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Research Article

On Nonlinear Boundary Value Problems for Functional Difference Equations with p-Laplacian

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Sufficient conditions for the existence of solutions of nonlinear boundary value problems for higher-order functional difference equations with p-Laplacian are established by making of continuation theorems. We allow f to be at most linear, superlinear, or sublinear in obtained results.

1. Introduction

The existence of solutions of boundary value problems for finite difference equations were studied by many authors, one may see the text books [1, 2], the papers [3–5] and the references therein. We present some representative ones, which are the motivations of this paper.

In papers [3, 4], using Krasnoselskii fixed point theorem and Leggett-Williams fixed point theorem, respectively, Karakostas studied the existence of three positive solutions of the problems consisting of the functional differential equation

$$[\Phi(x')]' + c(t) f(t, x(g_1(t)), x(g_2(t)), \ldots, x(g_n(t))) = 0, \quad \text{a.a. } t \in [0,1] \tag{1.1}$$

and one of the following three pairs of conditions:

$$x(0) - B_0(x'(0)) = 0, \qquad x(1) + B_1(x'(1)) = 0,$$

$$
\begin{gathered}
x(0) - B_0\big(x'(0)\big) = 0, \qquad x'(1) = 0, \\
x'(0) = 0, \qquad x(1) + B_1\big(x'(1)\big) = 0.
\end{gathered}
\tag{1.2}
$$

Here Φ is a Sup-Multiplicative-Like function, see [3, 4]. In [4] to get the main existence theorems the author assumes the validity of

(H1) for each $i = 0, 1$ the function B_i is continuous nondecreasing and such that $\alpha B_i(\alpha) \geq 0$ and at least one of the following:

(H2) $\limsup_{\alpha \to 0+}(B_0(\alpha)/\alpha) < +\infty$,

(H3) $\limsup_{\alpha \to 0+}(-B_1(-\alpha)/\alpha) < +\infty$,

(H4) $\sup_{\alpha>0}(B_0(\alpha)/\alpha) < +\infty$,

(H5) $\sup_{\alpha>0}(-B_1(-\alpha)/\alpha) < +\infty$.

The discrete simulation of BVPs studied in [3, 4] is as follows:

$$
\Delta[\Phi(\Delta x(k))] + c(k) f\big(k, x\big(g_1(k)\big), \dots, x\big(g_n(k)\big)\big) = 0, \quad k \in [0, T] \tag{1.3}
$$

subject to one of the following boundary conditions:

$$
\begin{gathered}
x(0) - B_0(\Delta x(0)) = 0, \qquad x(T+2) + B_1(\Delta x(T+1)) = 0, \\
x(0) - B_0(\Delta x(0)) = 0, \qquad \Delta x(T+1) = 0, \\
\Delta x(0) = 0, \qquad x(T+2) + B_1(\Delta x(T+1)) = 0.
\end{gathered}
\tag{1.4}
$$

The question follows: under what conditions above BVP (1.3) has solutions if (H1)–(H5) are not satisfied?

Particular significance lies in the fact that when a BVP is discretized, strange and interesting changes can occur in the solutions. For example, properties such as existence, uniqueness and multiplicity of solutions may not be shared between the continuous differential equation and its related discrete difference equation. Moreover, when investigating difference equations, as opposed to differential equations, basic ideas from calculus are not necessarily available to use, such as the intermediate value theorem, the mean value theorem and Rolle's theorem. Thus, new challenges are faced and innovation is required [5].

In recent paper [5], Liu studied the solvability of the following problem consisting of the higher-order functional difference equation and boundary conditions

$$
\begin{gathered}
\Delta^2 x(n) = f(n, x(n+1), x(n-\tau_1(n)), \dots, x(n-\tau_m(n))), \quad n \in [0, T-1], \\
a x(0) - b \Delta x(0) = 0, \qquad c x(T+1) + d \Delta x(T) = 0, \\
x(i) = \phi(i), \quad i \in [-\tau, -1], \\
x(i) = \psi(i), \quad i \in [T+2, T+\delta],
\end{gathered}
\tag{1.5}
$$

where $T \geq 1$, $a, b, c, d \in R$ with $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$, $\tau_i(n)$, $i = 1, \ldots, m$, are sequences,

$$\tau = \max\left\{0, \max_{n \in [0, T-1]} \{\tau_i(n)\} : i = 1, \ldots, m\right\},$$
$$\delta = -\min\left\{0, \min_{n \in [0, T-1]} \{\tau_i(n)\} : i = 1, \ldots, m\right\}, \tag{1.6}$$

$f(n, u)$ is continuous in $u = (x_0, \ldots, x_m, x_{m+1})$ for each n. Two cases, that is, $bc + ad + ac(T+1) = 0$ or $bc + ad + ac(T+1) \neq 0$ are considered in [6].

Motivated by [3–5], we study the nonlinear boundary value problems for higher-order functional difference equation with p-Laplacian, that is, the equation

$$\Delta\left[\phi(\Delta x(n))\right] = f(n, x(n+1), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))), \quad n \in [0, T-1], \tag{1.7}$$

subject to the following boundary conditions

$$x(0) - B_0(\Delta x(0)) = 0, \qquad B_1(x(T+1)) + \Delta x(T) = 0,$$
$$x(i) = \gamma(i), \quad i \in [-\tau, -1], \tag{1.8}$$
$$x(i) = \psi(i), \quad i \in [T+2, T+\delta],$$

where $[0, T-1] = \{0, 1, \ldots, T-1\}$, $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$ with $p > 1$, the inverse function is denoted by ϕ^{-1}, $T \geq 1$ an integer, B_0, B_1 are continuous and satisfy $xB_i(x) \geq 0$ for all $x \in R$, $i = 0, 1$, $\{\tau_i(n)\}$, $i = 1, \ldots, m$, are integer vectors,

$$\tau = \max\left\{0, \max_{n \in [0, T-1]} \{\tau_i(n)\} : i = 1, \ldots, m\right\},$$
$$\delta = -\min\left\{0, \min_{n \in [0, T-1]} \{\tau_i(n)\} : i = 1, \ldots, m\right\}, \tag{1.9}$$

$f(n, u)$ is continuous about $u = (x_0, \ldots, x_m)$ for each n. Boundary condition (1.8) is called nonlinear Sturm-Liouville type conditions.

The purposes of this paper are to establish sufficient conditions for the existence of at least one solutions of BVP (1.7)-(1.8). It is interesting that we allow that f to be sublinear, at most linear or superlinear. We do not need the assumptions (H2)–(H5) imposed on B_0, B_1.

This paper is organized as follows. In Section 2, we give the main results of this paper, and in Section 3, examples to illustrate the main results will be presented.

2. Main Results

To get existence results for solutions of BVP (1.7)-(1.8), we need the following fixed point theorem, which was used to solve multi-point boundary value problems for differential equations in many papers but not used to solve boundary value problems for difference equations.

Let X and Y be real Banach spaces, let $L : \operatorname{Dom} L \subset X \to Y$ be a Fredholm operator of index zero. If Ω is an open bounded subset of X, $\operatorname{Dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $N(\overline{\Omega})$ is bounded and compact.

Lemma 2.1 (see [6]). *Let X and Y be Banach spaces. Suppose $L : \operatorname{Dom} L \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L = \{0\}$, $N : X \to Y$ is L-compact on any open bounded subset of X. If $\Omega \subset X$ is an open bounded set with $0 \in \Omega$ and*

$$Lx \neq \lambda Nx, \quad \forall x \in \operatorname{Dom} L \cap \partial\Omega, \ \lambda \in [0,1], \tag{2.1}$$

then there is at least one $x \in \Omega$ so that $Lx = Nx$.

Let $X = R^{T+\tau+\delta+1} \times R^{T+1}$ be endowed with the norm

$$\|(x,y)\| = \max\left\{ \max_{n\in[1,T+\tau+\delta+1]} |x(n)|, \max_{n\in[0,T]} |y(n)| \right\} \quad \text{for } (x,y) \in X. \tag{2.2}$$

Let $Y = R^{T+1} \times R^{T} \times R^{2} \times R^{\tau} \times R^{\delta-1}$ be endowed with the norm

$$\|(u,v,a,b,s,t)\| = \max\left\{ \max_{n\in[0,T]} |u(n)|, \max_{n\in[0,T-1]} |v(n)|, |a|, |b|, \max_{n\in[-\tau,-1]} |s(n)|, \max_{n\in[T+2,T+\delta]} |t(n)| \right\} \tag{2.3}$$

for $(u,v,a,b,s,t) \in Y$.

It is easy to see that X and Y are real Banach spaces.

Let $L : X \to Y$, and

$$L\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{cases} \Delta x(n), & n \in [0,T] \\ \Delta y(n), & n \in [0,T-1] \\ x(1) & \\ y(T) & \\ x(i), & i \in [-\tau,-1] \\ x(j), & j \in [T+2,T+\delta], \end{cases} \tag{2.4}$$

and $N : X \rightarrow Y$ *by*

$$N\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{cases} \phi^{-1}(y(n)), & n \in [0,T] \\ f(n, x(n+1), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))), & n \in [0,T-1] \\ B_0(\phi^{-1}(y(0))) & \\ -\phi(B_1(x(T+1))) & \\ \gamma(i), & i \in [-\tau,-1] \\ \psi(j), & j \in [T+2, T+\delta] \end{cases} \tag{2.5}$$

for all $(x,y) \in X$.

Since f, B_0, B_1 *are continuous, it is easy to show that*

(i) $(x,y) \in \operatorname{Dom} L$ *is a solution of* $L(x,y) = N(x,y)$ *implies that* x *is a solution of BVP* (1.7)-(1.8),

(ii) $\operatorname{Ker} L = \{0\}$,

(iii) L *is a Fredholm operator of index zero and* N *is L-compact on each* $\overline{\Omega}$ *with* Ω *being an open bounded subset of* X.

Lemma 2.2. $(\sum_{i=1}^{m} a_i)^{\sigma} \leq l_{\sigma}^{m-1}(\sum_{i=1}^{m} a_i^{\sigma})$ *for all* $a_i \geq 0$ *and* $\sigma > 0$, *where* l_σ *is defined by* $l_\sigma = 2^{\sigma-1}$ *for* $\sigma \geq 1$ *and* $l_\sigma = 1$ *for* $\sigma \in (0,1)$.

Proof. We have the following cases. □

Case 1 ($m = 2$). Without loss of generality, suppose $a_1 \geq a_2$. Let

$$g(x) = (1+x)^{\sigma} - l_\sigma(1+x^{\sigma}), \quad x \in [1,+\infty), \tag{2.6}$$

then

$$g(1) = 2^{\sigma} - 2l_\sigma = \begin{cases} 2^{\sigma} - 2^{\sigma} = 0, & \sigma \geq 1, \\ 2^{\sigma+1} - 2 \leq 0, & \sigma \in (0,1) \end{cases} \tag{2.7}$$

and for $x \in [1,\infty)$, we get

$$g'(x) = \sigma x^{\sigma-1}\left[\left(1+\frac{1}{x}\right)^{\sigma-1} - l_\sigma\right] = \begin{cases} \sigma x^{\sigma-1}\left[\left(1+\dfrac{1}{x}\right)^{\sigma-1} - 2^{\sigma-1}\right] \leq 0, & \sigma \geq 1, \\ \sigma x^{\sigma-1}\left[\left(1+\dfrac{1}{x}\right)^{\sigma-1} - 1\right] \leq 0, & \sigma \in (0,1). \end{cases} \tag{2.8}$$

We get that $g(x) \leq g(1)$ for all $x \geq 1$ and so $(1+x)^{\sigma} \leq l_\sigma(1+x^{\sigma})$ for all $x \in [1,+\infty)$. Hence $(a_1+a_2)^{\sigma} = a_2^{\sigma}(1+(a_1/a_2))^{\sigma} \leq l_\sigma a_2^{\sigma}[1+(a_1/a_2)^{\sigma}) = l_\sigma(a_1^{\sigma}+a_2^{\sigma})$.

Case 2 ($m > 2$). It is easy to see that

$$\begin{aligned}\left(\sum_{i=1}^{m}a_i\right)^{\sigma} &= \left(a_1+\sum_{i=2}^{m}a_i\right)^{\sigma} \le l_\sigma\left(a_1^\sigma+\left(\sum_{i=2}^{m}a_i\right)^{\sigma}\right) \le l_\sigma\left[a_1^\sigma + l_\sigma\left(a_2^\sigma+\left(\sum_{i=3}^{m}a_i\right)^{\sigma}\right)\right]\\ &\le l_\sigma^2\left(a_1^\sigma+a_2^\sigma+\left(\sum_{i=3}^{m}a_i\right)^{\sigma}\right) \le \cdots \le l_\sigma^{m-1}\left(\sum_{i=1}^{m}a_i^\sigma\right).\end{aligned} \tag{2.9}$$

The proof is complete.

Theorem 2.3. *Suppose that*

(A) f, B_0, B_1 *are continuous, and* $xB_i(x) \ge 0$ *for all* $x \in R$ *and* $i = 0, 1$;

(B) *there exist numbers* $\beta > 0$, $\theta > 1$, *nonnegative sequences* $p_i(n), r(n)$ $(i = 0, \ldots, m)$, *functions* $g(n, x_0, \ldots, x_m)$, $h(n, x_0, \ldots, x_m)$ *such that*

$$\begin{aligned} f(n,x_0,\ldots,x_m) &= g(n,x_0,\ldots,x_m)+h(n,x_0,\ldots,x_m),\\ g(n,x_0,x_1,\ldots,x_m)x_0 &\ge \beta|x_0|^{\theta+1},\\ |h(n,x_0,\ldots,x_m)| &\le \sum_{i=0}^{m}p_i(n)|x_i|^{\theta}+r(n),\end{aligned} \tag{2.10}$$

for all $n \in \{1, \ldots, T\}$, $(x_0, x_1, \ldots, x_m) \in R^{m+1}$.

Then BVP (1.7)-(1.8) *has at least one solution if*

$$\|p_0\| + T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\| < \beta. \tag{2.11}$$

Proof. To apply Lemma 2.1, we divide the proof into two steps.

Let $\Omega_1 = \{(x, y) : L(x, y) = \lambda N(x, y), ((x, y), \lambda) \in [\operatorname{Dom} L] \times [0, 1]\}$. For $(x, y) \in \Omega_1$, we have $L(x, y) = \lambda N(x, y)$, $\lambda \in [0, 1]$, then

$$\begin{aligned} \Delta x(n) &= \lambda\phi^{-1}(y(n)), \quad n \in [0,T],\\ \Delta y(n) &= \lambda f(n, x(n+1), x(n-\tau_1(n)), \ldots, x(n-\tau_m(n))), \quad n \in [0, T-1],\\ x(0) &= \lambda B_0\left(\phi^{-1}(y(0))\right),\\ y(T) &= -\lambda\phi(B_1(x(T+1))),\\ x(i) &= \lambda\gamma(i), \quad i \in [-\tau, -1],\\ x(j) &= \lambda\psi(j), \quad j \in [T+2, T+\delta].\end{aligned} \tag{2.12}$$

Step 1. We will show that if $L(x,y)=\lambda N(x,y)$, for some $\lambda\in[0,1]$, then x is bounded. Indeed, we see that

$$\Delta\left[\phi(\Delta x(n))\right]x(n+1)=\lambda\phi(\lambda)f(n,x(n+1),x(n-\tau_1(n)),\dots,x(n-\tau_m(n)))x(n+1). \quad (2.13)$$

Since

$$x(0)=\lambda B_0\left(\phi^{-1}(y(0))\right),\qquad y(T)=-\lambda\phi(B_1(x(T+1))), \quad (2.14)$$

we get

$$x(0)=\lambda B_0\left(\frac{\Delta x(0)}{\lambda}\right),\qquad \frac{\Delta x(T)}{\lambda}=-\phi^{-1}(\lambda)B_1(x(T+1)). \quad (2.15)$$

It is easy to see from (2.12) and the definition of B_0, B_1 and ϕ that

$$\begin{aligned}
&\sum_{n=0}^{T-1}\Delta\left[\phi(\Delta x(n))\right]x(n+1)\\
&\quad=\sum_{n=0}^{T-1}\left[\phi(\Delta x(n+1))-\phi(\Delta x(n))\right]\left[x(n+2)-\Delta x(n+1)\right]\\
&\quad=\sum_{n=0}^{T-1}\left[\phi(\Delta x(n+1))x(n+2)-\phi(\Delta x(n))x(n+1)\right]-\sum_{n=0}^{T-1}\phi(\Delta x(n+1))\Delta x(n+1)\\
&\quad=\phi(\Delta x(T))x(T+1)-\phi(\Delta x(0))x(1)-\sum_{n=0}^{T-1}\phi(\Delta x(n+1))\Delta x(n+1)\\
&\quad=-\phi\left(\lambda\phi^{-1}(\lambda)B_1(x(T+1))\right)x(T+1)-\phi(\Delta x(0))\Delta x(0)-\phi(\Delta x(0))x(0)\\
&\qquad-\sum_{n=0}^{T-1}\phi(\Delta x(n+1))\Delta x(n+1)\\
&\quad=-\phi\left(\lambda\phi^{-1}(\lambda)B_1(x(T+1))\right)x(T+1)-\phi(\Delta x(0))\lambda B_0\left(\frac{\Delta x(0)}{\lambda}\right)\\
&\qquad-\sum_{n=0}^{T-1}\phi(\Delta x(n+1))\Delta x(n+1)-\phi(\Delta x(0))\Delta x(0)\\
&\quad\le 0.
\end{aligned} \quad (2.16)$$

So we get

$$\sum_{n=0}^{T-1}f(n,x(n+1),x(n-\tau_1(n)),\dots,x(n-\tau_m(n)))x(n+1)\le 0. \quad (2.17)$$

It follows from the assumptions that

$$\begin{aligned}
&\beta\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\\
&\quad\le\sum_{n=0}^{T-1}g(n,x(n+1),x(n-\tau_1(n)),\dots,x(n-\tau_m(n))x(n+1)\\
&\quad\le-\sum_{n=0}^{T-1}h(n,x(n+1),x(n-\tau_1(n)),\dots,x(n-\tau_m(n))x(n+1)\\
&\quad\le\sum_{n=0}^{T-1}|h(n,x(n+1),x(n-\tau_1(n)),\dots,x(n-\tau_m(n))||x(n+1)|\\
&\quad\le\sum_{n=0}^{T-1}p_0(n)|x(n+1)|^{\theta+1}+\sum_{i=1}^{m}\sum_{n=0}^{T-1}p_i(n)|x(n-\tau_i(n))|^{\theta}|x(n+1)|+\sum_{n=0}^{T-1}r(n)|x(n+1)|\\
&\quad\le\|p_0\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}+\sum_{i=1}^{m}\|p_i\|\sum_{n=0}^{T-1}|x(n-\tau_i(n))|^{\theta}|x(n+1)|+\|r\|\sum_{n=0}^{T-1}|x(n+1)|.
\end{aligned}\tag{2.18}$$

For $x_i\ge 0, y_i\ge 0$, we have Holder's inequality

$$\sum_{i=1}^{s}x_iy_i\le\left(\sum_{i=1}^{s}x_i^p\right)^{1/p}\left(\sum_{i=1}^{s}y_i^q\right)^{1/q},\quad \frac{1}{p}+\frac{1}{q}=1,\ q>0,\ p>0.\tag{2.19}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
&\beta\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\\
&\quad\le\|p_0\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}+\|r\|T^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\qquad+\sum_{i=1}^{m}\|p_i\|\left(\sum_{n=0}^{T-1}|x(n-\tau_i(n))|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\quad=\|p_0\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}+\|r\|T^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}\\
&\qquad+\sum_{i=1}^{m}\|p_i\|\left(\sum_{u\in\{n-\tau_i(n)-1:n=0,\dots,T-1\}}|x(u+1)|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}
\end{aligned}$$

$$
\begin{aligned}
&\le \|p_0\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1} + \|r\|T^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)} \\
&\quad + \left[T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{\theta/(\theta+1)}\right. \\
&\qquad + T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\left(\sum_{n=T}^{T+\delta}\left|\lambda\varphi(n+1)\right|^{\theta+1}\right)^{\theta/(\theta+1)} \\
&\qquad \left. +T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\left(\sum_{n=-\tau}^{-1}\left|\lambda\gamma(n+1)\right|^{\theta+1}\right)^{\theta/(\theta+1)}\right]\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)} \\
&\le \|p_0\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1} + \|r\|T^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)} \\
&\quad + T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1} \\
&\quad + T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\left(\sum_{n=T}^{T+\delta}\left|\varphi(n+1)\right|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)} \\
&\quad + T^{\theta/(\theta+1)}\sum_{i=1}^{m}\|p_i\|\left(\sum_{n=-\tau}^{-1}\left|\gamma(n+1)\right|^{\theta+1}\right)^{\theta/(\theta+1)}\left(\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right)^{1/(\theta+1)}.
\end{aligned}
\tag{2.20}
$$

It follows from (2.11) that there is $M_1 > 0$ such that $\sum_{u=0}^{T-1}|x(u+1)|^{\theta+1} \le M_1$.

Hence $|x(n+1)| \le M_1^{1/(\theta+1)}$ for all $n \in \{0,\dots,T-1\}$, which proves Step 1.

Step 2. We will show that the set Ω_1 is bounded. We first prove that there exists $k \in [0, T-1]$ such that $y(k)y(k+1) \le 0$. In fact, if $y(i) > 0$ for all $i \in [0,T]$, then

$$\Delta x(n) = \lambda\phi^{-1}(y(n)), \quad n \in [0,T] \tag{2.21}$$

implies that $x(i)$ is increasing on $[0,T+1]$, so $x(0) < x(T+1)$. Then assumption (A) and

$$x(0) = \lambda B_0\left(\phi^{-1}(y(0))\right), \qquad y(T) = -\lambda\phi(B_1(x(T+1))) \tag{2.22}$$

imply that $x(0) > 0$ and $x(T+1) < 0$. This contradicts $x(0) < x(T+1)$. If $y(i) < 0$ for all $i \in [0,T]$, the similar contradiction can be deduced. Hence there exists $k \in [0, T-1]$ such that $y(k)y(k+1) \le 0$.

It follows that there is a constant $\xi \in [k, k+1]$ such that

$$\frac{y(k-1)-y(k)}{k+1-k} = \frac{0-y(k)}{\xi-k}. \tag{2.23}$$

Then $|y(k)| \le |\Delta y(k)|$. Hence

$$\begin{aligned} \left|y(k)\right| &\le \left|\Delta y(k)\right| = \lambda\left|f(k, x(k+1), x(k-\tau_1(k)), \ldots, x(k-\tau_m(k)))\right| \\ &\le \max_{n\in[0,T-1],|x_i|\le M_1^{1/(\theta+1)},i=0,\ldots,m} \left|f(n, x_0, \ldots, x_m)\right| =: M_2. \end{aligned} \tag{2.24}$$

Then for $n \in [k+1, T]$ one sees that

$$\begin{aligned} \left|y(n)\right| &= \left|y(k) + \sum_{s=k}^{n-1}\Delta y(s)\right| \le M_2 + \sum_{s=k}^{n-1}\left|\Delta y(s)\right| \\ &\le M_2 + T \max_{n\in[0,T-1],|x_i|\le M_1^{1/(\theta+1)},i=0,\ldots,m} \left|f(n, x_0, \ldots, x_m)\right| = (T+1)M_2. \end{aligned} \tag{2.25}$$

For $n \in [0, k-1]$, we get

$$\begin{aligned} \left|y(n)\right| &= \left|y(k) - \sum_{s=n}^{k-1}\Delta y(s)\right| \le M_2 + \sum_{s=n}^{k-1}\left|\Delta y(s)\right| \\ &\le M_2 + T \max_{n\in[0,T-1],|x_i|\le M_1^{1/(\theta+1)},i=0,\ldots,m} \left|f(n, x_0, \ldots, x_m)\right| = (T+1)M_2. \end{aligned} \tag{2.26}$$

It follows that

$$\|y\| \le (T+1)M_2. \tag{2.27}$$

Then

$$|x(0)| = \left|\lambda B_0\left(\phi^{-1}(y(0))\right)\right| \le \left|B_0\left(\phi^{-1}(y(0))\right)\right| \le \max_{|x|\le(T+1)M_2}\left|B_0\left(\phi^{-1}(x)\right)\right|. \tag{2.28}$$

On the other hand, since $\Delta x(T) = \lambda\phi^{-1}(y(T))$ implies that

$$|\Delta x(T)| \le \phi^{-1}(\left|y(T)\right|) \le \phi^{-1}((T+1)M_2), \tag{2.29}$$

then

$$|x(T+1)| = |\Delta x(T) + x(T)| \le |\Delta x(T)| + |x(T)| \le \phi^{-1}((T+1)M_2) + M_1^{1/(\theta+1)}. \tag{2.30}$$

It follows that

$$\|x\| \leq \max\left\{\max_{i\in[T+2,T+\delta]}|\psi(i)|, \max_{i\in[-\tau,-1]}|\gamma(i)|, \phi^{-1}((T+1)M_2) + M_1^{1/(\theta+1)}\right\}. \tag{2.31}$$

Hence

$$\|(x,y)\| \leq \max\left\{(T+1)M_2, \max_{i\in[T+2,T+\delta]}|\psi(i)|, \max_{i\in[-\tau,-1]}|\gamma(i)|, \phi^{-1}((T+1)M_2) + M_1^{1/(\theta+1)}\right\}. \tag{2.32}$$

Step 2 is proved, namely Ω_1 is bounded. Let $\Omega \supset \overline{\Omega_1}$ be an open bounded subset of X centered at zero, it is easy to see that $L(x,y) \neq \lambda N(x,y)$ for all $(x,y) \in \partial\Omega$ and $\lambda \in (0,1)$. It follows from Lemma 2.1 that equation $L(x,y) = N(x,y)$ has at least one solution (x,y), then x is a solution of BVP (1.7)-(1.8). The proof is complete. □

3. An Example

In this section, we present an example to illustrate the main results in Section 2.

Example 3.1. Consider the following BVP:

$$\begin{gathered}
\Delta(\phi(\Delta x(n))) = \beta[x(n+1)]^{2k+1} + \sum_{i=1}^{m} p_i(n)[x(n-i)]^{2k+1} + r(n), \quad n \in [0, T-1], \\
x(0) - 5[\Delta x(0)]^5 = 0, \\
6[x(T+1)]^3 + \Delta x(T) = 0, \\
x(n) = \gamma(n), \quad n \in [-m, -1],
\end{gathered} \tag{3.1}$$

where $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$ with $p > 1$ a constant, $T \geq 1$ is an integer, $\beta > 0$, $p_i(n), r(n)$ are sequences. Corresponding to the assumptions of Theorem 2.3, we set

$$\begin{gathered}
g(n, x_0, \ldots, x_m) = \beta[x_0]^{2k+1}, \\
h(n, x_0, \ldots, x_m) = \sum_{i=1}^{m} p_i(n) x_i^{2k+1} + r(n), \qquad \theta = 2k+1, \\
B_0(x) = 5x^5, \qquad B_1(x) = 6x^3.
\end{gathered} \tag{3.2}$$

It is easy to see that assumptions (A) and (B) in Theorem 2.3 hold. It follows from Theorem 2.3 that (3.1) has at least one solution if $\|p_0\| + T^{\theta/(\theta+1)} \sum_{i=1}^{m} \|p_i\| < \beta$.

Remark 3.2. The BVP in Example 3.1 cannot be solved by the theorems in [3–5] since the difference equation in (3.1) is a p-Laplacian equation and the boundary conditions superlinear.

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