$, then $\nu(A_1) = 0$. By Proposition 2.13, if $\alpha > 0$ is small enough, then $\nu(A_2) = 0$ and $i(A_1) = i(A_2)$. Hence, by Theorem 3.1, (1.9) has at least one solution. In particular, if we choose $f_i(t) = \alpha(\sin t)^{2i}$ and $\alpha_i(n) \equiv \alpha_i \in \mathbb{R}$, $[\alpha_i, \alpha_i + \alpha] \cap \{\lambda_k\}_{k=1}^{T} = \emptyset$ for every $i = 1, 2, \ldots, d$, then $i(A_1) = i(A_2)$, $\nu(A_1) = \nu(A_2) = 0$. And hence (1.9) has at least one solution.

Theorem 3.4. *In assumption* (3.1) *if* $A(n,z) = A(n)$ *satisfying* $\nu(A) \neq 0$ *and*

$$(z, h(n,z)) \geq c_1|z|^{\alpha} - b_1, \qquad |h(n,z)| \leq c_2|z|^{\alpha-1} + b_2, \tag{3.16}$$

where c_1, c_2, b_1, b_2 *are all positive constants and* $1 \leq \alpha < 2$, *then* (1.9) *has at least one solution.*

Proof. From Proposition 2.13, there exists $\varepsilon > 0$ such that $i(A + \varepsilon I_d) = i(A) + \nu(A)$ and $\nu(A + \varepsilon I_d) = 0$ for any $A : \mathbb{Z}(1,T) \to \mathrm{GL}_s(\mathbb{R}^d)$. Denote $A_1 = A + \varepsilon I_d$, by Lemma 3.2, we only need to prove that the possible solutions to

$$\Delta^2 u(n-1) + \lambda A_1(n)u(n) + (1-\lambda)A(n)u(n) + (1-\lambda)h(n,u(n)) = 0, \quad n \in \mathbb{Z}(1,T),$$
$$u(0) = 0 = u(T+1) \tag{3.17}$$

are priori bounded with respect to the norm $\|\cdot\|$ in E. If not, there exist $\{u_k\} \subset E$, $\lambda_k \in (0,1)$ with $\|u_k\| \to \infty$ such that

$$\Delta^2 u_k(n-1) + \lambda_k A_1(n)u_k(n) + (1-\lambda_k)A(n)u_k(n) + (1-\lambda_k)h(n,u_k(n)) = 0, \quad n \in \mathbb{Z}(1,T),$$
$$u_k(0) = 0 = u_k(T+1). \tag{3.18}$$

Denote $v_k(n) = u_k(n)/\|u_k\|$, we may assume that $v_k \to v_0$ and $\lambda_k \to \lambda_0$. Hence, $v = v_0$ is a nontrivial solution to

$$\Delta^2 v_0(n-1) + \lambda_0 A_1(n)v_0(n) + (1-\lambda_0)A(n)v_0(n) = 0, \quad n \in \mathbb{Z}(1,T),$$
$$v_0(0) = 0 = v_0(T+1) \tag{3.19}$$

which implies that $\nu(A + \varepsilon\lambda_0 I_d) \neq 0$. We claim that $\lambda_0 = 0$. If not, $\lambda_0 \in (0,1]$, then $A < A + \lambda_0\varepsilon I_d \leq A + \varepsilon I_d$. From Proposition 2.7, we have $i(A + \varepsilon I_d) = i(A) + \nu(A) \leq i(A + \lambda_0\varepsilon I_d)$ and

$i(A+\varepsilon I_d) = i(A+\lambda_0\varepsilon I_d)$. However, from Proposition 2.7, we also have $i(A+\lambda_0\varepsilon I_d)+\nu(A+\lambda_0\varepsilon I_d) \le i(A+\varepsilon I_d)+\nu(A+\varepsilon I_d)$ which implies that $\nu(A+\lambda_0\varepsilon I_d) = 0$, a contradiction! Hence $\lambda_0 = 0$ and

$$\begin{gathered}\Delta^2 v_0(n-1) + A(n)v_0(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ v_0(0) = 0 = v_0(T+1).\end{gathered} \tag{3.20}$$

From (3.18), we have

$$\begin{aligned}&\sum_{n=1}^{T}\Big(\Delta^2 u_k(n-1) + A(n)u_k(n), v_0(\mathrm{n})\Big) \\ &\quad + \sum_{n=1}^{T}\varepsilon\lambda_k(u_k(n), v_0(n)) + \sum_{n=1}^{T}(1-\lambda_k)(h(n,u_k(n)), v_0(n)) = 0.\end{aligned} \tag{3.21}$$

Therefore, from (3.16), (3.20), and (3.21), for k large enough,

$$\begin{aligned}0 &\ge \sum_{n=1}^{T}(h(n,u_k(n)), v_0(n)) \\ &= \sum_{n=1}^{T}\left(h(n,u_k(n)), \frac{u_k(n)}{\|u_k\|}\right) + \sum_{n=1}^{T}(h(n,u_k(n)), v_0(n)-v_k(n)) \\ &\ge \|u_k\|^{-1}\sum_{n=1}^{T}(c_1|u_k(n)|^{\alpha} - b_1) - \|v_0 - v_k\|_{\infty}\sum_{n=1}^{T}\Big(c_2|u_k(n)|^{\alpha-1} + b_2\Big).\end{aligned} \tag{3.22}$$

Dividing $\|u_k\|^{\alpha-1}$ at both sides, we have

$$0 \ge \sum_{n=1}^{T}(c_1|v_k(n)|^{\alpha} - b_1\|u_k\|^{-\alpha}) - \|v_0 - v_k\|_{\infty}\sum_{n=1}^{T}\Big(c_2|v_k(n)|^{\alpha-1} + b_2\|u_k\|^{1-\alpha}\Big) \longrightarrow c_1\sum_{n=1}^{T}|v_0(n)|^{\alpha}. \tag{3.23}$$

This is a contradiction since $\|v_0\| \neq 0$ and $c_1 > 0$. The proof is complete. □

If $\nabla V(n,0) \equiv 0$, then $u \equiv 0$ is a solution to (1.9). As usual we call this solution the trivial solution. It is much regretted that we do not know if the solution we found is not the trivial one in Theorems 3.1 and 3.4. In the following, we will obtain the existence of nontrivial solutions to (1.9) by using Morse theory.

Theorem 3.5. *Assume the following*

(1) $V : \mathbb{Z}(1,T)\times\mathbb{R}^d \to \mathbb{R}$ *is* C^2 *with respect to the second variable, and*

$$\nabla V(n,z) = A_1(n)z + o(|z|), \quad \text{as } |z| \longrightarrow \infty \tag{3.24}$$

for every $n \in \mathbb{Z}(1,T)$ *with* $\nu(A_1) = 0$.

(2) $\nabla V(n,0) \equiv 0$, $A_0(n) = V''(n,0)$, *and* $i(A_1) \notin \mathbb{Z}(i(A_0), i(A_0) + \nu(A_0))$. *Then* (1.9) *has at least one nontrivial solution.*

(3) *Moreover, if we further assume that* $\nu(A_0) = 0$, $|i(A_1) - i(A_0)| \geq d$, *then* (1.9) *has at least two nontrivial solutions.*

To prove Theorem 3.5, we need some results on Morse theory. Let E be a real Hilbert space and $f \in C^1(E, \mathbb{R})$. As in [2], denote

$$f^c = \{u \in E \mid f(u) \leq c\}, \qquad \mathcal{K}_c = \{u \in E \mid f'(u) = 0,\ f(u) = c\} \tag{3.25}$$

for $c \in \mathbb{R}$. The following is the definition of the Palais-Smale condition (the (PS) condition for short).

Definition 3.6. The functional f satisfies the (PS) condition if any sequence $\{u_m\} \subset E$ such that $\{f(u_m)\}$ is bounded and $f'(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.

Let u_0 be an isolated critical point of f with $f(u_0) = c \in \mathbb{R}$, and let U be a neighborhood of u_0; the group

$$C_q(f, u_0) := H_q(f^c \cap U, f^c \cap U \setminus \{u_0\}), \quad q = 1, 2, \ldots \tag{3.26}$$

is called the qth critical group of f at u_0, where $H_q(A, B)$ denotes qth singular relative homology group of the pair (A, B) over a field F, which is defined to be quotient $H_q(A, B) = Z_q(A, B)/B_q(A, B)$, where $Z_q(A, B)$ is the qth singular relative closed chain group and $B_q(A, B)$ is the qth singular relative boundary chain group.

For any two regular values $a < b$, if $\mathcal{K} \cap f^{-1}[a, b] = \{u_1, u_2, \ldots, u_l\}$, denote $M_q = \sum_{i=1}^{l} \dim C_q(f, u_i)$ and $\beta_q = \dim H_q(f^b, f^a)$. The following results play an important role in proving Theorem 3.5; see [2] for the detailed proof.

Lemma 3.7. *Assume that* $f \in C^2(E, \mathbb{R})$ *satisfies the (PS) condition. Then one has the following Morse inequalities:*

$$M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq \beta_q - \beta_{q-1} + \cdots + (-1)^q \beta_0 \tag{3.27}$$

for $q = 0, 1, 2, \ldots$. *One also has the following Morse equality:*

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t) Q(t), \tag{3.28}$$

where $Q(t)$ *is a polynomial with nonnegative integer coefficients.*

Lemma 3.8. *Assume that $f \in C^2(E, \mathbb{R})$ and that u_0 is an isolated critical point of f with finite Morse index $\mu(u_0)$ and nullity $\nu(u_0)$. Thenone has the following.*

(1) *For any $q \notin \mathbb{Z}(\mu(u_0), \mu(u_0) + \nu(u_0))$, $C_q(f, u_0) \cong 0$.*

(2) *If u_0 is a nondegenerate critical point, then*

$$C_q(f, u_0) \cong \delta_{q,\mu(u_0)} F, \quad q = 0, 1, 2, \ldots. \tag{3.29}$$

(3) *If f has a minimal value at u_0, then*

$$C_q(f, x_0) \cong \delta_{q,0} F, \quad q = 0, 1, 2, \ldots. \tag{3.30}$$

(4) *If there exist integers $q_1 \neq q_2$ such that $C_{q_1}(f, u_0) \ncong 0$ and $C_{q_2}(f, u_0) \ncong 0$, then $|q_1 - q_2| \leq \nu(u_0) - 2$.*

Now, Define

$$f(u) = \frac{1}{2}\sum_{n=0}^{T} |\Delta u(n)|^2 - \sum_{n=1}^{T} V(n, u(n)), \quad u \in E. \tag{3.31}$$

Then the functional f is C^2 with

$$\begin{aligned} \langle f'(u), v \rangle &= \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)) - \sum_{n=1}^{T} (\nabla V(n, u(n)), v(n)) \\ &= -\sum_{n=1}^{T} \Big(\Delta^2 u(n-1) + \nabla V(n, u(n)), v(n) \Big), \quad \forall u, v \in E. \end{aligned} \tag{3.32}$$

So solutions to (1.9) are precisely the critical points of f.

Lemma 3.9. *Under assumptions of Theorem 3.5, there exist $R_0 > 0, R_1 > 0$ and $\tilde{f}$ satisfying the following conditions.*

(1) *$f'(u) = 0$ implies $\|u\| \leq R_0$.*

(2) *f and $\tilde{f}$ have the same critical set.*

(3) *If $\|u\| \geq R_1$, then $\tilde{f}(u) = (1/2)\langle Lu, u \rangle$.*

Proof. Define L and g on E as

$$\begin{aligned} \langle Lu, v \rangle &= \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)) - \sum_{n=1}^{T} (A_1(n) u(n), v(n)), \\ g(u) &= f(u) - \frac{1}{2} \langle Lu, u \rangle. \end{aligned} \tag{3.33}$$

Then

$$\langle f'(u), v\rangle = \langle Lu, v\rangle + \langle g'(u), v\rangle. \tag{3.34}$$

Assumption $\nu(A_1) = 0$ implies that L is invertible. Taking $\varepsilon_1 = \|L^{-1}\|^{-1}/2$, there exists $R_0 > 0$ such that if $\|u\| > R_0$, then

$$\|g'(u)\| < \varepsilon_1\|u\|. \tag{3.35}$$

So, as $\|u\| > R_0$, we have

$$\|f'(u)\| \geq \left\|L^{-1}\right\|^{-1}\|u\| - \|g'(u)\| > \varepsilon_1\|u\|, \tag{3.36}$$

that is, no critical point is outside the ball B_{R_0}.

To prove (2) and (3), let $\rho \in C^\infty(\mathbb{R}, [0,1])$ satisfy

$$\rho(t) = \begin{cases} 1, & t \leq 0, \\ 0, & t \geq 1 \end{cases} \tag{3.37}$$

with $0 \leq \rho(t) \leq 1$ and $\max|\rho'(t)| \leq 3/2$. Let $\varepsilon = \|L^{-1}\|^{-1}/5$, then there exists $c_1 \geq 0$ such that

$$\|g'(u)\| < \varepsilon\|u\| + c_1. \tag{3.38}$$

Hence,

$$\begin{aligned} |g(u)| &\leq \int_0^1 |\langle g'(su), u\rangle| ds + |g(0)| \\ &\leq \int_0^1 \left(\varepsilon s\|u\|^2 + c_1\|u\|\right) ds + |g(0)| \\ &= \frac{\varepsilon}{2}\|u\|^2 + c_1\|u\| + |g(0)|. \end{aligned} \tag{3.39}$$

Therefore, there exists $c_2 > 0$ such that

$$|g(u)| \leq \varepsilon\|u\|^2 + c_2, \quad \forall u \in E. \tag{3.40}$$

Define

$$R > \max\left\{1, R_0, \frac{c_1 + 3/2}{\epsilon}\right\}, \tag{3.41}$$

where R_0 is defined above and

$$\lambda = \max\{c_2, 1\} \cdot R. \tag{3.42}$$

If $R \le \|u\| \le \lambda + R$, then

$$\frac{c_2}{\lambda} < 1, \qquad \frac{\|u\|}{\lambda} \le 2. \tag{3.43}$$

Let

$$\rho_\lambda(t) = \rho\left(\frac{t - R}{\lambda}\right), \tag{3.44}$$

and define

$$\tilde{f}(u) = \frac{1}{2}\langle Lu, u\rangle + \rho_\lambda(\|u\|) g(u). \tag{3.45}$$

Then the function $\tilde{f}(u)$ satisfies (2) and (3). In fact,

$$\tilde{f}(u) = \begin{cases} f(u), & \|u\| \le R, \\ \frac{1}{2}\langle Lu, u\rangle, & \|u\| \ge \lambda + R. \end{cases} \tag{3.46}$$

The only thing we have to check is that $\tilde{f}'(u) \neq 0$ as $R \le \|u\| \le \lambda + R$. However,

$$\begin{aligned} \left\|\tilde{f}'(u)\right\| &= \left\| Lu + \frac{1}{\lambda}\rho'\left(\frac{\|u\| - R}{\lambda}\right) g(u) \frac{u}{\|u\|} + \rho\left(\frac{\|u\| - R}{\lambda}\right) g'(u) \right\| \\ &\ge \left\|L^{-1}\right\|^{-1} \|u\| - \frac{3}{2\lambda}\left(\varepsilon\|u\|^2 + c_2\right) - \left(\varepsilon\|u\| + c_1\right) \\ &= 5\varepsilon\|u\| - \varepsilon\|u\| - \frac{3\varepsilon}{2\lambda}\|u\|^2 - \left(c_1 + \frac{3c_2}{2\lambda}\right) \\ &\ge \varepsilon\|u\| - \left(c_1 + \frac{3}{2}\right) > 0. \end{aligned} \tag{3.47}$$

Hence, let $R_1 = \lambda + R$; the proof is completed. □

From Lemma 3.9, $f'(u) = 0$ if and only if $\tilde{f}'(u) = 0$. Thus, in order to find solutions to (1.9), it suffices to find the critical points of $\tilde{f}$. Moreover, $\tilde{f}$ satisfies the (PS) condition by Lemma 3.9.

Lemma 3.10. *Under the assumptions of Theorem 3.5, there exist a, b with $a < b$ such that the critical points of $\widetilde{f}$ belong to $\widetilde{f}^{-1}((a,b)) := \{u \mid a < \widetilde{f}(u) < b\}$ and*

$$H_q\left(\widetilde{f}^b, \widetilde{f}^a\right) \cong \delta_{q,i(A_1)} F. \tag{3.48}$$

Proof. Define

$$a < \min_{B_{2R_1}} \widetilde{f} - 1, \qquad b > \max_{B_{2R_1}} \widetilde{f} + 1, \qquad f_1(u) = \frac{1}{2}\langle Lu, u\rangle, \tag{3.49}$$

where a, b are finite. Noticing that all critical points of $\widetilde{f}$ lie in B_{R_0}, if u is a critical point of $\widetilde{f}$, then

$$a < \min_{B_{R_0}} \widetilde{f} \leq \widetilde{f}(u) \leq \max_{B_{R_0}} \widetilde{f} < b. \tag{3.50}$$

This implies that $\{u \mid a < \widetilde{f}(u) < b\}$ contains all critical points of $\widetilde{f}$. By the properties of the raltive singular homology group, we have $H_q(\widetilde{f}^b, \widetilde{f}^a) \cong H_q(f_1^b, f_1^a)$. However, $\nu(A_1) = 0$ implies that f_1 has only critical point 0 with Morse index $i(A_1)$. From Lemma 3.8 the conclusion holds. □

Proof of Theorem 3.5. (1) By Lemma 3.10 and the Morse inequalities, f must have a critical point u with $C_{i(A_1)}(f, u) \ncong 0$. Since $i(A_1) \notin \mathbb{Z}(i(A_0), i(A_0) + \nu(A_0))$, then from Lemma 3.8, we have

$$C_{i(A_1)}(f, 0) \cong 0. \tag{3.51}$$

And hence $u \neq 0$ is a critical point of f; that is, (1.9) has at least one nontrivial solution.

(2) Since $\nu(A_0) = 0$, we have

$$C_q(f, 0) \cong \delta_{q,i(A_0)} F, \quad q = 0, 1, 2, \ldots. \tag{3.52}$$

Now we assume that $|i(A_1) - i(A_0)| \geq d$ and that u is the only nonzero critical point of f. Then from Morse equality, we have

$$t^{i(A_0)} + \sum_{q=0}^{\infty} \dim C_q(f, u) t^q = t^{i(A_1)} + (1 + t) Q(t). \tag{3.53}$$

We necessarily have $\dim C_{i(A_1)}(f,u) \geq 1$, and

$$\text{either } \dim C_{i(A_0)-1}(f,u) \geq 1, \quad \text{or } \dim C_{i(A_0)+1}(f,u) \geq 1. \tag{3.54}$$

(i) If $\dim C_{i(A_0)-1}(f,u) \geq 1$, then by assumption we have $i(A_0) - 1 \neq i(A_1)$. Since the nullity of u is less or equal to d, from Lemma 3.8, we have

$$|i(A_0) - 1 - i(A_1)| \leq d - 2 \tag{3.55}$$

which implies that $|i(A_0) - i(A_1)| \leq d - 1$. This is impossible since $|i(A_0) - i(A_1)| \geq d$.

(ii) If $\dim C_{i(A_0)+1}(f,u) \geq 1$, then similar to the above proof we have $|i(A_0) - i(A_1)| \leq d - 1$, also a contradiction.

Therefore, f has at least two nonzero critical points and hence (1.9) has at least two nontrivial solutions. □

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