

Research Article

Global Exponential Stability of Antiperiodic Solutions for Discrete-Time Neural Networks with Mixed Delays and Impulses

Xiaofeng Chen and Qiankun Song

Department of Mathematics, Chongqing Jiaotong University, Chongqing 400074, China

Correspondence should be addressed to Qiankun Song, qiankunsong@163.com

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The problem on global exponential stability of antiperiodic solution is investigated for a class of impulsive discrete-time neural networks with time-varying discrete delays and distributed delays. By constructing an appropriate Lyapunov-Krasovskii functional, and using the contraction mapping principle and the matrix inequality techniques, a new delay-dependent criterion for checking the existence, uniqueness, and global exponential stability of anti-periodic solution is derived in linear matrix inequalities (LMIs). Two simulation examples are given to show the effectiveness of the proposed result.

1. Introduction

Over the past decades, delayed neural networks have found successful applications in many areas such as signal processing, pattern recognition, associative memories, and optimization solvers. In such applications, the qualitative analysis of the dynamical behaviors is a necessary step for the practical design of neural networks [1]. Many important results on the dynamical behaviors have been reported for delayed neural networks, see [1–5] and the references therein for some recent publications. Although neural networks are mostly studied in the continuous-time setting, they are often discretized for experimental or computational purposes. The dynamic characteristics of discrete-time neural networks have been extensively investigated, for example, see [6–10] and the references cited therein.

Impulsive differential equations are mathematical apparatus for simulation of process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, and so forth [11, 12]. Consequently, many neural networks with impulses have been studied extensively, and a great deal of the

literature is focused on the existence and stability of an equilibrium point [13–17]. In [18, 19], the authors discussed the existence and global exponential stability of periodic solution of a class of neural networks with impulse.

The study of antiperiodic solutions for nonlinear differential equations is closely related to the study of periodic solutions, and it was initiated by Okochi in 1988 [20]. Arising from problems in applied sciences, it is well-known that the existence and stability of antiperiodic solutions plays a key role in characterizing the behavior of nonlinear differential equations as a special periodic solution [21, 22]. As pointed out in [23], antiperiodic solutions arise naturally in the mathematical modeling of various physical processes. For example, antiperiodic trigonometric polynomials are often studied in interpolation problems [24, 25], the signal transmission process of neural networks can often be described as an antiperiodic process [26], and antiperiodic wavelets are discussed in [27]. During the past twenty years antiperiodic problems of nonlinear differential equations have been extensively studied by many authors, for example, see [28–30] and references therein. Recently, the problem of antiperiodic solutions for neural networks with or without time delays has received considerable research interest, see for example [26, 31–38] and references therein. In [31, 32], using some analysis skills and Lyapunov method, the authors studied the existence and exponential stability of antiperiodic solutions for shunting inhibitory cellular neural networks with time-varying discrete delays or distributed delays. In [33], some sufficient conditions have been established for checking the existence and exponential stability of antiperiodic solutions of high-order Hopfield neural networks with time-varying delays. In [34], the recurrent neural networks with time-varying delays and continuously distributed delays have been considered, and some sufficient conditions for the existence and exponential stability of the antiperiodic solutions have been given. In [35, 36], the existence and exponential stability of antiperiodic solutions have been studied for Cohen-Grossberg neural networks with time-varying delays and continuously distributed delays. In [37], the authors obtained some sufficient conditions to ensure existence and exponential stability of the antiperiodic solutions for a class of Hopfield neural networks with periodic impulses. In [38], several sufficient conditions are established for the existence and global exponential stability of antiperiodic solutions to impulsive shunting inhibitory cellular neural networks with distributed delays on time scale.

However, to the best of our knowledge, there are few papers published on the existence and exponential stability of antiperiodic solutions for discrete-time neural networks with mixed delays and impulses. Motivated by the above discussion, the objective of this paper is to study the existence and exponential stability of antiperiodic solutions for impulsive discrete-time recurrent neural networks with time-varying discrete delays and distributed delays. Under more general description on the activation functions, we obtain a new criterion for checking the existence, uniqueness, and global exponential stability of antiperiodic solution, which can be checked numerically using the effective LMI toolbox in MATLAB. Two simulation examples are given to show the effectiveness and less conservatism of the proposed criteria.

Notations

The notation used here is quite standard. \mathbb{R}^n and $\mathbb{R}^{m \times n}$ denote, respectively, the n -dimensional Euclidean space and the set of all $m \times n$ real matrices. The superscript “T” denotes matrix transposition. The notation $X \geq Y$ (resp., $X > Y$) means that X and Y are symmetric matrices, and that $X - Y$ is positive semidefinite (resp., positive definite). $\|\cdot\|$ is the Euclidean

norm in \mathbb{R}^n . If A is a matrix, denote by $\lambda_{\max}(A)$ (resp., $\lambda_{\min}(A)$) the largest (resp., smallest) eigenvalue of A . For integers a, b , and $a < b$, $N[a, b]$ denotes the discrete interval given by $N[a, b] = \{a, a+1, \dots, b\}$. $\mathcal{C}(N[-\tau, 0], \mathbb{R}^n)$ denotes the set of all functions $\phi : N[-\tau, 0] \rightarrow \mathbb{R}^n$. In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. Model Description and Preliminaries

In this paper, we consider the following impulsive discrete-time neural networks with time-varying discrete delays and distributed delays

$$\begin{aligned} x_i(k+1) &= a_i x_i(k) + \sum_{j=1}^n b_{ij} g_j(x_j(k)) + \sum_{j=1}^n c_{ij} g_j(x_j(k - \tau_1(k))) \\ &\quad + \sum_{j=1}^n d_{ij} \sum_{m=k}^{k+\tau_2-1} g_j(x_j(m - \tau_2)) + I_i(k), \quad k \neq k_r \\ x_i(k_r + 1) &= x_i(k_r) + e_{ir}(x_i(k_r)), \quad r = 1, 2, \dots \end{aligned} \quad (2.1)$$

or, in an equivalent vector form

$$\begin{aligned} x(k+1) &= Ax(k) + Bg(x(k)) + Cg(x(k - \tau_1(k))) \\ &\quad + D \sum_{m=k}^{k+\tau_2-1} g(x(m - \tau_2)) + I(k), \quad k \neq k_r \\ x(k_r + 1) &= x(k_r) + e_r(x(k_r)), \quad r = 1, 2, \dots \end{aligned} \quad (2.2)$$

for $k = 1, 2, \dots$, where $x(k) = (x_1(k), x_2(k), \dots, x_n(k))^T \in \mathbb{R}^n$, $x_i(k)$ is the state of the i th neuron at time k ; $g(x(k)) = (g_1(x_1(k)), g_2(x_2(k)), \dots, g_n(x_n(k)))^T \in \mathbb{R}^n$, $g_j(x_j(k))$ denotes the activation function of the j th neuron at time k ; $I(k) = (I_1(k), I_2(k), \dots, I_n(k))^T \in \mathbb{R}^n$, $I_i(k)$ represents the external input on the i th neuron at time k ; the positive integer $\tau_1(k)$ corresponds to the transmission delay and satisfies $\check{\tau} \leq \tau_1(k) \leq \hat{\tau}$ ($\check{\tau}$ and $\hat{\tau}$ are known integers such that $\hat{\tau} > \check{\tau} \geq 0$); the positive integer τ_2 describes the distributed time delay; $A = \text{diag}(a_1, a_2, \dots, a_n)$, a_i ($0 \leq a_i < 1$) describes the rate with which the i th neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $B = (b_{ij})_{n \times n}$ is the connection weight matrix, $C = (c_{ij})_{n \times n}$ and $D = (d_{ij})_{n \times n}$ are the delayed connection weight matrices; k_r are the impulse instants satisfying $0 = k_0 < k_1 < \dots < k_r < k_{r+1} < \dots$ and $k_{r+1} - k_r > 1$; $e_r : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($r = 1, 2, \dots$) denotes a sequence of jump operators.

Remark 2.1. The delay term $\sum_{m=k}^{k+\tau_2-1} g(x(m - \tau_2))$ in the system (2.2) is the so-called distributed delay in the discrete-time setting, which can be regarded as the discretization of the integral form $\int_{t-\tau_2}^t g(x(s)) ds$ for the continuous-time system.

The initial condition associated with system (2.2) is given by

$$x(s) = \phi(s), \quad \phi \in C(N[-\tau, 0], \mathbb{R}^n), \quad (2.3)$$

where $\tau = \max\{\hat{\tau}, \tau_2\}$.

Throughout this paper, we make the following assumptions.

(H1) $g(u)$ and $e_r(u)$ are odd functions, $\tau_1(k)$ and $I(k)$ are ω -periodic function and ω -antiperiodic function, respectively, that is,

$$\begin{aligned} g(-u) &= -g(u), \quad e_r(-u) = -e_r(u), \quad u \in \mathbb{R}^n, \quad r = 1, 2, \dots, \\ \tau_1(k) &= \tau_1(k + \omega), \quad I(k) = -I(k + \omega), \quad k = 1, 2, \dots \end{aligned} \quad (2.4)$$

And there exists a positive integer p such that

$$k_{r+p} = k_r + \omega, \quad e_{r+p}(u) = e_r(u), \quad u \in \mathbb{R}^n, \quad r = 1, 2, \dots \quad (2.5)$$

(H2) There exist constants \check{l}_j and \hat{l}_j ($j = 1, 2, \dots, n$) such that

$$\check{l}_j \leq \frac{g_j(\alpha_1) - g_j(\alpha_2)}{\alpha_1 - \alpha_2} \leq \hat{l}_j, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \quad \alpha_1 \neq \alpha_2. \quad (2.6)$$

(H3) There exist constants \check{h}_i and \hat{h}_i ($i = 1, 2, \dots, n$) such that

$$\check{h}_i \leq \frac{e_{ir}(\alpha_1) - e_{ir}(\alpha_2)}{\alpha_1 - \alpha_2} \leq \hat{h}_i, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \quad \alpha_1 \neq \alpha_2, \quad r = 1, 2, \dots \quad (2.7)$$

Definition 2.2. The antiperiodic solution $x^*(k)$ of system (2.2) with (2.3) is said to be globally exponentially stable if there exist two positive constants $M > 0$ and $0 < \varepsilon < 1$ such that

$$\|x(k) - x^*(k)\| \leq M\varepsilon^k \sup_{k \in N[-\tau, 0]} \|\phi(k) - \phi^*(k)\|, \quad (2.8)$$

for all $k = 1, 2, \dots$, where $x(k)$ is any solution of system (2.2) with (2.3), $\phi(k)$ and $\phi^*(k)$ are the initial functions of solutions $x(k)$ and $x^*(k)$, respectively.

Lemma 2.3. Let $M \in \mathbb{R}^{n \times n}$ be positive definite matrix, $\alpha_i \in \mathbb{R}^n$ ($i = 1, 2, \dots, m$). Then the following inequality holds:

$$\left[\sum_{i=1}^m \alpha_i \right]^T M \left[\sum_{i=1}^m \alpha_i \right] \leq m \sum_{i=1}^m \alpha_i^T M \alpha_i. \quad (2.9)$$

The proof of Lemma 2.3 can be carried out by following a similar line as in [10], and hence it is omitted.

$$\Omega_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -3\mathcal{R}_3 & 2\mathcal{R}_3 & 0 & \mathcal{R}_3 \\ * & * & * & * & * & -\mathcal{R}_2 - 2\mathcal{R}_3 & \mathcal{R}_2 & 0 \\ * & * & * & * & * & * & -\mathcal{R}_1 - \mathcal{R}_2 & 0 \\ * & * & * & * & * & * & * & -\mathcal{R}_3 \end{bmatrix},$$

(3.4)

with

$$\begin{aligned} \Pi_{11} &= A\rho A - \rho + Q_1 + Q_2 + Q_3 \\ &\quad + (1 + \hat{\tau} - \check{\tau})(Q_4 - 2L_1S_1 + 2L_2S_2) \\ &\quad + (A - E)\mathcal{R}(A - E) - \mathcal{R}_1 - L_3\mathcal{T}_1, \\ \Pi_{12} &= A\rho B + (1 + \hat{\tau} - \check{\tau})(S_1 - S_2) + (A - E)\mathcal{R}B + L_4\mathcal{T}_1, \\ \Pi_{13} &= A\rho C + (A - E)\mathcal{R}C, \\ \Pi_{14} &= A\rho D + (A - E)\mathcal{R}D, \\ \Pi_{22} &= B^T\rho B + B^T\mathcal{R}B + (1 + \hat{\tau} - \check{\tau})Q_5 + \tau_2Q_6 - \mathcal{T}_1, \\ \Pi_{23} &= B^T\rho C + B^T\mathcal{R}C, \\ \Pi_{24} &= B^T\rho D + B^T\mathcal{R}D, \\ \Pi_{33} &= C^T\rho C + C^T\mathcal{R}C - Q_5 - \mathcal{T}_2, \\ \Pi_{34} &= C^T\rho D + C^T\mathcal{R}D, \\ \Pi_{35} &= S_2 - S_1 + L_4\mathcal{T}_2, \\ \Pi_{44} &= D^T\rho D + D^T\mathcal{R}D - \frac{1}{\tau_2}Q_6, \\ \Pi_{55} &= -Q_4 + 2L_1S_1 - 2L_2S_2 - L_3\mathcal{T}_2, \\ \Xi_{11} &= H_1\rho + \rho^T H_1 + H_2\rho H_2 + Q_1 + Q_2 + Q_3 \\ &\quad + (1 + \hat{\tau} - \check{\tau})(Q_4 - 2L_1S_1 + 2L_2S_2) \\ &\quad + H_2\mathcal{R}H_2 - \mathcal{R}_1 - L_3\mathcal{T}_1, \\ \Xi_{12} &= (1 + \hat{\tau} - \check{\tau})(S_1 - S_2) + L_4\mathcal{T}_1, \\ \Xi_{22} &= (1 + \hat{\tau} - \check{\tau})Q_5 + \tau_2Q_6 - \mathcal{T}_1, \end{aligned}$$

$$\begin{aligned}
\Xi_{33} &= -Q_5 - \mathcal{T}_2, \\
\Xi_{35} &= \Pi_{35}, \\
\Xi_{44} &= -\frac{1}{\tau_2} Q_6, \\
\Xi_{55} &= \Pi_{55}, \\
\mathcal{R} &= \check{\tau}^2 \mathcal{R}_1 + (\bar{\tau} - \check{\tau})^2 \mathcal{R}_2 + (\hat{\tau} - \bar{\tau})^2 \mathcal{R}_3.
\end{aligned} \tag{3.5}$$

Proof. Let $\mathcal{X} = \{\phi \mid \phi \in C(N[-\tau, 0], \mathbb{R}^n)\}$. For $\phi \in \mathcal{X}$, define

$$\|\phi\| = \sup_{s \in [-\tau, 0]} \|\phi(s)\|, \tag{3.6}$$

then \mathcal{X} is a Banach space with the topology of uniform convergence.

For $\phi, \psi \in \mathcal{X}$, let $x(k, \phi)$ and $x(k, \psi)$ be the solutions of system (2.2) with initial values ϕ and ψ , respectively. Define

$$x^{(k)}(\phi)(t) = x(k+t, \phi), \quad \forall t \in N[-\tau, 0], k = 1, 2, \dots, \tag{3.7}$$

and then $x^{(k)}(\phi) \in \mathcal{X}$ for all $k = 1, 2, \dots$. It follows from system (2.2) that

$$\begin{aligned}
x(k+1, \phi) - x(k+1, \psi) &= A[x(k, \phi) - x(k, \psi)] + B[g(x(k, \phi)) - g(x(k, \psi))] \\
&\quad + C[g(x(k - \tau_1(k), \phi)) - g(x(k - \tau_1(k), \psi))] \\
&\quad + D \sum_{m=1}^{\tau_2} [g(x(k-m, \phi)) - g(x(k-m, \psi))], \quad k \neq k_r \\
x(k_r+1, \phi) - x(k_r+1, \psi) &= x(k_r, \phi) - x(k_r, \psi) \\
&\quad + e_r(x(k_r, \phi)) - e_r(x(k_r, \psi)), \quad r = 1, 2, \dots
\end{aligned} \tag{3.8}$$

Letting

$$\begin{aligned}
y(k) &= x(k, \phi) - x(k, \psi), \\
f(k) &= g(x(k, \phi)) - g(x(k, \psi)), \\
u_r(y(k_r)) &= e_r(x(k_r, \phi)) - e_r(x(k_r, \psi)),
\end{aligned} \tag{3.9}$$

system (3.8) can then be simplified as

$$\begin{aligned} y(k+1) &= Ay(k) + Bf(k) + Cf(k - \tau_1(k)) \\ &\quad + D \sum_{m=1}^{\tau_2} f(k-m), \quad k \neq k_r \\ y(k_r+1) &= y(k_r) + u_r(y(k_r)), \quad r = 1, 2, \dots \end{aligned} \quad (3.10)$$

Defining $\eta(k) = y(k+1) - y(k)$, we consider the following Lyapunov-Krasovskii functional candidate for system (3.10) as

$$V(k) = \sum_{i=1}^8 V_i(k), \quad (3.11)$$

where

$$\begin{aligned} V_1(k) &= y^T(k) \rho y(k), \\ V_2(k) &= \sum_{j=k-\bar{\tau}}^{k-1} y^T(j) Q_1 y(j) + \sum_{j=k-\check{\tau}}^{k-1} y^T(j) Q_2 y(j) + \sum_{j=k-\bar{\tau}}^{k-1} y^T(j) Q_3 y(j), \\ V_3(k) &= \sum_{j=k-\tau_1(k)}^{k-1} y^T(j) Q_4 y(j) + \sum_{m=k-\bar{\tau}+1}^{k-\check{\tau}} \sum_{j=m}^{k-1} y^T(j) Q_4 y(j), \\ V_4(k) &= \sum_{j=k-\tau_1(k)}^{k-1} f^T(j) Q_5 f(j) + \sum_{m=k-\bar{\tau}+1}^{k-\check{\tau}} \sum_{j=m}^{k-1} f^T(j) Q_5 f(j), \\ V_5(k) &= \sum_{m=1}^{\tau_2} \sum_{j=k-m}^{k-1} f^T(j) Q_6 f(j), \\ V_6(k) &= 2 \sum_{j=k-\tau_1(k)}^{k-1} (f(j) - L_1 y(j))^T S_1 y(j) \\ &\quad + 2 \sum_{m=k-\bar{\tau}+1}^{k-\check{\tau}} \sum_{j=m}^{k-1} (f(j) - L_1 y(j))^T S_1 y(j), \\ V_7(k) &= 2 \sum_{j=k-\tau_1(k)}^{k-1} (L_2 y(j) - f(j))^T S_2 y(j) \\ &\quad + 2 \sum_{m=k-\bar{\tau}+1}^{k-\check{\tau}} \sum_{j=m}^{k-1} (L_2 y(j) - f(j))^T S_2 y(j), \end{aligned}$$

$$\begin{aligned}
V_8(k) &= \check{\tau} \sum_{m=k-\check{\tau}}^{k-1} \sum_{j=m}^{k-1} \eta^T(j) \mathcal{R}_1 \eta(j) + (\bar{\tau} - \check{\tau}) \sum_{m=k-\bar{\tau}}^{k-\check{\tau}-1} \sum_{j=m}^{k-1} \eta^T(j) \mathcal{R}_2 \eta(j) \\
&\quad + (\hat{\tau} - \bar{\tau}) \sum_{m=k-\hat{\tau}}^{k-\bar{\tau}-1} \sum_{j=m}^{k-1} \eta^T(j) \mathcal{R}_3 \eta(j).
\end{aligned} \tag{3.12}$$

We proceed by considering two possible cases of $k \neq k_r$ and $k = k_r$.

Case 1 ($k \neq k_r$ ($r = 1, 2, \dots$)). Calculating the difference of $V_i(k)$ ($i = 1, 2, \dots, 8$) along the system (3.10), by Lemma 2.3 we have

$$\begin{aligned}
\Delta V_1(k) &= y^T(k+1) \rho y(k+1) - y^T(k) \rho y(k) \\
&= y^T(k) \left(A^T \rho A - \rho \right) y(k) + 2y^T(k) A^T \rho B f(k) \\
&\quad + 2y^T(k) A^T \rho C f(k - \tau_1(k)) + 2y^T(k) A^T \rho D \sum_{m=1}^{\tau_2} f(k-m) \\
&\quad + f^T(k) B^T \rho B f(k) + 2f^T(k) B^T \rho C f(k - \tau_1(k)) \\
&\quad + 2f^T(k) B^T \rho D \sum_{m=1}^{\tau_2} f(k-m) + f^T(k - \tau_1(k)) C^T \rho C f(k - \tau_1(k)) \\
&\quad + 2f^T(k - \tau_1(k)) C^T \rho D \sum_{m=1}^{\tau_2} f(k-m) \\
&\quad + \left[\sum_{m=1}^{\tau_2} f(k-m) \right]^T D^T \rho D \sum_{m=1}^{\tau_2} f(k-m),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\Delta V_2(k) &= y^T(k) (Q_1 + Q_2 + Q_3) y(k) - y^T(k - \bar{\tau}) Q_1 y(k - \bar{\tau}) \\
&\quad - y^T(k - \check{\tau}) Q_2 y(k - \check{\tau}) - y^T(k - \hat{\tau}) Q_3 y(k - \hat{\tau}),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\Delta V_3(k) &= \sum_{j=k+1-\tau_1(k+1)}^{k-\check{\tau}} y^T(j) Q_4 y(j) + \sum_{j=k-\check{\tau}+1}^{k-1} y^T(j) Q_4 y(j) + y^T(k) Q_4 y(k) \\
&\quad - \sum_{j=k-\tau_1(k)+1}^{k-1} y^T(j) Q_4 y(j) - y^T(k - \tau_1(k)) Q_4 y(k - \tau_1(k)) \\
&\quad + (\hat{\tau} - \check{\tau}) y^T(k) Q_4 y(k) - \sum_{j=k-\hat{\tau}+1}^{k-\check{\tau}} y^T(j) Q_4 y(j) \\
&\leq (1 + \hat{\tau} - \check{\tau}) y^T(k) Q_4 y(k) - y^T(k - \tau_1(k)) Q_4 y(k - \tau_1(k)),
\end{aligned} \tag{3.15}$$

$$\Delta V_4(k) \leq (1 + \hat{\tau} - \check{\tau})f^T(k)Q_5f(k) - f^T(k - \tau_1(k))Q_5f(k - \tau_1(k)), \quad (3.16)$$

$$\begin{aligned} \Delta V_5(k) &= \sum_{m=1}^{\tau_2} \left[f^T(k)Q_6f(k) - f^T(k - m)Q_6f(k - m) \right], \\ &\leq \tau_2 f^T(k)Q_6f(k) - \frac{1}{\tau_2} \left[\sum_{m=1}^{\tau_2} f(k - m) \right]^T Q_6 \left[\sum_{m=1}^{\tau_2} f(k - m) \right], \end{aligned} \quad (3.17)$$

$$\begin{aligned} \Delta V_6(k) &\leq 2(1 + \hat{\tau} - \check{\tau})(f(k) - L_1y(k))^T \mathcal{S}_1y(k) \\ &\quad - 2(f(k - \tau_1(k)) - L_1y(k - \tau_1(k)))^T \mathcal{S}_1y(k - \tau_1(k)), \end{aligned} \quad (3.18)$$

$$\begin{aligned} \Delta V_7(k) &\leq 2(1 + \hat{\tau} - \check{\tau})(L_2y(k) - f(k))^T \mathcal{S}_2y(k) \\ &\quad - 2(L_2y(k - \tau_1(k)) - f(k - \tau_1(k)))^T \mathcal{S}_2y(k - \tau_1(k)), \end{aligned} \quad (3.19)$$

$$\begin{aligned} \Delta V_8(k) &= \eta^T(k)\mathcal{R}\eta(k) - \check{\tau} \sum_{j=k-\check{\tau}}^{k-1} \eta^T(j)\mathcal{R}_1\eta(j) \\ &\quad - (\bar{\tau} - \check{\tau}) \sum_{j=k-\bar{\tau}}^{k-\check{\tau}-1} \eta^T(j)\mathcal{R}_2\eta(j) - (\hat{\tau} - \bar{\tau}) \sum_{j=k-\hat{\tau}}^{k-\bar{\tau}-1} \eta^T(j)\mathcal{R}_3\eta(j). \end{aligned} \quad (3.20)$$

From the definition of $\eta(k)$ and (3.10), we have

$$\begin{aligned} \eta^T(k)\mathcal{R}\eta(k) &= y^T(k)(A - E)^T \mathcal{R}(A - E)y(k) + 2y^T(k)(A - E)^T \mathcal{R}Bf(k) \\ &\quad + 2y^T(k)(A - E)^T \mathcal{R}Cf(k - \tau_1(k)) + 2y^T(k)(A - E)^T \mathcal{R}D \sum_{m=1}^{\tau_2} f(k - m) \\ &\quad + f^T(k)B^T \mathcal{R}Bf(k) + 2f^T(k)B^T \mathcal{R}Cf(k - \tau_1(k)) \\ &\quad + 2f^T(k)B^T \mathcal{R}D \sum_{m=1}^{\tau_2} f(k - m) + f^T(k - \tau_1(k))C^T \mathcal{R}Cf(k - \tau_1(k)) \\ &\quad + 2f^T(k - \tau_1(k))C^T \mathcal{R}D \sum_{m=1}^{\tau_2} f(k - m) \\ &\quad + \left[\sum_{m=1}^{\tau_2} f(k - m) \right]^T D^T \mathcal{R}D \sum_{m=1}^{\tau_2} f(k - m). \end{aligned} \quad (3.21)$$

From Lemma 2.3, it can be shown that the following inequality holds:

$$\begin{aligned} -\check{\tau} \sum_{j=k-\check{\tau}}^{k-1} \eta^T(j)\mathcal{R}_1\eta(j) &\leq - \left[\sum_{j=k-\check{\tau}}^{k-1} \eta(j) \right]^T \mathcal{R}_1 \left[\sum_{j=k-\check{\tau}}^{k-1} \eta(j) \right] \\ &= \begin{bmatrix} y(k) \\ y(k - \check{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_1 & \mathcal{R}_1 \\ * & -\mathcal{R}_1 \end{bmatrix} \begin{bmatrix} y(k) \\ y(k - \check{\tau}) \end{bmatrix}. \end{aligned} \quad (3.22)$$

When $\check{\tau} \leq \tau_1(k) \leq \bar{\tau}$, let $a(k) = (\tau_1(k) - \check{\tau}) / (\bar{\tau} - \check{\tau})$. Then $0 \leq a(k) \leq 1$. It is easy to get that

$$\begin{aligned}
-(\bar{\tau} - \check{\tau}) \sum_{j=k-\bar{\tau}}^{k-\check{\tau}-1} \eta^T(j) \mathcal{R}_2 \eta(j) &= -(\bar{\tau} - \check{\tau}) \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta^T(j) \mathcal{R}_2 \eta(j) - (\bar{\tau} - \check{\tau}) \sum_{j=k-\tau_1(k)}^{k-\check{\tau}-1} \eta^T(j) \mathcal{R}_2 \eta(j) \\
&\leq -(\bar{\tau} - \tau_1(k)) \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta^T(j) \mathcal{R}_2 \eta(j) \\
&\quad - a(k) (\bar{\tau} - \tau_1(k)) \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta^T(j) \mathcal{R}_2 \eta(j) \\
&\quad - (1 - a(k)) (\tau_1(k) - \check{\tau}) \sum_{j=k-\tau_1(k)}^{k-\check{\tau}-1} \eta^T(j) \mathcal{R}_2 \eta(j) \\
&\quad - (\tau_1(k) - \check{\tau}) \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta^T(j) \mathcal{R}_2 \eta(j) \\
&\leq - \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta^T(j) \mathcal{R}_2 \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta(j) \\
&\quad - a(k) \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta^T(j) \mathcal{R}_2 \sum_{j=k-\bar{\tau}}^{k-\tau_1(k)-1} \eta(j) \\
&\quad - (1 - a(k)) \sum_{j=k-\tau_1(k)}^{k-\check{\tau}-1} \eta^T(j) \mathcal{R}_2 \sum_{j=k-\tau_1(k)}^{k-\check{\tau}-1} \eta(j) \\
&\quad - \sum_{j=k-\tau_1(k)}^{k-\check{\tau}-1} \eta^T(j) \mathcal{R}_2 \sum_{j=k-\tau_1(k)}^{k-\check{\tau}-1} \eta(j) \\
&= \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \bar{\tau}) \\ y(k - \check{\tau}) \end{bmatrix}^T \begin{bmatrix} -2\mathcal{R}_2 & \mathcal{R}_2 & \mathcal{R}_2 \\ * & -\mathcal{R}_2 & 0 \\ * & * & -\mathcal{R}_2 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \bar{\tau}) \\ y(k - \check{\tau}) \end{bmatrix} \\
&\quad + a(k) \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \bar{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_2 & \mathcal{R}_2 \\ * & -\mathcal{R}_2 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \bar{\tau}) \end{bmatrix} \\
&\quad + (1 - a(k)) \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \check{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_2 & \mathcal{R}_2 \\ * & -\mathcal{R}_2 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \check{\tau}) \end{bmatrix}, \tag{3.23}
\end{aligned}$$

$$-(\hat{\tau} - \bar{\tau}) \sum_{j=k-\bar{\tau}}^{k-\hat{\tau}-1} \eta^T(j) \mathcal{R}_3 \eta(j) \leq \begin{bmatrix} y(k - \bar{\tau}) \\ y(k - \hat{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_3 & \mathcal{R}_3 \\ * & -\mathcal{R}_3 \end{bmatrix} \begin{bmatrix} y(k - \bar{\tau}) \\ y(k - \hat{\tau}) \end{bmatrix}. \tag{3.24}$$

For positive diagonal matrices $\tau_1 > 0$ and $\tau_2 > 0$, we can get from assumption (H2) that

$$0 \leq \begin{bmatrix} y(k) \\ f(k) \end{bmatrix}^T \begin{bmatrix} -L_3\tau_1 & L_4\tau_1 \\ * & -\tau_1 \end{bmatrix} \begin{bmatrix} y(k) \\ f(k) \end{bmatrix}, \quad (3.25)$$

$$0 \leq \begin{bmatrix} y(k - \tau_1(k)) \\ f(k - \tau_1(k)) \end{bmatrix}^T \begin{bmatrix} -L_3\tau_2 & L_4\tau_2 \\ * & -\tau_2 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ f(k - \tau_1(k)) \end{bmatrix}. \quad (3.26)$$

Denoting

$$\alpha(k) = \begin{bmatrix} y^T(k), f^T(k), f^T(k - \tau_1(k)), \sum_{m=1}^{\tau_2} f^T(k - m), \\ y^T(k - \tau_1(k)), y^T(k - \bar{\tau}), y^T(k - \check{\tau}), y^T(k - \hat{\tau}) \end{bmatrix}^T, \quad (3.27)$$

it follows from (3.13)–(3.26) that

$$\begin{aligned} \Delta V(k) &= \sum_{i=1}^8 \Delta V_i(k) \\ &\leq y^T(k) [A\rho A - \rho + Q_1 + Q_2 + Q_3 \\ &\quad + (1 + \hat{\tau} - \check{\tau})(Q_4 - 2L_1S_1 + 2L_2S_2) \\ &\quad + (A - E)\mathcal{R}(A - E) - \mathcal{R}_1 - L_3\tau_1] y(k) \\ &\quad + 2y^T(k) [A\rho B + (1 + \hat{\tau} - \check{\tau})(S_1 - S_2) + (A - E)\mathcal{R}B + L_4\tau_1] f(k) \\ &\quad + 2y^T(k) [A\rho C + (A - E)\mathcal{R}C] f(k - \tau_1(k)) \\ &\quad + 2y^T(k) [A\rho D + (A - E)\mathcal{R}D] \sum_{m=1}^{\tau_2} f(k - m) \\ &\quad + f^T(k) [B^T\rho B + B^T\mathcal{R}B + (1 + \hat{\tau} - \check{\tau})Q_5 + \tau_2Q_6 - \tau_1] f(k) \\ &\quad + 2f^T(k) [B^T\rho C + B^T\mathcal{R}C] f(k - \tau_1(k)) \\ &\quad + 2f^T(k) [B^T\rho D + B^T\mathcal{R}D] \sum_{m=1}^{\tau_2} f(k - m) \\ &\quad + f^T(k - \tau_1(k)) [C^T\rho C + C^T\mathcal{R}C - Q_5 - \tau_2] f(k - \tau_1(k)) \\ &\quad + 2f^T(k - \tau_1(k)) [C^T\rho D + C^T\mathcal{R}D] \sum_{m=1}^{\tau_2} f(k - m) \end{aligned}$$

$$\begin{aligned}
& + \left[\sum_{m=1}^{\tau_2} f(k-m) \right]^T \left[D^T \rho D + D^T \mathcal{R} D - \frac{1}{\tau_2} Q_6 \right] \left[\sum_{m=1}^{\tau_2} f(k-m) \right] \\
& + 2y^T(k) \mathcal{R}_1 y^T(k - \check{\tau}) \\
& + 2f^T(k - \tau_1(k)) [S_2 - S_1 + L_4 \mathcal{T}_2] y(k - \tau_1(k)) \\
& + y^T(k - \tau_1(k)) [-Q_4 + 2L_1 S_1 - 2L_2 S_2 - L_3 \mathcal{T}_2 - 3\mathcal{R}_2] y(k - \tau_1(k)) \\
& + 2y^T(k - \tau_1(k)) [\mathcal{R}_2 + a(k) \mathcal{R}_2] y(k - \bar{\tau}) \\
& + 2y^T(k - \tau_1(k)) [\mathcal{R}_2 + (1 - a(k)) \mathcal{R}_2] y(k - \check{\tau}) \\
& - y^T(k - \bar{\tau}) [Q_1 + (1 + a(k)) \mathcal{R}_2 + \mathcal{R}_3] y(k - \bar{\tau}) \\
& + 2y^T(k - \bar{\tau}) \mathcal{R}_3 y(k - \hat{\tau}) \\
& - y^T(k - \check{\tau}) [Q_2 + \mathcal{R}_1 + \mathcal{R}_2 + (1 - a(k)) \mathcal{R}_2] y(k - \check{\tau}) \\
& - y^T(k - \hat{\tau}) [Q_3 + \mathcal{R}_3] y(k - \hat{\tau}) \\
& = \alpha^T(k) [a(k)(\Pi + \Omega_1) + (1 - a(k))(\Pi + \Omega_2)] \alpha(k).
\end{aligned} \tag{3.28}$$

When $\bar{\tau} \leq \tau_1(k) \leq \hat{\tau}$, let $b(k) = (\hat{\tau} - \tau_1(k)) / (\hat{\tau} - \bar{\tau})$. Then $0 \leq b(k) \leq 1$. In the similitude of the proof of inequality (3.23), we have

$$\begin{aligned}
-(\hat{\tau} - \bar{\tau}) \sum_{j=k-\bar{\tau}}^{k-\bar{\tau}-1} \eta^T(j) \mathcal{R}_3 \eta(j) & \leq \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \hat{\tau}) \\ y(k - \bar{\tau}) \end{bmatrix}^T \begin{bmatrix} -2\mathcal{R}_3 & \mathcal{R}_3 & \mathcal{R}_3 \\ * & -\mathcal{R}_3 & 0 \\ * & * & -\mathcal{R}_3 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \hat{\tau}) \\ y(k - \bar{\tau}) \end{bmatrix} \\
& + b(k) \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \hat{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_3 & \mathcal{R}_3 \\ * & -\mathcal{R}_3 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \hat{\tau}) \end{bmatrix} \\
& + (1 - b(k)) \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \bar{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_3 & \mathcal{R}_3 \\ * & -\mathcal{R}_3 \end{bmatrix} \begin{bmatrix} y(k - \tau_1(k)) \\ y(k - \bar{\tau}) \end{bmatrix}, \\
-(\bar{\tau} - \check{\tau}) \sum_{j=k-\bar{\tau}}^{k-\check{\tau}-1} \eta^T(j) \mathcal{R}_2 \eta(j) & \leq \begin{bmatrix} y(k - \check{\tau}) \\ y(k - \bar{\tau}) \end{bmatrix}^T \begin{bmatrix} -\mathcal{R}_2 & \mathcal{R}_2 \\ * & -\mathcal{R}_2 \end{bmatrix} \begin{bmatrix} y(k - \check{\tau}) \\ y(k - \bar{\tau}) \end{bmatrix}.
\end{aligned} \tag{3.29}$$

By using similar method in (3.28), it follows from (3.13) to (3.22), and (3.25) to (3.26), and (3.29) that

$$\Delta V(k) \leq \alpha^T(k) [b(k)(\Pi + \Omega_3) + (1 - b(k))(\Pi + \Omega_4)] \alpha(k). \tag{3.30}$$

Case 2 ($k = k_r$ ($r = 1, 2, \dots$)). Note that the inequalities from (3.13) to (3.26) except (3.13) and (3.21) hold for $k = k_r$. Calculating $\Delta V_1(k_r)$ and $\eta^T(k_r)\mathcal{R}\eta(k_r)$ along the system (3.10), we have

$$\begin{aligned}\Delta V_1(k_r) &= \mathbf{y}^T(k_r + 1)\mathcal{D}\mathbf{y}(k_r + 1) - \mathbf{y}^T(k_r)\mathcal{D}\mathbf{y}(k_r) \\ &= [\mathbf{y}(k_r) + \mathbf{u}_r(\mathbf{y}(k_r))]^T \mathcal{D}[\mathbf{y}(k_r) + \mathbf{u}_r(\mathbf{y}(k_r))] - \mathbf{y}^T(k_r)\mathcal{D}\mathbf{y}(k_r) \\ &\leq \mathbf{y}^T(k_r)(2H_1\mathcal{D} + H_2\mathcal{D}H_2)\mathbf{y}(k_r) \\ \eta^T(k_r)\mathcal{R}\eta(k_r) &= \mathbf{u}_r^T(\mathbf{y}(k_r))\mathcal{R}\mathbf{u}_r(\mathbf{y}(k_r)) \leq \mathbf{y}^T(k_r)H_2\mathcal{R}H_2\mathbf{y}(k_r).\end{aligned}\tag{3.31}$$

By using similar method in Case 1, we can obtain that

$$\Delta V(k) = \alpha^T(k)[a(k)(\Xi + \Omega_1) + (1 - a(k))(\Xi + \Omega_2)]\alpha(k)\tag{3.32}$$

or

$$\Delta V(k) = \alpha^T(k)[b(k)(\Xi + \Omega_3) + (1 - b(k))(\Xi + \Omega_4)]\alpha(k).\tag{3.33}$$

Combining the above discussions in Case 1 and 2, we obtain from (3.2), (3.3), (3.28), (3.30), (3.32), and (3.33) that

$$\Delta V(k) \leq -\gamma_1 \|\alpha(k)\|^2 \leq -\gamma_1 \|\mathbf{y}(k)\|^2,\tag{3.34}$$

where $\gamma_1 = \min_{i=1,2,3,4} \{\lambda_{\min}(-\Pi - \Omega_i), \lambda_{\min}(-\Xi - \Omega_i)\} > 0$.

From the definition of $V(k)$, it is easy to verify that

$$V(k) \leq \lambda_{\max}(\mathcal{D}) \|\mathbf{y}(k)\|^2 + \gamma_2 \sum_{j=k-\tau}^{k-1} \|\mathbf{y}(j)\|^2,\tag{3.35}$$

where

$$\begin{aligned}\gamma_2 &= \lambda_{\max}(\mathcal{Q}_1) + \lambda_{\max}(\mathcal{Q}_2) + \lambda_{\max}(\mathcal{Q}_3) + (1 + \hat{\tau} - \check{\tau}) \left[\lambda_{\max}(\mathcal{Q}_4) \right. \\ &\quad \left. + \lambda_{\max}(\mathcal{Q}_5)\lambda_{\max}(L^T L) + 4\lambda_{\max}(L\mathcal{S}_1) + 4\lambda_{\max}(L\mathcal{S}_2) \right] + \tau_2 \lambda_{\max}(\mathcal{Q}_6) \\ &\quad + 4\check{\tau}^2 \lambda_{\max}(\mathcal{R}_1) + 4(\bar{\tau} - \check{\tau})^2 \lambda_{\max}(\mathcal{R}_2) + 4(\hat{\tau} - \bar{\tau})^2 \lambda_{\max}(\mathcal{R}_3),\end{aligned}\tag{3.36}$$

where $L = \text{diag}(\max\{|\check{l}_1|, |\hat{l}_1|\}, \max\{|\check{l}_2|, |\hat{l}_2|\}, \dots, \max\{|\check{l}_n|, |\hat{l}_n|\})$.

For any scalar $\rho \geq 1$, it follows from (3.34) and (3.35) that

$$\begin{aligned} \rho^{i+1}V(i+1) - \rho^iV(i) &= \rho^{i+1}\Delta V(i) + \rho^i(\rho-1)V(i) \\ &\leq \left[\rho^i(\rho-1)\lambda_{\max}(\mathcal{D}) - \gamma_1\rho^{i+1} \right] \|y(i)\|^2 \\ &\quad + \gamma_2\rho^i(\rho-1) \sum_{j=i-\tau}^{i-1} \|y(j)\|^2. \end{aligned} \quad (3.37)$$

Summing up both sides of (3.37) from 0 to $k-1$ with respect to i , we have

$$\begin{aligned} \rho^kV(k) - V(0) &\leq [(\rho-1)\lambda_{\max}(\mathcal{D}) - \gamma_1\rho] \sum_{i=0}^{k-1} \rho^i \|y(i)\|^2 \\ &\quad + \gamma_2(\rho-1) \sum_{i=0}^{k-1} \sum_{j=i-\tau}^{i-1} \rho^i \|y(j)\|^2. \end{aligned} \quad (3.38)$$

Note that

$$\begin{aligned} \sum_{i=0}^{k-1} \sum_{j=i-\tau}^{i-1} \rho^i \|y(j)\|^2 &= \left(\sum_{j=-\tau}^{-1} \sum_{i=0}^{j+\tau} + \sum_{j=0}^{k-1-\tau} \sum_{i=j+1}^{j+\tau} + \sum_{j=k-\tau}^{k-2} \sum_{i=j+1}^{k-1} \right) \rho^i \|y(j)\|^2 \\ &\leq \tau\rho^\tau \sum_{j=-\tau}^{-1} \|y(j)\|^2 + \tau\rho^\tau \sum_{j=0}^{k-1-\tau} \rho^j \|y(j)\|^2 + \tau\rho^\tau \sum_{j=k-\tau}^{k-2} \rho^j \|y(j)\|^2 \\ &\leq \tau^2\rho^\tau \sup_{s \in N[-\tau,0]} \|y(s)\|^2 + \tau\rho^\tau \sum_{j=0}^{k-2} \rho^j \|y(j)\|^2. \end{aligned} \quad (3.39)$$

From (3.35), we have

$$V(0) \leq [\lambda_{\max}(\mathcal{D}) + \gamma_2\tau] \sup_{s \in N[-\tau,0]} \|y(s)\|^2. \quad (3.40)$$

It follows from (3.38)–(3.40) that

$$\rho^kV(k) \leq K_1(\rho) \sup_{s \in N[-\tau,0]} \|y(s)\|^2 + K_2(\rho) \sum_{j=0}^{k-1} \rho^j \|y(j)\|^2, \quad (3.41)$$

where

$$\begin{aligned} K_1(\rho) &= \lambda_{\max}(\mathcal{D}) + \gamma_2\tau + \gamma_2(\rho-1)\tau^2\rho^\tau, \\ K_2(\rho) &= (\rho-1)\lambda_{\max}(\mathcal{D}) - \gamma_1\rho + \gamma_2(\rho-1)\tau\rho^\tau. \end{aligned} \quad (3.42)$$

Since $K_2(1) < 0$, by the continuity of function $K_2(\rho)$, we can choose a scalar $\xi > 1$ such that $K_2(\xi) < 0$. Obviously, $K_1(\xi) > 0$. From (3.41), we obtain

$$\xi^k V(k) \leq K_1(\xi) \sup_{s \in N[-\tau, 0]} \|y(s)\|^2. \quad (3.43)$$

From the definition of $V(k)$, we have

$$V(k) \geq \lambda_{\min}(\mathcal{D}) \|y(k)\|^2. \quad (3.44)$$

Let $M = \sqrt{K_1(\xi)/\lambda_{\min}(\mathcal{D})}$, $\varepsilon = \sqrt{1/\xi}$, then $M > 0$, $0 < \varepsilon < 1$. It follows from (3.43) and (3.44) that

$$\|y(k)\| \leq M\varepsilon^k \sup_{s \in N[-\tau, 0]} \|y(s)\|. \quad (3.45)$$

That is

$$\|x(k, \phi) - x(k, \psi)\| \leq M\varepsilon^k \|\phi - \psi\|. \quad (3.46)$$

We can choose a positive integer N such that

$$M\varepsilon^{N\omega} \leq \frac{1}{4}. \quad (3.47)$$

Define a Poincaré mapping $\Gamma : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\Gamma(\phi) = -x^{(\omega)}(\phi). \quad (3.48)$$

Then, we can derive from (3.46) and (3.47) that

$$\|\Gamma^{(N)}(\phi) - \Gamma^{(N)}(\psi)\| \leq \frac{1}{4} \|\phi - \psi\|, \quad (3.49)$$

which shows that $\Gamma^{(N)}$ is a contraction mapping and therefore there exists a unique fixed point $\phi^* \in \mathcal{X}$ of $\Gamma^{(N)}$, which is also the unique fixed point of Γ such that

$$\Gamma(\phi^*) = \phi^*. \quad (3.50)$$

Therefore,

$$-x(\omega + k, \phi^*) = \phi^*(k), \quad \forall k \in N[-\tau, 0]. \quad (3.51)$$

Let $x(k, \phi^*)$ be the solution of system (2.2) through $(0, \phi^*)$. From assumption (H1) we know that $-x(k + \omega, \phi^*)$ is also a solution of system (2.2). It follows from (3.51) that $-x(k + \omega, \phi^*)$ is also through $(0, \phi^*)$. By the uniqueness of solution we can know

$$x(k, \phi^*) = -x(k + \omega, \phi^*), \quad (3.52)$$

for $k = 1, 2, \dots$, which indicates that $x(k, \phi^*)$ is exactly one ω -antiperiodic solution of system (2.2). To this end, it is easy to see that all other solutions converge exponentially to it as $k \rightarrow +\infty$. The proof is completed. \square

Remark 3.2. The conditions are dependent on both the lower bound and upper bound of delays. It has been shown that the delay-dependent stability conditions are generally less conservative than the delay-independent ones, especially when the size of the delay is small.

Remark 3.3. In this paper, the model includes both discrete and distributed delays simultaneously, and can be used to describe some well-known neural networks owing to its generality. In [38], only one kind of delay has been considered, which is a special case of neural networks with mixed delays. Furthermore, in [38], the activations were assumed to be bounded functions, while the boundedness condition is removed in this paper.

4. Examples

In this section, some examples and numerical simulations are provided to illustrate our results.

Example 4.1. Consider a discrete-time neural networks (2.2) with two neurons, where

$$\begin{aligned} A &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, & B &= \begin{bmatrix} 0.06 & 0.01 \\ -0.06 & 0.1 \end{bmatrix}, & C &= \begin{bmatrix} 0.1 & -0.05 \\ -0.05 & -0.02 \end{bmatrix}, & D &= \begin{bmatrix} -0.02 & 0.02 \\ -0.05 & 0.07 \end{bmatrix}, \\ g_1(u) &= \arctan(2u), & g_2(u) &= -\arctan(4u), & I_1(k) &= -3 \sin\left(\frac{k\pi}{8}\right), \\ I_2(k) &= 4 \cos\left(\frac{k\pi}{8}\right), & \tau_1(k) &= 3 + \sin\left(\frac{k\pi}{4}\right), & \tau_2 &= 2, \\ k_r &= 8r - 3, & e_{1r}(u) &= e_{2r}(u) = 0.08 \sin(u) - 0.68u, & r &= 1, 2, \dots \end{aligned} \quad (4.1)$$

It can be verified that assumptions (H1), (H2), and (H3) are satisfied with $\check{l}_1 = 0$, $\hat{l}_1 = 2$, $\check{l}_2 = -4$, $\hat{l}_2 = 0$, $\check{\tau} = 2$, $\hat{\tau} = 4$, $\omega = 8$, $\check{h}_1 = \check{h}_2 = -0.76$, $\hat{h}_1 = \hat{h}_2 = -0.6$. Thus, $L_1 = \text{diag}(0, -4)$, $L_2 = \text{diag}(2, 0)$, $L_3 = \text{diag}(0, 0)$, $L_4 = \text{diag}(1, -2)$, $H_1 = \text{diag}(-0.6, -0.6)$,

$H_2 = \text{diag}(0.76, 76)$. By the MATLAB Control Toolbox, we find a solution to the LMI in (3.2) as follows:

$$\begin{aligned}
\rho &= \begin{bmatrix} 3.5412 & -0.8274 \\ -0.8274 & 8.5025 \end{bmatrix}, & \mathcal{R}_1 &= \begin{bmatrix} 0.0254 & -0.0149 \\ -0.0149 & 0.0106 \end{bmatrix}, & \mathcal{R}_2 &= \begin{bmatrix} 0.1931 & -0.0932 \\ -0.0932 & 0.1155 \end{bmatrix}, \\
\mathcal{R}_3 &= \begin{bmatrix} 0.1941 & -0.0937 \\ -0.0937 & 0.1158 \end{bmatrix}, & Q_1 &= \begin{bmatrix} 0.1396 & -0.0553 \\ -0.0553 & 0.0807 \end{bmatrix}, & Q_2 &= \begin{bmatrix} 0.1762 & -0.0693 \\ -0.0693 & 0.0917 \end{bmatrix}, \\
Q_3 &= \begin{bmatrix} 0.1877 & -0.0771 \\ -0.0771 & 0.0986 \end{bmatrix}, & Q_4 &= \begin{bmatrix} 0.0230 & -0.0077 \\ -0.0077 & 0.0222 \end{bmatrix}, & Q_5 &= \begin{bmatrix} 0.1197 & 0.0554 \\ 0.0554 & 0.0274 \end{bmatrix}, \\
Q_6 &= \begin{bmatrix} 0.1370 & -0.0709 \\ -0.0709 & 0.1201 \end{bmatrix}, & \tau_1 &= \begin{bmatrix} 2.5687 & 0 \\ 0 & 0.8572 \end{bmatrix}, & \tau_2 &= \begin{bmatrix} 1.2461 & 0 \\ 0 & 0.0252 \end{bmatrix}, \\
S_1 &= \begin{bmatrix} 0.0056 & 0 \\ 0 & 0.0102 \end{bmatrix}, & S_2 &= \begin{bmatrix} 0.0756 & 0 \\ 0 & 0.0034 \end{bmatrix}.
\end{aligned} \tag{4.2}$$

Therefore, by Theorem 3.1, we know that system (2.2) with above given parameters has exactly one 8-antiperiodic solution and all other solutions of the system converge exponentially to it as $k \rightarrow +\infty$, which is further verified by the simulation given in Figure 1.

Example 4.2. Consider a discrete-time neural networks (2.2) with three neurons, where

$$\begin{aligned}
A &= \begin{bmatrix} 0.15 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, & B &= \begin{bmatrix} 0.05 & 0.04 & -0.04 \\ 0.08 & 0.04 & -0.04 \\ -0.04 & -0.08 & 0.04 \end{bmatrix}, \\
C &= \begin{bmatrix} -0.04 & -0.02 & 0.06 \\ 0.09 & -0.02 & 0.02 \\ 0.06 & 0.01 & 0.01 \end{bmatrix}, & D &= \begin{bmatrix} -0.02 & 0.05 & 0.02 \\ -0.05 & 0.02 & -0.01 \\ 0.02 & -0.05 & -0.02 \end{bmatrix}, \\
g_1(u) &= g_2(u) = g_3(u) = \sin(2u) - u, & I_1(k) &= -2 \sin\left(\frac{k\pi}{12}\right), \\
I_2(k) &= I_3(k) = \cos\left(\frac{k\pi}{12}\right), & \tau_1(k) &= 2 + \cos\left(\frac{k\pi}{6}\right), & \tau_2 &= 2, \\
k_r &= 12 \left\lfloor \frac{r-1}{2} \right\rfloor + 3(-1)^r + 7, & e_{1r}(u) &= e_{2r}(u) = e_{3r}(u) = -u, & r &= 1, 2, \dots
\end{aligned} \tag{4.3}$$

It can be verified that assumptions (H1), (H2), and (H3) are satisfied with $\check{l}_1 = \check{l}_2 = \check{l}_3 = -3$, $\hat{l}_1 = \hat{l}_2 = \hat{l}_3 = 1$, $\check{\tau} = 1$, $\hat{\tau} = 3$, $\omega = 12$, $\check{h}_1 = \check{h}_2 = \check{h}_3 = -1$, $\hat{h}_1 = \hat{h}_2 = \hat{h}_3 = -1$. Thus, $L_1 = \text{diag}(-3, -3, -3)$, $L_2 = \text{diag}(1, 1, 1)$, $L_3 = \text{diag}(-3, -3, -3)$, $L_4 = \text{diag}(-1, -1, -1)$,

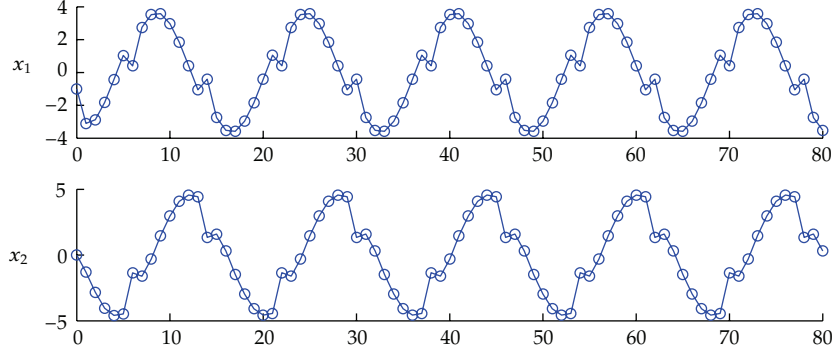


Figure 1: State responses of the discrete-time neural networks with initial conditions $(x_1(s), x_2(s))^T = (-1, 0)^T$, $s \in N[-4, 0]$.

$H_1 = \text{diag}(-1, -1, -1)$, $H_2 = \text{diag}(1, 1, 1)$. By the MATLAB Control Toolbox, we find a solution to the LMI in (3.3) as follows:

$$\begin{aligned}
 \rho &= \begin{bmatrix} 41.8020 & -4.3473 & 4.5690 \\ -4.3473 & 30.0757 & 0.2682 \\ 4.5690 & 0.2682 & 25.8887 \end{bmatrix}, & \mathcal{R}_1 &= \begin{bmatrix} 0.4005 & -0.2958 & 0.1172 \\ -0.2958 & 0.8118 & 0.2942 \\ 0.1172 & 0.2942 & 0.6825 \end{bmatrix}, \\
 \mathcal{R}_2 &= \begin{bmatrix} 0.5178 & -0.2326 & -0.1006 \\ -0.2326 & 0.6843 & 0.2146 \\ -0.1006 & 0.2146 & 0.6101 \end{bmatrix}, & \mathcal{R}_3 &= \begin{bmatrix} 0.4881 & -0.2381 & -0.1038 \\ -0.2381 & 0.7032 & 0.2240 \\ -0.1038 & 0.2240 & 0.6280 \end{bmatrix}, \\
 \mathcal{Q}_1 &= \begin{bmatrix} 1.0470 & -0.1515 & 0.4772 \\ -0.1515 & 0.6361 & -0.1655 \\ 0.4772 & -0.1655 & 1.0423 \end{bmatrix}, & \mathcal{Q}_2 &= \begin{bmatrix} 1.0644 & -0.1626 & 0.4402 \\ -0.1626 & 0.6310 & -0.1597 \\ 0.4402 & -0.1597 & 0.9820 \end{bmatrix}, \\
 \mathcal{Q}_3 &= \begin{bmatrix} 1.2250 & -0.1486 & 0.6129 \\ -0.1486 & 0.8285 & -0.1864 \\ 0.6129 & -0.1864 & 1.2586 \end{bmatrix}, & \mathcal{Q}_4 &= \begin{bmatrix} 1.7504 & -0.3124 & 0.7392 \\ -0.3124 & 0.3251 & -0.4540 \\ 0.7392 & -0.4540 & 0.9975 \end{bmatrix}, & (4.4) \\
 \mathcal{Q}_5 &= \begin{bmatrix} 0.2637 & 0.0305 & -0.0300 \\ 0.0305 & 0.0846 & -0.1291 \\ -0.0300 & -0.1291 & 0.2397 \end{bmatrix}, & \mathcal{Q}_6 &= \begin{bmatrix} 0.5803 & -0.1963 & 0.1481 \\ -0.1963 & 0.8232 & 0.3296 \\ 0.1481 & 0.3296 & 0.3495 \end{bmatrix}, \\
 \mathcal{T}_1 &= \begin{bmatrix} 3.4923 & 0 & 0 \\ 0 & 3.4632 & 0 \\ 0 & 0 & 2.3878 \end{bmatrix}, & \mathcal{T}_2 &= \begin{bmatrix} 0.9595 & 0 & 0 \\ 0 & 0.4567 & 0 \\ 0 & 0 & 0.3671 \end{bmatrix}, \\
 \mathcal{S}_1 &= \begin{bmatrix} 0.6575 & 0 & 0 \\ 0 & 0.4853 & 0 \\ 0 & 0 & 0.2337 \end{bmatrix}, & \mathcal{S}_2 &= \begin{bmatrix} 0.3709 & 0 & 0 \\ 0 & 0.0163 & 0 \\ 0 & 0 & 0.0507 \end{bmatrix}.
 \end{aligned}$$

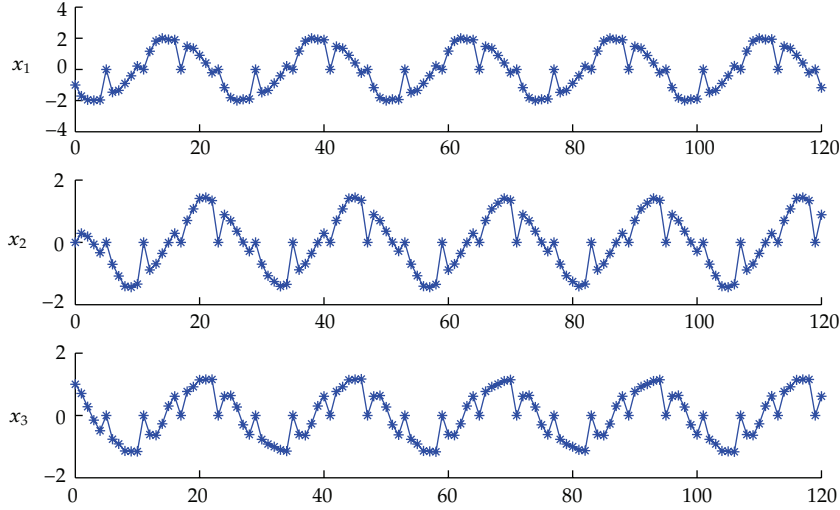


Figure 2: State responses of the discrete-time neural networks with initial conditions $(x_1(s), x_2(s), x_3(s))^T = (-1, 0, 1)^T, s \in N[-3, 0]$.

Therefore, by Theorem 3.1, we know that system (2.2) with the above-given parameters has exactly one 12-antiperiodic solution and all other solutions of the system converge exponentially to it as $k \rightarrow +\infty$, which is further verified by the simulation given in Figure 2.

5. Conclusions

In this paper, the discrete-time neural networks with mixed delays and impulses have been studied. A delay-dependent LMI criterion for the existence and global exponential stability of antiperiodic solutions has been established by constructing an appropriate Lyapunov-Krasovskii functional, and using the contraction mapping principle and the matrix inequality techniques. Moreover, two examples are given to illustrate the effectiveness of the results.

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