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Research Article

Dynamic Properties of Coupled Maps

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Dynamic properties are investigated in the coupled system of three maps with symmetric nearest neighbor coupling and periodic boundary conditions. The dynamics of the system is controlled by certain coupling parameters. We show that, for some values of the parameters, the system exhibits nontrivial collective behavior, such as multiple bifurcations, and chaos. We give computer simulations to support the theoretical predictions.

1. Introduction

Coupled maps is one of the most interesting topics on spatial extended systems. Bifurcation and chaos of coupled maps generated by iterated maps of nonlinear difference equations have attracted considerable attention in both theoretical studies and practical applications [1]. When many identical maps are coupled to a larger system (coupled map lattices or CML), the system can exhibit nontrivial collective behavior, such as stability, multiple bifurcations, and chaos [2–7]. In this paper, we extend this work by considering three-dimensional discrete dynamical systems which arise from three coupled one-dimensional maps with delay. In general, delayed coupled maps of three maps with symmetric nearest neighbor coupling and periodic boundary conditions can be described as

$$\begin{aligned} x_{n+1} &= ax_n + \alpha f(x_n) + \beta[x_{n-k} - y_{n-k}] + \beta[x_{n-k} - z_{n-k}], \\ y_{n+1} &= ay_n + \alpha f(y_n) + \beta[y_{n-k} - z_{n-k}] + \beta[y_{n-k} - x_{n-k}], \\ z_{n+1} &= az_n + \alpha f(z_n) + \beta[z_{n-k} - x_{n-k}] + \beta[z_{n-k} - y_{n-k}], \end{aligned} \tag{1.1}$$

where $\alpha, \beta,$ and a are parameters.

Symmetry breaking Hopf bifurcations from steady states to periodic orbits can occur in systems with some symmetry. The coexistence of more chaotic attractors in phase space is phenomenon that has been proven to exist in many fields of science. There are many researches of equivariant bifurcations of ODEs. We refer the readers to the study by Golubitsky et al. in [8]. But, until now, there are fewer papers to discuss equivariant bifurcation problems in maps, which motivates us to write this paper. The goal of this paper is to investigate how parameters affect coupled maps with delay (1.1) by using the symmetric groups theory of Chossat and Golubitsky [9].

Accordingly, the paper is organized as follows. In Sections 2 and 3, we show that the structure of system (1.1) can be represented by a dihedral group D_3. The generalized center subspace is invariant under the action of the symmetry group, and the center manifold reduction can be performed in such a way that the reduced equations commute with the restricted action of the symmetry group. We obtain some important results about the stability and chaos and spontaneous bifurcations of multiple branches of periodic solutions and their spatiotemporal patterns, which describe the oscillatory mode of each oscillator. Finally, some numerical simulations are carried out to support the analysis results, and the existence of chaotic attractors is also exhibited numerically.

2. D_3-Equivariant and Linear Stability of Coupled Maps

Assume that $X_n = (x_n, y_n, z_n)^T$. Equation (1.1) can be rewritten as

$$X_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} X_n + \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} f((X_n)_1) \\ f((X_n)_2) \\ f((X_n)_3) \end{pmatrix} + \begin{pmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{pmatrix} \begin{pmatrix} (X_{n-k})_1 \\ (X_{n-k})_2 \\ (X_{n-k})_3 \end{pmatrix}. \tag{2.1}$$

Throughout this section, to establish the main results for (1.1), we make the following hypothesis on the activation functions in (2.1):

(H_1) : $f : R \to R$ is a C^1-smooth function with $f(0) = 0$, $f'(0) = 1$.

Under the assumptions (H_1), the origin (0,0,0) is an equilibrium of (1.1). Linearizing system (2.1) about origin (0,0,0) gives the following linear system:

$$Y_{n+1} = \begin{pmatrix} a+\alpha & 0 & 0 \\ 0 & a+\alpha & 0 \\ 0 & 0 & a+\alpha \end{pmatrix} Y_n + \begin{pmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{pmatrix} Y_{n-k}. \tag{2.2}$$

Lemma 2.1. *Both systems* (2.1) *and* (2.2) *are D_3-equivariant where D_3 is the dihedral group of order* 6.

Proof. D_3 can be generated by matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.3}$$

Hence,

$$D_3 = \left\{ I_3, P, P^2, Q, QP, QP^2 \right\}. \tag{2.4}$$

For all $G \in D_3$, we have

$$\begin{aligned}
&G\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} X_n + \begin{pmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{pmatrix} X_{n-k} + \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} f((X_n)_1) \\ f((X_n)_2) \\ f((X_n)_3) \end{pmatrix} \right) \\
&\quad = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} GX_n + \begin{pmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{pmatrix} GX_{n-k} + \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} f((GX_n)_1) \\ f((GX_n)_2) \\ f((GX_n)_3) \end{pmatrix}, \\
&G\left(\begin{pmatrix} a+\alpha & 0 & 0 \\ 0 & a+\alpha & 0 \\ 0 & 0 & a+\alpha \end{pmatrix} Y_n + \begin{pmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{pmatrix} Y_{n-k} \right) \\
&\quad = \begin{pmatrix} a+\alpha & 0 & 0 \\ 0 & a+\alpha & 0 \\ 0 & 0 & a+\alpha \end{pmatrix} GY_n + \begin{pmatrix} 2\beta & -\beta & -\beta \\ -\beta & 2\beta & -\beta \\ -\beta & -\beta & 2\beta \end{pmatrix} GY_{n-k}.
\end{aligned} \tag{2.5}$$

For discussing the linear stability and Hopf bifurcations of (2.1), we need to consider the characteristic equation for (2.2). We can induce the method of Zhang and Zheng [10] to mark $M_{n+1} = AM_n$, where

$$\begin{gathered}
M_n = \left(x_n, x_{n-1}, x_{n-2}, \dots, x_{n-k}, y_n, y_{n-1}, y_{n-2}, \dots, y_{n-k}, z_n, z_{n-1}, z_{n-2}, \dots, z_{n-k}\right)^T, \\
A = \begin{pmatrix} R_1 & R_2 & R_2 \\ R_2 & R_1 & R_2 \\ R_2 & R_2 & R_1 \end{pmatrix}_{(3k+3)\times(3k+3)}, \\
R_1 = \begin{pmatrix} a+\alpha & 0 & \cdots & \cdots & 0 & 2\beta \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{(k+1)\times(k+1)}, \\
R_2 = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & -\beta \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{(k+1)\times(k+1)}.
\end{gathered} \tag{2.6}$$

The characteristic equation for (2.2) is given by

$$|\lambda E - A| = \left[\lambda^{k+1} - (a+\alpha)\lambda^k\right]\left[\lambda^{k+1} - (a+\alpha)\lambda^k - 3\beta\right]^2 = \Delta_1\Delta_2^2 = 0, \tag{2.7}$$

where

$$\begin{aligned} \Delta_1 &= \lambda^{k+1} - (a+\alpha)\lambda^k, \\ \Delta_2 &= \lambda^{k+1} - (a+\alpha)\lambda^k - 3\beta. \end{aligned} \tag{2.8}$$

The equation $\Delta_1 = 0$ has a k-fold root $\lambda = 0$ and a simple root $\lambda = a + \alpha$. Assuming that $a + \alpha < 1$, we only need to analyze the distribution of the roots of $\Delta_2 = 0$.

In what follows, the analysis on the distribution of the roots to (2.7) is based on the conclusion given in [10]: the sum of the order of the zeros of (2.7) can change only if a zero appears or accesses the unit circle as parameters $a, \alpha,$ and β are varied.

□

Theorem 2.2. *Let $a + \alpha < 1$.*

(1) *The zero solution of* (2.1) *is local asymptotically stable if $(a,\alpha) \in E = \{(a,\alpha) \mid [(a+\alpha-1)^2 - 9\beta^2] \geq 2|a+\alpha|, a+\alpha < 1, a, \alpha \in R\}$.*

(2) *Equation* (2.1) *has equivariant bifurcation at the origin:*

  (1°) *if $\beta_j = [1+(a+\alpha)^2 - 2(a+\alpha)\cos\omega]^{1/2}/3$ $(j = 0, 1, \ldots, k-1)$, then D_3-equivariant Hopf bifurcation occurs;*

  (2°) *if $a + \alpha + 3\beta = 1$, then the D_3-equivariant pitchfork bifurcation occurs;*

  (3°) *if $1 + a + \alpha = (-1)^{k+1}3\beta$, then the D_3-equivariant doubling bifurcation occurs.*

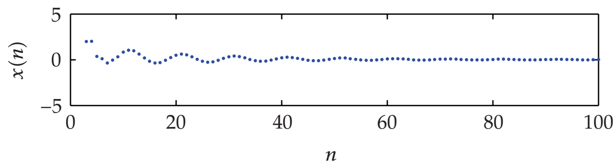
Proof. Consider $\Delta_2 = 0$.

(1) It is clears that $\Delta_2 = 0$ has roots $\lambda = 1$ if $a+\alpha+3\beta = 1$ and $\lambda = -1$ if $1+a+\alpha = (-1)^{k+1}3\beta$.

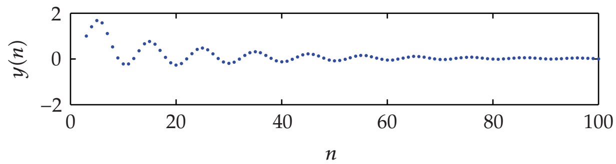
(2) It is observed that, if $\beta = 0$, then $\Delta_2 = 0$ has simple root $\lambda_1 = a + \alpha$ with $|\lambda_1| < 1$ and $\lambda_2 = 0$ with multiplies k.

(3) Let $\lambda = e^{i\omega}$ be a root of $\Delta_2 = 0$, then we have

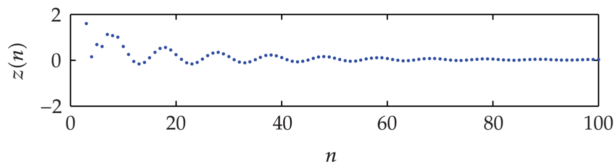
$$\begin{gathered} \beta_j = \frac{\left[1+(a+\alpha)^2 - 2(a+\alpha)\cos\omega\right]^{1/2}}{3} \quad (j = 0, 1, \ldots, k-1), \\ \sin(k+1)\omega - e^{-a}\sin k\omega = 0, \\ \omega \in \left(\frac{j\pi}{k}, \frac{(j+1)\pi}{k+1}\right) \quad \text{for } j = 0, 1, \ldots, k-1. \end{gathered} \tag{2.9}$$



(a)



(b)



(c)

Figure 1: The origin is local asymptotically stable for parameters $a = 0.5,\ \alpha = 0.48,$ and $\beta = -0.18$.

(4) One, moreover, has

$$\left.\frac{d\lambda}{d\beta}\right|_{\beta=\beta_j} = \frac{2(k+1)\sin\omega - 2\sin(k+1)\omega\cos\omega}{\sin k\omega\left|(k+1)e^{i\omega k} - (a+\alpha)ke^{i\omega(k+1)}\right|^2} > 0. \tag{2.10}$$

Using the theorem of [10], the conclusions are obtained. □

3. Multiple Bifurcations

Recently, there has been an increasing interest in the multistability problem in dynamical systems theory. The purpose of this section is to explore the coexistence of multiple stable patterns such as multiple periodic orbits of maps (2.1).

Let $\beta_j = ([1+(a+\alpha)^2 - 2(a+\alpha)\cos\omega]^{1/2}/3)$ $(j = 0, 1, \ldots, k-1)$. We will consider the periodic solutions of (2.1).

Assume that

$$g : R^{2n} \times R \longrightarrow R^{2n} \tag{3.1}$$

and that $(Dg)_{(0,0)}$ has eigenvalues $e^{\pm 2\pi i\theta}$, each with multiplicity n, where $\theta \neq 0, 1/2$.

Denote by SP_T the subspace of P_T consisting of all T-periodic solution of (2.1) under condition (2.4). Let Σ be a subgroup and let (Σ, SP_T) be fixed-point subspace of SP_T.

Lemma 3.1 (see [8]). *Let Σ be a subgroup such that* $\dim\mathrm{Fix}(\Sigma, SP_T) = 2$. *Assume that the eigenvalues cross the unit circle with nonzero speed. Then generically there exists a unique branch*

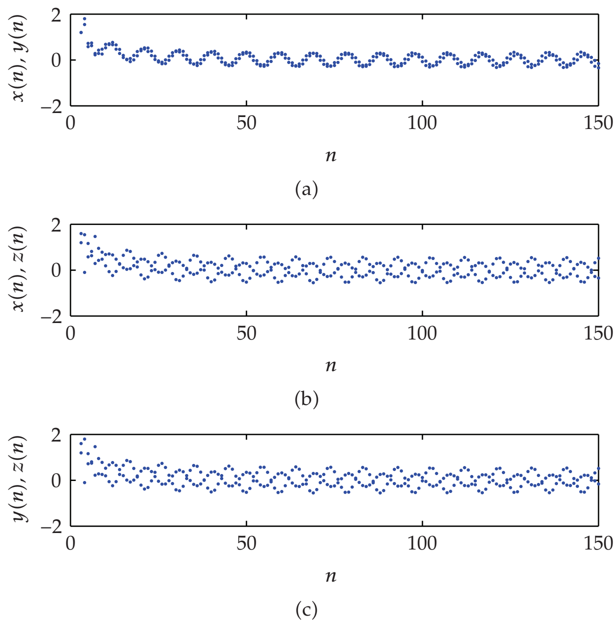


Figure 2: Completely symmetrical periodic solution of (x_n, y_n); antisymmetrical periodic solutions (x_n, z_n), (y_n, z_n).

of g-invariant circles emanating from the trivial fixed point $x = 0$, and this branch is tangent to $\mathrm{Fix}(\Sigma) \subset R^{2n} \times R$ *at $x = 0$.*

Consider the subgroup of D_3:

$$\begin{gathered}
\Sigma_1 = \left\{ I_9, P, P^2 \right\}, \ \Sigma_1 = (P, \theta(t)) \left\{ P, \theta(\omega t) = \omega t - \frac{2\pi}{3} \right\}, \\
\Sigma_2 = \{ I_9, Q \}, \ \Sigma_2 = (Q, \theta(t)) \{ Q, \theta(\omega t) = \omega t \}, \\
\Sigma_3 = \{ I_9, QP \}, \ \Sigma_3 = (QP, \theta(t)) \{ QP, \theta(\omega t) = \omega t - \pi \}.
\end{gathered} \tag{3.2}$$

It is clear that

$$\begin{aligned}
&(1) \quad \dim\mathrm{Fix}(\Sigma_1 x, SP_T) = 2, \\
&(2) \quad \dim\mathrm{Fix}(\Sigma_2 x, SP_T) = 2, \\
&(3) \quad \dim\mathrm{Fix}(\Sigma_3 x, SP_T) = 2.
\end{aligned} \tag{3.3}$$

System (2.1) is equivariant with respect to the D_3-action where the subgroups Σ_1, Σ_2, and Σ_3 of D_3 act by permutation (sending x_i to x_{i+1}) and by interchanging (sending x_i to x_{3-i}).

Applying Lemma 3.1 of the symmetric Hopf bifurcation theorem for maps, eight branches of asynchronous periodic solutions are obtained due to their corresponding two-dimensional fixed-point subspace. More precisely, we have the following theorem.

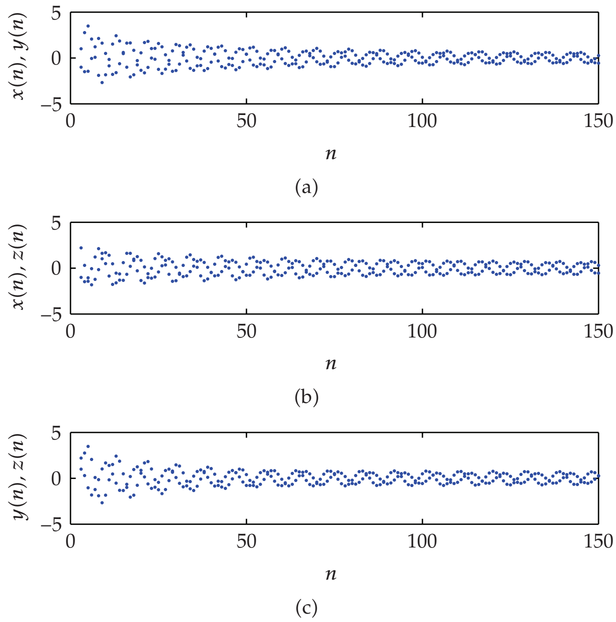


Figure 3: Three waveforms have $2\pi/3\omega$ out of phase.

Theorem 3.2. *Assume that β_j is defined as in* (2.9)*. When near $\beta = \beta_j$, there exist eight branches of asynchronous periodic solutions of period P_T near $2\pi/\omega$, bifurcated simultaneously from the zero solution of system* (2.1)*, and these are*

(1) *two phase-locked oscillations, $x_i(n) = x_{i-1}(n \pm P_T/3) = x_{i-1}(n \pm 2\pi/3\omega)$;*

(2) *three mirror-reflecting waves, $x_i(n) = x_j(n) \neq x_k(n)$;*

(3) *three standing, $x_j(n) = x_i(n + P_T/2) = x_i(n + \pi/\omega)$.*

To illustrate analytical results found, let us consider the following example:

$$\begin{aligned} x_{n+1} &= ax_n + \alpha\tanh(x_n) + \beta(x_{n-2} - y_{n-2}) + \beta(x_{n-2} - z_{n-2}), \\ y_{n+1} &= ay_n + \alpha\tanh(y_n) + \beta(y_{n-2} - z_{n-2}) + \beta(y_{n-2} - x_{n-2}), \\ z_{n+1} &= az_n + \alpha\tanh(z_n) + \beta(z_{n-2} - x_{n-2}) + \beta(z_{n-2} - y_{n-2}). \end{aligned} \tag{3.4}$$

Firstly, we choose parameters such that (2.9) met. When $a = 0.5$, and $\alpha = 0.48$, $\beta = -0.18$, we have the result that the origin of (3.4) is local asymptotically stable. See Figure 1.

Next, one possible choice of the parameters is $a = 0.5$, $\alpha = 0.48$, and $\beta = -0.21$. In this case, multiple branches of asynchronous periodic solutions appear. These solutions are shown in Figures 2-3.

It is shown that, in Figures 1, 2, and 3 for different values of parameters, system (3.4) exhibits its different dynamics. At first, the trivial solution is local asymptotically stable, then loses its stability, and several different periodic patterns described by the equivariant Hopf bifurcation can be observed, which depend on different values of parameters.

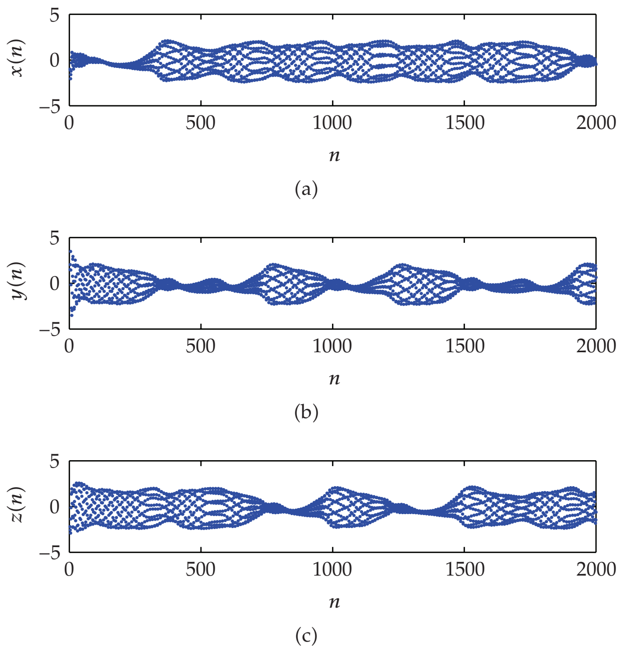


Figure 4: Time series for $(n, x_n); (n, y_n); (n, z_n)$.

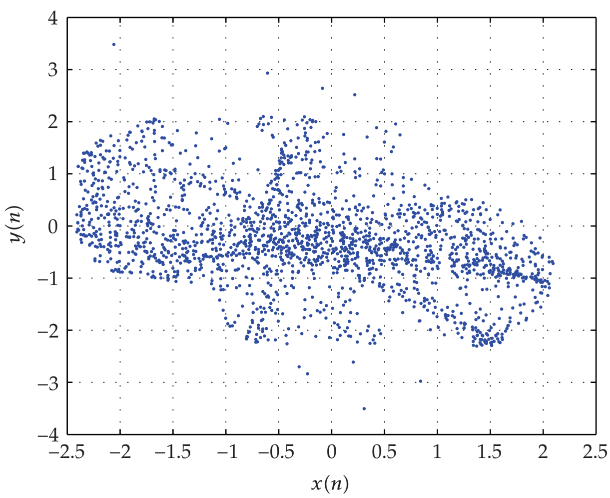


Figure 5: Phase portrait of the chaotic attractor (x_n, y_n).

4. Chaos and Lyapunov Exponents

The presence of chaos in discrete dynamic systems is well known. In this section, the numerical method is used to indicate the chaotic behavior of system (3.4). Analytic approach is usually limited to some simple cases in studying nonlinear systems, and numerical simulation by computer is more efficient sometimes. Therefore, we study the chaos of (3.4) by numerical methods.

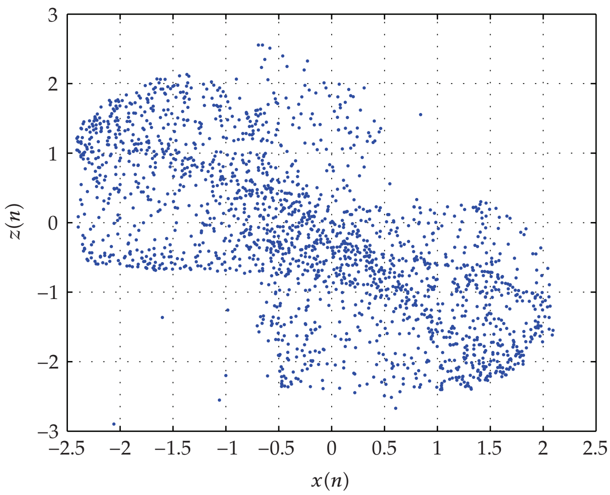


Figure 6: Phase portrait of the chaotic attractor (x_n, z_n).

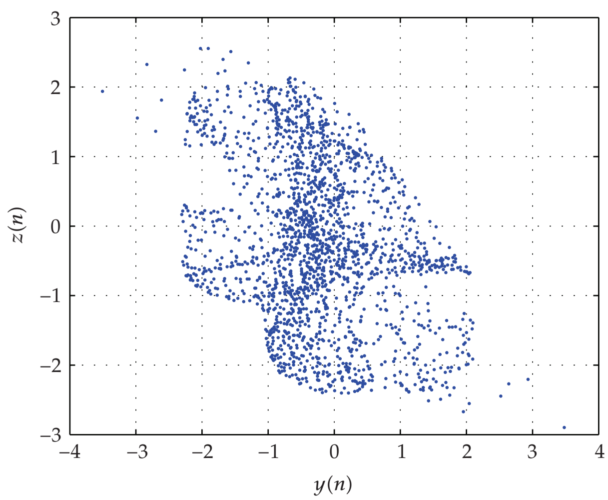


Figure 7: Phase portrait of the chaotic attractor (y_n, z_n).

We notice that, when $a = 0.6$, $\alpha = 0.66$, and $\beta = -0.21$, system (3.4) has coexistence of chaotic attractors. See Figures 4, 5, 6, 7.

Lyapunov exponents are a quantitative measure for distinguishing among the various types of orbits based upon their sensitive dependence on the initial conditions and are used to determine the stability of any steady-state behavior, including chaotic solutions.

The Lyapunov exponents for $a = 0.6$, $\alpha = 0.66$, and $\beta = -0.21$ are calculated by means of Matlab and are illustrated in Figure 8.

5. Conclusions

In this paper we have shown a rich dynamics in a new class coupled maps described by (1.1). In particular, we have demonstrated that system (1.1) with very simple connection

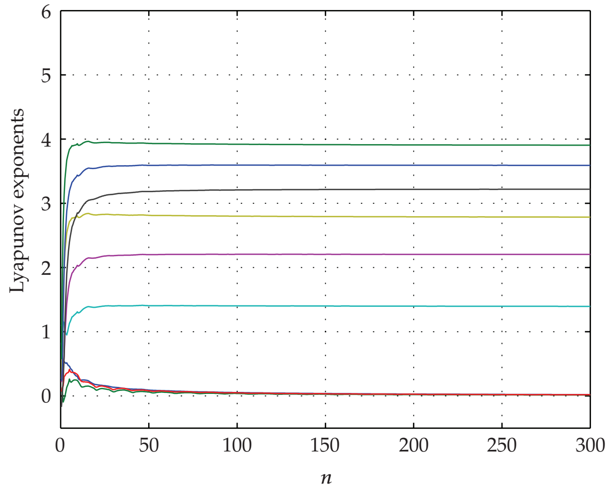


Figure 8: Lyapunov exponents of system (3.4) when $a = 0.6$, $\alpha = 0.66$, $\beta = -0.21$.

matrices can exhibit stability-to Multiple bifurcations-to-chaos when a parameter varies. These properties are similar to those in symmetric delayed differential equations. It is expected that finding more such simple coupled maps would be helpful for studying the role of symmetry in discrete system.

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