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Research Article

Periodic Solutions to a Third-Order Conditional Difference Equation over the Integers

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This paper studies a third-order conditional difference equation which is a generalization from the literature. We investigate this equation by transforming it into a first-order system. Finally it is proved that the equation has no period-two (or three) integer solutions. Besides, its all period-four (and five) integer solutions are derived under appropriate rational parameters.

1. Introduction

As mentioned in [1], difference equations appear naturally as a discrete analogue and as a numerical solution of differential and delay differential equations having applications in various scientific branches, such as biology, ecology, physics, economy, technics, and sociology. The stability, asymptotic behavior, and periodic property of solutions to difference equations had been widely investigated, such as [2–14]. Recently, the study of max-type difference equation attracted a considerable attention, for example, [7, 11, 15–25]. Although max-type difference equations are relatively simple in form, it is unfortunately extremely difficult to understand thoroughly the behavior of their solutions. The max operator arises naturally in certain models in automatic control theory. On the other hand, there exists another kind of difference equations called *conditional difference equations*, which also have simple forms, but it is difficult to understand clearly the behavior of their solutions.

From [2, 5], we know that the following conditional difference equation

$$a_n = \begin{cases} \dfrac{a_{n-1} + a_{n-2}}{2}, & \text{if } 2 \mid (a_{n-1} + a_{n-2}), \\ a_{n-1} + a_{n-2}, & \text{otherwise,} \end{cases} \tag{1.1}$$

with positive initial integers $a_0,\ a_1$, has the property that each positive integer solution (a_n) is either stationary or unbounded.

Later, Ladas [26] gave the conjecture (also mentioned in [5]) that all solutions to the conditional difference equation

$$a_n = \begin{cases} \dfrac{a_{n-1} + a_{n-2}}{3}, & \text{if } 3 \mid (a_{n-1} + a_{n-2}), \\ a_{n-1} + a_{n-2}, & \text{otherwise}, \end{cases} \tag{1.2}$$

are unbounded except for certain obvious periodic solutions, such as the solutions $1, 1, 2, 1, 1, 2, \ldots$ or $7, 14, 7, 7, 14, 7, \ldots$. For such kind of conditional difference equations, any solution which is not eventually periodic must be unbounded; hence, the only problem is to classify the periodic solutions. However, this problem seems extremely difficult.

Equations such as (1.1) and (1.2) have many other generalizations, see [6, 26]. For instance, Clark [3] studied periodic solutions to $a_n = \lceil c a_{n-1} \rceil - a_{n-2}$ for various real c.

Greene and Niedzielski [5] considered a generalization of (1.1) and (1.2) and studied the following conditional difference equation:

$$a_n = \begin{cases} r(a_{n-1} + a_{n-2}), & \text{if } r(a_{n-1} + a_{n-2}) \in \mathbb{Z}, \\ a_{n-1} + a_{n-2}, & \text{otherwise}, \end{cases} \tag{1.3}$$

where r is some fixed rational number. However, they pointed out that it was also very hard to characterize the periodic solutions to (1.3). Hence, they addressed a different, easier question, and then did some research.

Motivated by the above works, in this paper, we study the following conditional third-order difference equation, which is another generalization of (1.1), (1.2), and (1.3):

$$y_n = \begin{cases} r(y_{n-1} + y_{n-2} + y_{n-3}), & \text{if } r(y_{n-1} + y_{n-2} + y_{n-3}) \in \mathbb{Z}, \\ y_{n-1} + y_{n-2} + y_{n-3}, & \text{otherwise}, \end{cases} \tag{1.4}$$

where r is some appropriate rational number. We study this equation by transforming it into a first-order system. It is eventually proved that the equation has no period-two (or three) positive integer solutions. Besides, its all period-four (and five) positive integer solutions are derived under appropriate rational parameters. The main results are presented in Section 4.

2. Auxiliary Results

For the convenience of investigation, higher-order difference equations are usually converted into first- or lower-order difference equation systems. As we all know, second-order difference equations can be transformed into first-order difference equation system. First, we define a mapping $\mathcal{F} : \mathbb{Z}^3 \to \mathbb{Z}^3$ such that

$$\mathcal{F}(\mathbf{x}) = \begin{cases} \mathbf{D}\mathbf{x}, & \text{if } r(x + y + z) \in \mathbb{Z}, \\ \mathbf{C}\mathbf{x}, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\mathbf{x} = (x, y, z)^T \in \mathbb{Z}^3$, r is a rational parameter, and

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & r & r \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.2}$$

Applying the defined mapping $\mathcal{F}(\mathbf{x})$, the third-order difference equation (1.4) can be converted into a corresponding first-order difference equation system as follows. Let $\mathbf{x}_n = (x_n, y_n, z_n)^T \in \mathbb{Z}^3$, $n \in \mathbb{N}_0$ and $\mathbf{x}_n = \mathcal{F}^n(\mathbf{x}_0)$, $n \in \mathbb{N}$, then we have the following first-order difference equation system:

$$\mathbf{x}_{n+1} = \mathcal{F}(\mathbf{x}_n) = \begin{cases} \mathbf{D}\mathbf{x}_n, & \text{if } r(x_n + y_n + z_n) \in \mathbb{Z}, \\ \mathbf{C}\mathbf{x}_n, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $\mathbf{x}_0 = (y_0, y_1, y_2)^T$, y_0, y_1, y_2 are initial integers in (1.4). Note that the condition $r(x_n + y_n + z_n) \in \mathbb{Z}$ can be replaced by the equivalent matrix product condition $r((1, 1, 1)\mathbf{x}_n) \in \mathbb{Z}$.

For instance, when $r = 1/5$, then the periodic solution $(1, 1, 1, 3, 1, 1, 1, 3, \ldots)$ to (1.4) corresponds to the following periodic solution to system (2.3):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ldots. \tag{2.4}$$

Given an initial vector $\mathbf{x}_0$, each subsequent $\mathbf{x}_n$ can be got from $\mathbf{x}_0$ via a formula $\mathbf{x}_n = \mathbf{L}_n\mathbf{x}_0$, where $\mathbf{L}_n$ is an appropriate product of n matrices, each of which is $\mathbf{C}$ or $\mathbf{D}$. Hence, for the above example, if $\mathbf{x}_0 = (1, 1, 1)^T$, then $\mathbf{x}_1 = \mathbf{C}\mathbf{x}_0$, $\mathbf{x}_2 = \mathbf{D}\mathbf{x}_1 = \mathbf{D}\mathbf{C}\mathbf{x}_0$, $\mathbf{x}_3 = \mathbf{D}^2\mathbf{C}\mathbf{x}_0$, $\mathbf{x}_4 = \mathbf{D}^3\mathbf{C}\mathbf{x}_0$, and so forth. Note that the matrices multiply $\mathbf{x}_0$ from right to left and that $\mathbf{x}_4 = \mathbf{x}_0$, so $\mathbf{D}^3\mathbf{C}\mathbf{x}_0 = \mathbf{x}_0$. We have the following obvious result.

Lemma 2.1. *If system* (2.3) *has a periodic solution with period k and $\mathbf{x}$ is a vector in that periodic solution, then there exists a corresponding matrix $\mathbf{L}$, which is a product of k matrices, each of which is $\mathbf{C}$ or $\mathbf{D}$, such that $\mathbf{x}$ is an eigenvector of $\mathbf{L}$ with eigenvalue 1.*

The proof of Lemma 2.1 is simple; here we point out that the converse of Lemma 2.1 is not true. The problem is that the sum of the entries in some $\mathbf{x}_i$ may be divisible by the denominator of r at a time, when multiplication by $\mathbf{C}$ is called for.

Next, we present the following linear algebra facts, for example, [27, Chapter 7], (similar results presented in [5]), followed by the form most convenient to us.

Lemma 2.2. *(i) If $\mathbf{v}$ is an eigenvector with eigenvalue λ for a matrix $\mathbf{L}$, then $\mathbf{v}$ is an eigenvector with eigenvalue λ^k for $\mathbf{L}^k$.*

(ii) The characteristic polynomial for a 3×3 matrix $\mathbf{L}$ has the form

$$f_{\mathbf{L}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{L}) = \lambda^3 - \operatorname{tr}(\mathbf{L})\lambda^2 + (\det(\mathbf{L}_{1,1}) + \det(\mathbf{L}_{2,2}) + \det(\mathbf{L}_{3,3}))\lambda - \det(\mathbf{L}), \tag{2.5}$$

where $\operatorname{tr}(\mathbf{L})$ *is the trace of* $\mathbf{L}$.

(iii) The characteristic polynomials for $\mathbf{LM}$ *and* $\mathbf{ML}$ *are identical for any* $n \times n$ *matrices* $\mathbf{L}$ *and* $\mathbf{M}$. *In particular,* $\operatorname{tr}(\mathbf{LM}) = \operatorname{tr}(\mathbf{ML})$.

(iv) The trace is linear. That is, $\operatorname{tr}(\alpha\mathbf{L} + \beta\mathbf{M}) = \alpha \operatorname{tr}(\mathbf{L}) + \beta \operatorname{tr}(\mathbf{M})$, *for any* $n \times n$ *matrices* $\mathbf{L}, \mathbf{M}$ *and scalars* α, β.

(v) Every matrix satisfies its characteristic polynomial. In particular, for a 3×3 *matrices* $\mathbf{L}$, *one has*

$$\mathbf{L}^3 - \operatorname{tr}(\mathbf{L})\mathbf{L}^2 + (\det(\mathbf{L}_{1,1}) + \det(\mathbf{L}_{2,2}) + \det(\mathbf{L}_{3,3}))\mathbf{L} - \det(\mathbf{L})\mathbf{I} = \mathbf{0}. \tag{2.6}$$

Let the sequences $(H_n)_{n\geq 0}$, $(J_n)_{n\geq 0}$ be two solutions to the difference equation

$$y_n = y_{n-1} + y_{n-2} + y_{n-3}, \quad n \geq 3 \tag{2.7}$$

with initial values $H_0 = H_1 = 0$, $H_2 = 1$ and $J_0 = J_2 = 1$, $J_1 = 0$, respectively.

Then, we get the following result about the matrix $\mathbf{C}$ in (2.2).

Lemma 2.3. *For the matrix* $\mathbf{C}$ *in* (2.2), *one has*

$$\mathbf{C}^n = \begin{pmatrix} H_{n-1} & J_{n-1} & H_n \\ H_n & J_n & H_{n+1} \\ H_{n+1} & J_{n+1} & H_{n+2} \end{pmatrix}, \tag{2.8}$$

for $n \in \mathbb{N}$.

Proof. This result can be proved by induction. By (2.2) and the definitions of the sequences $(H_n)_{n\geq 0}$, $(J_n)_{n\geq 0}$, we have

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} H_0 & J_0 & H_1 \\ H_1 & J_1 & H_2 \\ H_2 & J_2 & H_3 \end{pmatrix}. \tag{2.9}$$

Therefore, (2.8) holds for $n = 1$. Now, we assume that (2.8) holds for $n = \omega$, $\omega \in \mathbb{N}$. In the following, it suffices to prove that (2.8) holds for $n = \omega + 1$. By (2.2) and the associative law of matrix multiplication, we have

$$\begin{aligned} \mathbf{C}^{\omega+1} = \mathbf{C}\mathbf{C}^{\omega} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} H_{\omega-1} & J_{\omega-1} & H_{\omega} \\ H_{\omega} & J_{\omega} & H_{\omega+1} \\ H_{\omega+1} & J_{\omega+1} & H_{\omega+2} \end{pmatrix} \\ &= \begin{pmatrix} H_{\omega} & J_{\omega} & H_{\omega+1} \\ H_{\omega+1} & J_{\omega+1} & H_{\omega+2} \\ H_{\omega+2} & J_{\omega+2} & H_{\omega+3} \end{pmatrix}. \end{aligned} \tag{2.10}$$

Therefore, (2.8) holds for $k = \omega + 1$. The proof is complete. □

Given a 3×3 matrix $\mathbf{L} = \mathbf{N}_1\mathbf{N}_2 \cdots \mathbf{N}_k$, $k \in \mathbb{N}$, where each $\mathbf{N}_j$ is either $\mathbf{C}$ or $\mathbf{D}$, by Lemma 2.2 we define the following polynomial:

$$\wp_{\mathbf{L}}(r) = -\det(\mathbf{I} - \mathbf{L}) = -1 + \operatorname{tr}(\mathbf{L}) + \det(\mathbf{L}) - (\det(\mathbf{L}_{1,1}) + \det(\mathbf{L}_{2,2}) + \det(\mathbf{L}_{3,3})), \tag{2.11}$$

where $\mathbf{I}$ is the identity matrix, and seek $\mathbf{L}$ for which $\wp_{\mathbf{L}}(r)$ has rational zeros.

3. Some Other Properties

In this section, several properties of $\wp_{\mathbf{L}}(r)$ defined in (2.11) are derived. They are used to derive restrictions on values of r that allow periodic solutions to (1.4).

Lemma 3.1. *For the polynomial $\wp_{\mathbf{L}}(r)$ in* (2.11), *if $\wp_{\mathbf{L}}(r) = 0$, then $\wp_{\mathbf{L}^k}(r) = 0$, for each $k \in \mathbb{N}$.*

Proof. Since $\wp_{\mathbf{L}^k}(r) = -\det(\mathbf{I}-\mathbf{L}^k) = -\det(\mathbf{I}-\mathbf{L})\det(\mathbf{L}^{k-1}+\mathbf{L}^{k-1}+\cdots+\mathbf{I})$, then $\wp_{\mathbf{L}^k}(r) = \wp_{\mathbf{L}}(r)q(r)$ for some polynomial

$$q(r) = \det\left(\mathbf{L}^{k-1} + \mathbf{L}^{k-1} + \cdots + \mathbf{I}\right), \tag{3.1}$$

and the result follows. □

Lemma 3.2. *If* $\mathbf{C} = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$, $\mathbf{E} = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & x & x \end{smallmatrix}\right)$, *then* $\operatorname{tr}(\mathbf{C}^{n_1}\mathbf{E}\mathbf{C}^{n_2}\mathbf{E}\cdots\mathbf{C}^{n_k}\mathbf{E})$ *is a polynomial in x of degree k with nonnegative integer coefficients, with leading coefficient $\prod_{j=1}^{k} H_{n_j+3}$, where $n_j \in \mathbb{N}$.*

Proof. By Lemma 2.3, we get that

$$\begin{aligned}
\mathbf{C}^n\mathbf{E} &= \begin{pmatrix} xH_n & xH_n + H_{n-1} & xH_n + J_{n-1} \\ xH_{n+1} & xH_{n+1} + H_n & xH_{n+1} + J_n \\ xH_{n+2} & xH_{n+2} + H_{n+1} & xH_{n+2} + J_{n+1} \end{pmatrix} \\
&= x\begin{pmatrix} H_n & H_n & H_n \\ H_{n+1} & H_{n+1} & H_{n+1} \\ H_{n+2} & H_{n+2} & H_{n+2} \end{pmatrix} + \begin{pmatrix} 0 & H_{n-1} & J_{n-1} \\ 0 & H_n & J_n \\ 0 & H_{n+1} & J_{n+1} \end{pmatrix} \\
&= x(H_n, H_{n+1}, H_{n+2})^T\mathbf{1} + \mathbf{F}_n,
\end{aligned} \tag{3.2}$$

where $\mathbf{1}$ represents the vector $(1, 1, 1)$, and

$$\mathbf{F}_n = \begin{pmatrix} 0 & H_{n-1} & J_{n-1} \\ 0 & H_n & J_n \\ 0 & H_{n+1} & J_{n+1} \end{pmatrix}. \tag{3.3}$$

Hence,

$$
\begin{aligned}
&\mathbf{C}^{n_1}\mathbf{E}\mathbf{C}^{n_2}\mathbf{E}\cdots\mathbf{C}^{n_k}\mathbf{E} \\
&\quad= \left(x(H_{n_1}, H_{n_1+1}, H_{n_1+2})^T\mathbf{1} + \mathbf{F}_{n_1}\right)\cdots\left(x(H_{n_k}, H_{n_k+1}, H_{n_k+2})^T\mathbf{1} + \mathbf{F}_{n_k}\right) \\
&\quad= x^k(H_{n_1}, H_{n_1+1}, H_{n_1+2})^T\mathbf{1}(H_{n_2}, H_{n_2+1}, H_{n_2+2})^T\mathbf{1}\cdots(H_{n_k}, H_{n_k+1}, H_{n_k+2})^T\mathbf{1} \\
&\qquad+\cdots+\mathbf{F}_{n_1}\mathbf{F}_{n_2}\cdots\mathbf{F}_{n_k} \\
&\quad= x^k\prod_{j=1}^{k}\left(H_{n_j}, H_{n_j+1}, H_{n_j+2}\right)^T\mathbf{1}+\cdots+\prod_{j=1}^{\mathrm{k}}\mathbf{F}_{n_j}.
\end{aligned}
\tag{3.4}
$$

To obtain the leading coefficient, we may induct on the simple calculation

$$
\left((a,b,c)^T\mathbf{1}\right)\left((d,e,f)^T\mathbf{1}\right) = (d+e+f)\left((a,b,c)^T\mathbf{1}\right)
\tag{3.5}
$$

to show that the trace of the product is the product of the traces, such as the following. Easily, the trace of each individual matrix is H_{n_j+3}, $j = 1, 2, \ldots, k$. By (2.7) and (3.5), we have that

$$
\begin{aligned}
&\prod_{j=1}^{k}\left(H_{n_j}, H_{n_j+1}, H_{n_j+2}\right)^T\mathbf{1} \\
&\quad= (H_{n_k} + H_{n_k+1} + H_{n_k+2})\prod_{j=1}^{k-1}\left(H_{n_j}, H_{n_j+1}, H_{n_j+2}\right)^T\mathbf{1} \\
&\quad= H_{n_k+3}\prod_{j=1}^{k-1}\left(H_{n_j}, H_{n_j+1}, H_{n_j+2}\right)^T\mathbf{1} = \cdots \\
&\quad= H_{n_k+3}H_{n_{k-1}+3}\cdots H_{n_2+3}(H_{n_1}, H_{n_1+1}, H_{n_1+2})^T\mathbf{1} \\
&\quad= \prod_{j=2}^{k}H_{n_j+3}(H_{n_1}, H_{n_1+1}, H_{n_1+2})^T\mathbf{1}.
\end{aligned}
\tag{3.6}
$$

Thus the leading coefficient is

$$
\prod_{j=2}^{k}H_{n_j+3}(H_{n_1} + H_{n_1+1} + H_{n_1+2}) = \prod_{j=1}^{k}H_{n_j+3}.
\tag{3.7}
$$

The proof is complete. □

4. Periodic Solutions

In this section, we prove that (1.4) has no periodic solution with prime period two or three and derive all periodic solutions to (1.4) with prime period four and five.

Table 1

$\mathbf{L}_2$	$\wp_{\mathbf{L}_2}(r)$	Roots of $\wp_{\mathbf{L}_2}(r) = 0$
$\mathbf{D}^2$	$3r^2 + 2r - 1$	$1/3,\ -1$
$\mathbf{DC}$	$4r$	0
$\mathbf{CD}$	$4r$	0
$\mathbf{C}^2$	4	\

Theorem 4.1. *Suppose that r is a rational number and $r \notin \mathbb{Z}$, then* (1.4) *has no periodic solutions with prime period two.*

Proof. Suppose that $\mathbf{x} \in \mathbb{Z}^3$ be a vector of a period-two solution to system (2.3), then, for some matrix $\mathbf{L}_2$, we have $\mathbf{x} = \mathbf{L}_2\mathbf{x}$; here the matrix $\mathbf{L}_2$ has four possible cases $\mathbf{C}^2$, $\mathbf{CD}$, $\mathbf{DC}$ and $\mathbf{D}^2$.

Take $\mathbf{L}_2 = \mathbf{D}^2$, for example, through some calculations, we get that

$$\mathbf{D}^2 = \begin{pmatrix} 0 & 0 & 1 \\ r & r & r \\ r^2 & r(r+1) & r(r+1) \end{pmatrix}, \qquad \wp_{\mathbf{D}^2}(r) = 3r^2 + 2r - 1. \tag{4.1}$$

By solving the equation $\wp_{\mathbf{D}^2}(r) = 3r^2 + 2r - 1 = 0$, we get two real roots $r_1 = 1/3$ and $r_2 = -1$. Through similar calculations, we can get Table 1.

The only value of r which may lead to period-two solutions to system (2.3) is $r = 1/3$ since $r \notin \mathbb{Z}$. Hence, by solving the following matrix equation (with $r = 1/3$)

$$\left(\mathbf{D}^2 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.2}$$

we get its all integer solutions $\mathbf{x} = (t, t, t)^T$, $t \in \mathbb{Z}$. Obviously, the solutions are equilibrium points which contradicts the assumption. The proof is complete. □

Theorem 4.2. *Suppose that r is a rational number and $r \notin \mathbb{Z}$, then* (1.4) *has no periodic solutions with prime period three.*

Proof. Assume that $\mathbf{x} \in \mathbb{Z}^3$ be a vector of a period-three solution to system (2.3), then we get $\mathbf{x} = \mathbf{L}_3\mathbf{x}$ where $\mathbf{L}_3$ is a matrix product of three matrices, each of which is $\mathbf{C}$ or $\mathbf{D}$.

Obviously, $\mathbf{L}_3$ has eight possible cases which can be divided into four categories such as three $\mathbf{C}$s, two $\mathbf{C}$s and one $\mathbf{D}$, one $\mathbf{C}$ and two $\mathbf{D}$s, and three $\mathbf{D}$s. By (2.2) and through certain calculations, the matrices $\mathbf{C}^2\mathbf{D}$, $\mathbf{CDC}$, $\mathbf{DC}^2$ are similar, and; thus, they generate the same polynomials $\wp_{\mathbf{L}_3}(r)$, so do the matrices $\mathbf{CD}^2$, $\mathbf{DCD}$, $\mathbf{D}^2\mathbf{C}$.

Table 2

$\mathbf{L}_3$	$\wp_{\mathbf{L}_3}(r)$	Roots of $\wp_{\mathbf{L}_3}(r) = 0$
$\mathbf{D}^3$	$3r - 1$	$1/3$
1**D**2**C**s	$r + 1$	-1
1**C**2**D**s	$2r$	0
$\mathbf{C}^3$	2	\

Table 3

$\mathbf{L}_4$	$\wp_{\mathbf{L}_4}(r)$	Roots of $\wp_{\mathbf{L}_4}(r) = 0$
$\mathbf{D}^4$	$3r^4 + 6r^2 + 8r^3 - 1$	13, −1(triple)
3**D**s1**C**	$5r^3 + 9r^2 + 3r - 1$	15, −1(double)
2**D**s2**C**s	$8r^2 + 8r$	−1(double)
1**D**3**C**s	$12r + 4$	$-1/3$
$\mathbf{C}^4$	16	\

In the following, we take $\mathbf{L}_3 = \mathbf{D}^3$, for example. Through some calculations, we have

$$\mathbf{D}^3 = \begin{pmatrix} r & r & r \\ r^2 & \mathrm{r}(r+1) & r(r+1) \\ r^2(r+1) & r^2(r+2) & r\left(r^2+2r+1\right) \end{pmatrix}, \qquad (4.3)$$
$$\wp_{\mathbf{D}^3}(r) = 3r - 1.$$

By solving the equation $\wp_{\mathbf{D}^3}(r) = 3r - 1 = 0$, we get the only root $r = 1/3$. Through similar calculations, we can obtain Table 2.

The only value of r which may lead to period-three solutions to system (2.3) is $r = 1/3$ since $r \notin \mathbb{Z}$. Hence, by solving the following matrix equation (with $r = 1/3$)

$$\left(\mathbf{D}^3 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \qquad (4.4)$$

we get its all integer solutions $\mathbf{x} = (t,t,t)^T, t \in \mathbb{Z}$. Obviously, the solutions are equilibrium points which contradicts the assumption. The proof is complete. □

In the following, denote by $\mathcal{S}(k,r)$ the set of initial values (y_0, y_1, y_2) which lead to period-k solutions to (1.4) with the parameter r.

Theorem 4.3. *Suppose that r is a rational number and $r \notin \mathbb{Z}$, then* (1.4) *has periodic solutions with prime period four if and only if $r = -1/3$ or $1/5$. Moreover, $\mathcal{S}(4,-1/3) = \{(t,t,t),(-t,t,t),(t,-t,t),(t,t,-t) \mid t \in \mathbb{Z}, 3 \nmid t\}$, and $\mathcal{S}(4,1/5) = \{(t,t,3t),(t,3t,t),(3t,t,t),(t,t,t) \mid t \in \mathbb{Z}, 5 \nmid t\}$.*

Proof. Let $\mathbf{x} \in \mathbb{Z}^3$ be a vector of a period-four solution to system (2.3), then we get $\mathbf{x} = \mathbf{L}_4\mathbf{x}$, where $\mathbf{L}_4$ is an appropriate product of four matrices, each of which is **C** or **D**. Through similar calculation to those in Theorems 4.1 and 4.2, Table 3 is derived.

The only values of r which possibly lead to period-four solutions to system (2.3) are $r = 1/3, -1/3$, or $1/5$ because of $r \notin \mathbb{Z}$.

Case 1 ($r = 1/3$). By solving the following matrix equation

$$\left(\mathbf{D}^4 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.5}$$

we get its all integer solutions $\mathbf{x} = (t, t, t)^T,\ t \in \mathbb{Z}$. Obviously, the solutions are equilibrium points.

Case 2 ($r = -1/3$). In this case, the matrix $\mathbf{L}_4$ has four possible cases $\mathbf{C}^3\mathbf{D}, \mathbf{C}^2\mathbf{D}\mathbf{C}, \mathbf{C}\mathbf{D}\mathbf{C}^2$, and $\mathbf{D}\mathbf{C}^3$.

By solving the matrix equation

$$\left(\mathbf{C}^3\mathbf{D} - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.6}$$

we get its all integer solutions $\mathbf{x} = (t, t, t)^T,\ t \in \mathbb{Z}$. On the condition of that $3 \nmid t$, then we can verify that the initial vector $\mathbf{x}_0 = (t, t, t)^T$ leads to a period-four solution to system (2.3), such as the following:

$$\begin{pmatrix} t \\ t \\ t \end{pmatrix}, \begin{pmatrix} t \\ t \\ -t \end{pmatrix}, \begin{pmatrix} t \\ -t \\ t \end{pmatrix}, \begin{pmatrix} -t \\ t \\ t \end{pmatrix}, \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \ldots. \tag{4.7}$$

Similarly, for the following three matrix equations

$$\left(\mathbf{C}^2\mathbf{D}\mathbf{C} - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \qquad \left(\mathbf{C}\mathbf{D}\mathbf{C}^2 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \qquad \left(\mathbf{D}\mathbf{C}^3 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.8}$$

we derive its all integer solutions $\mathbf{x} = (-t, t, t)^T,\ \mathbf{x} = (t, -t, t)^T,\ \mathbf{x} = (t, t, -t)^T,\ t \in \mathbb{Z}$, respectively.

Note that, on the condition $3 \nmid t$, the initial vectors $\mathbf{x}_0 = (-t, t, t)^T, (t, -t, t)^T$, or $(t, t, -t)^T$ also lead to period-four solutions to system (2.3).

Case 3 ($r = 1/5$). In this case, the matrix $\mathbf{L}_4$ also has four possible cases $\mathbf{C}\mathbf{D}^3, \mathbf{D}\mathbf{C}\mathbf{D}^2, \mathbf{D}^2\mathbf{C}\mathbf{D}$, and $\mathbf{D}^3\mathbf{C}$. By solving the matrix equation

$$\left(\mathbf{C}\mathbf{D}^3 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.9}$$

we get its all integer solutions $\mathbf{x} = (t, t, 3t)^T, t \in \mathbb{Z}$. On the condition of that $5 \nmid t$, then we can verify that the initial vector $\mathbf{x}_0 = (t, t, 3t)^T$ leads to a period-four solution to system (2.3), such as the following:

$$\begin{pmatrix} t \\ t \\ 3t \end{pmatrix}, \begin{pmatrix} t \\ 3t \\ t \end{pmatrix}, \begin{pmatrix} 3t \\ t \\ t \end{pmatrix}, \begin{pmatrix} t \\ t \\ t \end{pmatrix}, \begin{pmatrix} t \\ t \\ 3t \end{pmatrix}, \ldots. \tag{4.10}$$

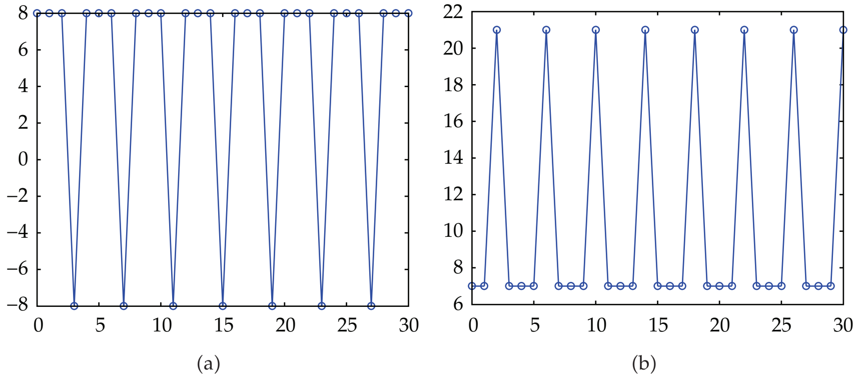

Figure 1: Equation (1.4). (a) $r = -1/3$. Initial values $y_0 = 8$, $y_1 = 8$, $y_2 = 8$ $(t = 8)$. (b) $r = 1/5$. Initial values $y_0 = 7$, $y_1 = 7$, $y_2 = 21$ $(t = 7)$.

Table 4

$\mathbf{L}_5$	$p_{\mathbf{L}_5}(r)$	Rational roots of $p_{\mathbf{L}_5}(r) = 0$
$\mathbf{D}^5$	$3r^5 + 5r^4 + 10r^3 + 5r^2 - 1$	$1/3$
4**D**s1**C**	$4r^4 + 8r^3 + 9r^2 + 2r - 1$	\
3**D**s2**C**s	$6r^3 + 10r^2 + 7r - 1$	\
2**D**s3**C**s	$9r^2 + 12r + 1$	\
1**D**4**C**s	$15r + 7$	$-7/15$
$\mathbf{C}^5$	22	\

Obviously, on the condition $5 \nmid t$, the initial vectors $\mathbf{x}_0 = (t, 3t, t)^T$, $(3t, t, t)^T$, or $(t, t, t)^T$ (which are integer solutions to appropriate matrix equations corresponding to $\mathbf{DCD}^2$, $\mathbf{D}^2\mathbf{CD}$, $\mathbf{D}^3\mathbf{C}$, resp.) also lead to period-four solutions to system (2.3). The proof is complete. □

To illustrate the results, we give two orbits (see the following Figure 1) of period-four integer solutions to (1.4) of the particular cases $r = -1/3$, $t = 8$ and $r = 1/5$, $t = 7$ in Theorem 4.3.

Theorem 4.4. *Suppose that r is a rational number and $r \notin \mathbb{Z}$, then* (1.4) *has periodic solutions with prime period five if and only if $r = -7/15$. Moreover, $\mathcal{S}(5, -7/15) = \{(5t, t, 9t), (t, 9t, -7t), (9t, -7t, 3t), (-7t, 3t, 5t), (3t, 5t, t) \mid t \in \mathbb{Z}, 3 \nmid t, 5 \nmid t\}$.*

Proof. Let $\mathbf{x} \in \mathbb{Z}^3$ be a vector of a period-five solution to system (2.3), then we get $\mathbf{x} = \mathbf{L}_5\mathbf{x}$ where $\mathbf{L}_5$ is an appropriate product of five matrices, each of which is **C** or **D**. Through similar calculation to those in Theorems 4.1 and 4.2, Table 4 is obtained.

Apparently, the only values of r which possibly lead to period-three solutions to system (2.3) are $r = 1/3$ or $-7/15$ because of $r \notin \mathbb{Z}$.

Case 1 $(r = 1/3)$. By solving the matrix equation

$$\left(\mathbf{D}^5 - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.11}$$

we get its all integer solutions $\mathbf{x} = (t, t, t)^T$, $t \in \mathbb{Z}$. Obviously, the solutions are equilibrium points.

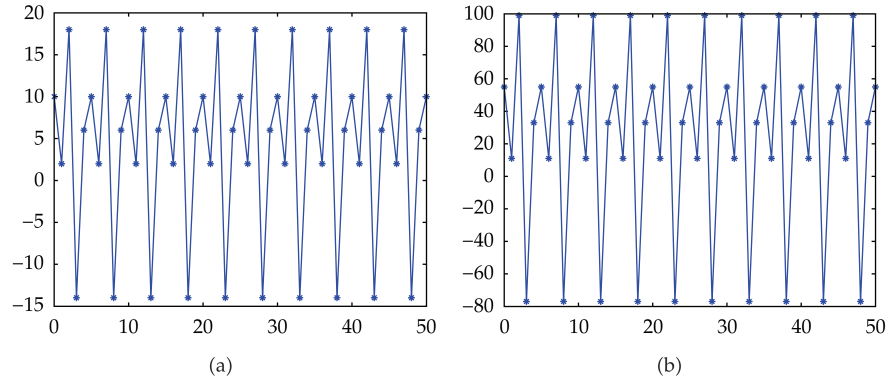

Figure 2: Equation (1.4) $r = -7/15$. (a): Initial values $y_0 = 10$, $y_1 = 2$, $y_2 = 18$ $(t = 2)$; (b): Initial values $y_0 = 55$, $y_1 = 11$, $y_2 = 99$ $(t = 11)$.

Case 2 ($r = -7/15$). In this case, the matrix $\mathbf{L}_5$ also has five possible cases $\mathbf{C}^4\mathbf{D}$, $\mathbf{C}^3\mathbf{D}\mathbf{C}$, $\mathbf{C}^2\mathbf{D}\mathbf{C}^2$, $\mathbf{C}\mathbf{D}\mathbf{C}^3$, and $\mathbf{D}\mathbf{C}^4$. Solve the matrix equation

$$\left(\mathbf{C}^4\mathbf{D} - \mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{4.12}$$

we get its all integer solutions $\mathbf{x} = (5t, t, 9t)^T$, $t \in \mathbb{Z}$. On the conditions that $3 \nmid t, 5 \nmid t$, then we can verify that the initial vector $\mathbf{x}_0 = (5t, t, 9t)^T$ leads to a period-four solution to system (2.3), such as the following:

$$\begin{pmatrix} 5t \\ t \\ 9t \end{pmatrix}, \begin{pmatrix} t \\ 9t \\ -7t \end{pmatrix}, \begin{pmatrix} 9t \\ -7t \\ 3t \end{pmatrix}, \begin{pmatrix} -7t \\ 3t \\ 5t \end{pmatrix}, \begin{pmatrix} 3t \\ 5t \\ t \end{pmatrix}, \begin{pmatrix} 5t \\ t \\ 9t \end{pmatrix}, \ldots. \tag{4.13}$$

Obviously, on the conditions $3 \nmid t, 5 \nmid t$, the initial vectors $\mathbf{x}_0 = (t, 9t, -7t)^T$, $(9t, -7t, 3t)^T$, $(-7t, 3t, 5t)^T$, or $(3t, 5t, t)^T$ (which are integer solutions to appropriate matrix equations corresponding to $\mathbf{D}\mathbf{C}^4$, $\mathbf{C}\mathbf{D}\mathbf{C}^3$, $\mathbf{C}^2\mathbf{D}\mathbf{C}^2$, and $\mathbf{C}^3\mathbf{D}\mathbf{C}$, resp.) also lead to period-four solutions to system (2.3). The proof is complete. □

To illustrate the results, we give two orbits (see Figure 2) of period-five integer solutions to (1.4) of the particular cases $t = 2, 11$ in Theorem 4.4.

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