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Research Article

Existence of Unbounded Solutions for a Third-Order Boundary Value Problem on Infinite Intervals

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We generalize the unbounded upper and lower solution method to a third-order ordinary differential equation on the half line subject to the Sturm-Liouville boundary conditions. By using such techniques and the Schäuder fixed point theorem, some criteria are presented for the existence of solutions and positive ones to the problem discussed.

1. Introduction

Boundary value problems on infinite intervals, arising from the study of radially symmetric solutions of nonlinear elliptic equation [1], have received much attention in recent years. Because the infinite interval is noncompact, the discussion about BVPs on the half-line is more complicated. There have been many existence results for some boundary value problems of differential equations on the half line. The main methods are the extension of continuous solutions on the corresponding finite intervals under a diagonalization process, fixed point theorems in special Banach space or in special Fréchet space; see [1–12] and the references therein.

The method of upper and lower solutions is a powerful technique to deal with the existence of boundary value problems (BVPs). In many cases, when given one pair of well-ordered lower and upper solution, nonlinear BVPs always have at least one solution in the closed interval. To obtain this kind of result, we can employ topological degree theory, the monotone iterative technique, or critical theory. For details, we refer the reader to see [1–4, 7, 9, 12–14] and therein.

When the method of upper and lower solution is applied to the infinite interval problems, diagonalization process is always used; see [2, 3, 7]. For example, in [3], Agarwal and O'Regan discussed a Sturm-Liouville boundary value problem of second-order differential equation:

$$\begin{gathered}
\frac{1}{p(t)}\left(p(t)y'(t)\right)' = q(t)f\left(t, y(t)\right), \quad t \in (0,+\infty), \\
-a_0 y(0) + b_0 \lim_{t\to 0^+} p(t)y'(t) = c_0, \quad \text{or} \quad \lim_{t\to 0^+} p(t)y'(t) = 0, \\
y(t) \text{ bounded on } [0,+\infty), \quad \text{or} \quad \lim_{t\to+\infty} y(t) = 0,
\end{gathered} \tag{1.1}$$

where $a_0 > 0, b_0 \geq 0$. General existence criteria were obtained to guarantee the existence of bounded solutions. The methods used therein were based on a diagonalization arguments and existence results of appropriate boundary value problems on finite intervals.

In [12], Yan et al. investigated the boundary value problem

$$\begin{gathered}
y''(t) + \Phi(t)f\left(t, y(t), y'(t)\right) = 0, \quad t \in (0,+\infty), \\
ay(0) - by'(0) = y_0 \geq 0, \qquad \lim_{t\to+\infty} y'(t) = k > 0,
\end{gathered} \tag{1.2}$$

where $a > 0, b > 0$. By using the upper and lower solutions method and the fixed point theorem, the authors presented sufficient conditions for the existence of unbounded positive solutions. In [9], Lian and the coauthors discussed further the existence of the unbounded solutions.

There are many results of third-order boundary value problems on finite interval; see [14, 15] and the references therein. However, there has been few papers concerned with the upper and lower solutions technique for the boundary value problems of third-order differential equation on infinite intervals. In this paper, we aim to investigate a general Sturm-Liouville boundary value problem for third-order differential equation on the half line

$$\begin{gathered}
u'''(t) + \phi(t)f\left(t, u(t), u'(t), u''(t)\right) = 0, \quad t \in (0,+\infty), \\
u(0) = A, \qquad u'(0) - au''(0) = B, \qquad u''(+\infty) = C,
\end{gathered} \tag{1.3}$$

where $\phi : (0,+\infty) \to (0,+\infty)$, $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ are continuous, $a > 0$, $A, B, C \in \mathbb{R}$. The methods mainly depend on the unbounded upper and lower solutions method and topological degree theory. The nonlinear is admitted to involve in the high-order derivatives under the considerations of the Nagumo condition. The solutions obtained can be unbounded in this paper. The results obtained in this paper generalize those in [4].

2. Preliminaries

We present here some definitions and lemmas which are essential in the proof of the main results.

Definition 2.1. A function $\alpha \in C^2[0,+\infty) \cap C^3(0,+\infty)$ is called a lower solution of BVP (1.3) if

$$\begin{aligned} &\alpha'''(t) + \phi(t) f\left(t, \alpha(t), \alpha'(t), \alpha''(t)\right) \geq 0, \quad t \in (0,+\infty), \\ &\alpha(0) \leq A, \qquad \alpha'(0) - a\alpha''(0) \leq B, \qquad \alpha''(+\infty) < C. \end{aligned} \tag{2.1}$$

Similarly, a function $\beta \in C^2[0,+\infty) \cap C^3(0,+\infty)$ is called an upper solution of BVP (1.3) if

$$\begin{aligned} &\beta'''(t) + \phi(t) f\left(t, \beta(t), \beta'(t), \beta''(t)\right) \leq 0, \quad t \in (0,+\infty), \\ &\beta(0) \geq A, \qquad \beta'(0) - a\beta''(0) \geq B, \qquad \beta''(+\infty) > C. \end{aligned} \tag{2.2}$$

Definition 2.2. Given a positive function $\phi \in C(0,+\infty)$ and a pair of functions $\alpha, \beta \in C^1[0,+\infty)$ satisfying $\alpha(0) \leq \beta(0)$ and $\alpha'(t) \leq \beta'(t), t \in [0,+\infty)$; a function $f : [0,+\infty) \times R^3 \to R$ is said to satisfy the Nagumo condition with respect to the pair of functions α, β, if there exist positive functions $\psi, h \in C[0,+\infty)$ satisfying $\int_0^{+\infty} \psi(s)\phi(s)ds < +\infty, \int^{+\infty} s/h(s)ds = +\infty$ such that

$$\left| f(t,x,y,z) \right| \leq \psi(t) h(|z|) \tag{2.3}$$

holds for all $0 \leq t < +\infty, \alpha(t) \leq x \leq \beta(t),\ \alpha'(t) \leq y \leq \beta'(t)$, and $z \in \mathbb{R}$.

Let $v_0(t) = 1 + t^2, v_1(t) = 1 + t, v_2(t) = 1$ and consider the space X defined by

$$\mathrm{X} = \left\{ x \in C^2[0,+\infty), \lim_{t \to +\infty} \frac{x^{(i)}(t)}{v_i(t)} \text{ exist, } i = 0,1,2 \right\}, \tag{2.4}$$

with the norm $\|x\| = \max\{\|x\|_0, \|x'\|_1, \|x''\|_2\}$, where $\|x\|_i = \sup_{t\in[0,+\infty)} |x(t)/v_i(t)|$. By the standard arguments, we can prove that $(\mathrm{X}, \|\cdot\|)$ is a Banach space.

Lemma 2.3. *If $e \in L^1[0,+\infty)$, then the BVP of third-order linear differential equation*

$$\begin{aligned} &u'''(t) + e(t) = 0, \quad t \in (0,+\infty), \\ &u(0) = A, \qquad u'(0) - au''(0) = B, \qquad u''(+\infty) = C, \end{aligned} \tag{2.5}$$

has a unique solution in X. Moreover this solution can be expressed as

$$u(t) = p(t) + \int_0^{+\infty} G(t,s) e(s) ds, \tag{2.6}$$

where

$$\begin{aligned} &p(t) = A + (aC + B)t + \frac{C}{2}t^2, \\ &G(t,s) = \begin{cases} at + st - \dfrac{1}{2}s^2, & 0 \leq s \leq t < +\infty, \\ \dfrac{1}{2}t^2 + at, & 0 \leq t \leq s < +\infty. \end{cases} \end{aligned} \tag{2.7}$$

Proof. It is easy to verify that (2.6) satisfies BVP (2.5). Now we show the uniqueness. Suppose u is a solution of (2.5). Let $v = u'$, then we have

$$\begin{gathered} v''(t) + e(t) = 0, \quad t \in (0, +\infty), \\ v(0) - av'(0) = B, \quad v'(+\infty) = C. \end{gathered} \tag{2.8}$$

By a direct calculation, we obtain the general solution of the above equation:

$$\begin{aligned} v(t) &= c_1 + c_2 t + \int_0^t \int_\tau^{+\infty} e(s)\,ds\,d\tau \\ &= c_1 + c_2 t + \int_0^t s e(s)\,ds + t \int_t^{+\infty} e(s)\,ds. \end{aligned} \tag{2.9}$$

Substituting this to the boundary condition, we arrive at

$$\begin{gathered} c_1 = aC + B + a \int_0^{+\infty} e(s)\,ds, \\ c_2 = C. \end{gathered} \tag{2.10}$$

Therefore, (2.8) has a unique solution

$$v(t) = aC + B + Ct + \int_0^{+\infty} g(t,s) e(s)\,ds, \tag{2.11}$$

where

$$g(t,s) = \begin{cases} a + s, & 0 \le s \le t < +\infty, \\ a + t, & 0 \le t \le s < +\infty. \end{cases} \tag{2.12}$$

Furthermore, $u' = v, u(0) = A$, so

$$\begin{aligned} u(t) &= u(0) + \int_0^t v(\tau)\,d\tau \\ &= A + \int_0^t \left[aC + B + C\tau + \int_0^{+\infty} g(\tau,s) e(s)\,ds \right] d\tau \\ &= p(t) + \int_0^{+\infty} \left[\int_0^t g(\tau,s)\,d\tau \right] e(s)\,ds \\ &= p(t) + \int_0^{+\infty} G(t,s) e(s)\,ds. \end{aligned} \tag{2.13}$$

The proof is complete. □

Theorem 2.4 (see [1]). *Let $M \subset C_\infty = \{x \in C[0,+\infty), \lim_{t\to+\infty} x(t)$ exists$\}$. Then M is relatively compact if the following conditions hold:*

(a) *all functions from M are uniformly bounded;*

(b) *all functions from M are equicontinuous on any compact interval of $[0,+\infty)$;*

(c) *all functions from M are equiconvergent at infinity; that is, for any given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $|f(t) - f(+\infty)| < \epsilon$, for all $t > T$ and $f \in M$.*

From the above results, we can obtain the following general criteria for the relative compactness of subsets in $C[0,+\infty)$.

Theorem 2.5. *Given $n+1$ continuous functions ρ_i satisfying $\rho_i \geq \varepsilon > 0, i = 0,1,\ldots,n$ with ε a positive constant. Let $M \subset C_\infty^n = \{x \in C^n[0,+\infty),\ \lim_{t\to+\infty}\rho_i(t)x^{(i)}(t)$ exists, $i = 0,1,2,\ldots,n\}$. Then M is relatively compact if the following conditions hold:*

(a) *all functions from M are uniformly bounded;*

(b) *the functions from $\{y_i : y_i = \rho_i x^{(i)}, x \in M\}$ are equicontinuous on any compact interval of $[0,+\infty)$, $i = 0,1,2,\ldots,n$;*

(c) *the functions from $\{y_i : y_i = \rho_i x^{(i)}, x \in M\}$ are equiconvergent at infinity, $i = 0,1,2,\ldots,n$.*

Proof. Set $M_i = \{y_i : y_i = \rho_i x^{(i)}, x^{(i)} \in M\}$, then $M_i \subset C_\infty, i = 0,1,2\ldots,n$. From conditions (a)–(c), we have M_i is relatively compact in C_∞. Therefore, for any sequence $\{y_{i,m}\}_{m=1}^{\infty} \subset M_i$, it has a convergent subsequence. Without loss of generality, we denote it this sequence. Then there exists $y_{i,0} \in M_i$ such that

$$y_{i,m} = \rho_i x_m^{(i)} \longrightarrow y_{i,0}, \quad m \longrightarrow +\infty,\ i = 0,1,2,\ldots,n. \tag{2.14}$$

Set $x_{i,0} = (1/\rho_i)\ y_{i,0}$, then $x_m^{(i)} \to x_{i,0}, i = 0,1,2,\ldots,n$. Noticing that all functions from M are uniformly continuous, we can obtain that $x_{i,0} = x_{0,0}^{(i)}, i = 1,2,\ldots,n$. So M is relatively compact. □

3. Main Results

In this section, we present the existence criteria for the existence of solutions and positive solutions of BVP (1.3). We first cite conditions (H_1) and (H_2) here.

(H_1) :

(1) BVP (1.3) has a pair of upper and lower solutions β, α in X with $\alpha'(t) \leq \beta'(t), t \in [0,+\infty)$;

(2) $f \in C([0,+\infty) \times \mathbb{R}^3, \mathbb{R})$ satisfies the Nagumo condition with respect to α and β.

(H_2) : For any $0 \leq t < +\infty, \alpha'(t) \leq y \leq \beta'(t)$ and $z \in \mathbb{R}$, it holds

$$f(t,\alpha(t),y,z) \leq f(t,x,y,z) \leq f(t,\beta(t),y,z), \quad \text{as } \alpha(t) \leq x \leq \beta(t). \tag{3.1}$$

Lemma 3.1. *Suppose condition (H_1) holds. And suppose further that the following condition holds:*

(H_3) *there exists a constant* $\gamma > 1$ *such that* $\sup_{0\leq t<+\infty}(1+t)^{\gamma}\phi(t)\varphi(t) < +\infty$.

If u *is a solution of* (1.3) *satisfying*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \alpha'(t) \leq u'(t) \leq \beta'(t), \quad t \in [0,+\infty), \tag{3.2}$$

then there exists a constant $R > 0$ *(without relations to* u*) such that* $\|u''\|_2 \leq R$.

Proof. Let $\delta > 0$ and $R > C$,

$$\eta \geq \max\left\{\sup_{t\in[\delta,+\infty)} \frac{\beta'(t)-\alpha'(0)}{t}, \sup_{t\in[\delta,+\infty)} \frac{\beta'(0)-\alpha'(t)}{t}\right\} \tag{3.3}$$

such that

$$\int_{\eta}^{R} \frac{s}{h(s)} ds \geq M\left(\sup_{t\in[0,+\infty)} \frac{\beta'(t)}{(1+t)^{\gamma}} - \inf_{t\in[0,+\infty)} \frac{\alpha'(t)}{(1+t)^{\gamma}} + \frac{\gamma}{\gamma-1} \cdot \sup_{t\in[0,+\infty)} \frac{\beta'(t)}{1+t}\right), \tag{3.4}$$

where C is the nonhomogeneous boundary value, $M = \sup_{0\leq t<+\infty}(1+t)^{\gamma}\phi(t)\varphi(t)$, then $|u''(t)| \leq R, t \in [0,+\infty)$. If it is untrue, we have the following three cases. □

Case 1. Consider

$$|u''(t)| > \eta, \quad \forall t \in [0,+\infty). \tag{3.5}$$

Without loss of generality, we suppose $u''(t) > \eta, t \in [0,+\infty)$. While for any $T \geq \delta$,

$$\frac{\beta'(T)-\alpha'(0)}{T} \geq \frac{u'(T)-u'(0)}{T} = \frac{1}{T}\int_0^T u''(s)ds > \eta \geq \frac{\beta'(T)-\alpha'(0)}{T}, \tag{3.6}$$

which is a contraction. So there must exist $t_0 \in [0,+\infty)$ such that $|u''(t_0)| \leq \eta$.

Case 2. Consider

$$\left|u''(t)\right| \leq \eta, \quad \forall t \in [0, +\infty). \tag{3.7}$$

Just take $R = \eta$ and we can complete the proof.

Case 3. There exists $[t_1, t_2] \subset [0, +\infty)$ such that $|u''(t_1)| = \eta, |u''(t)| > \eta, t \in (t_1, t_2]$ or $|u''(t_2)| = \eta, |u''(t)| > \eta, t \in [t_1, t_2)$.

Suppose that $u''(t_1) = \eta, u''(t) > \eta, t \in (t_1, t_2]$. Obviously,

$$\begin{aligned}
\int_{u''(t_1)}^{u''(t_2)} \frac{s}{h(s)} ds &= \int_{t_1}^{t_2} \frac{u''(s)}{h(u''(s))} u'''(s) ds \\
&= \int_{t_1}^{t_2} \frac{-\phi(s) f(s, u(s), u'(s), u''(s)) u''(s)}{h(u''(s))} ds \\
&\leq \int_{t_1}^{t_2} u''(s)\phi(s)\psi(s) ds \leq M \int_{t_1}^{t_2} \frac{u''(s)}{(1+s)^{\gamma}} ds \\
&= M\left(\int_{t_1}^{t_2} \left(\frac{u'(s)}{(1+s)^{\gamma}}\right)' ds + \int_{t_1}^{t_2} \frac{\gamma u'(s)}{(1+s)^{1+\gamma}} ds\right) \\
&\leq M\left(\sup_{t\in[0,+\infty)} \frac{\beta'(t)}{(1+t)^{\gamma}} - \inf_{t\in[0,+\infty)} \frac{\alpha'(t)}{(1+t)^{\gamma}} + \sup_{t\in[0,+\infty)} \frac{\beta'(t)}{1+t} \int_0^{+\infty} \frac{\gamma}{(1+s)^{\gamma}} ds\right) \\
&\leq \int_{\eta}^{R} \frac{s}{h(s)} ds
\end{aligned} \tag{3.8}$$

concludes that $u''(t_2) \leq R$. For t_1 and t_2 are arbitrary, we have $u''(t) \leq \max\{R, \eta\} = R, t \in [0, +\infty)$.

Similarly if $u''(t_1) = -\eta, u''(t) < -\eta, t \in (t_1, t_2]$, we can also obtain that $u''(t) > -R, t \in [0, +\infty)$.

Thus there exists $R > 0$, just related with α, β, and h, such that $\|u''\|_2 \leq R$.

Remark 3.2. Condition (H_3) is necessary for an a priori estimation of u'' in Lemma 3.1. Because the upper and lower solutions are in X, $\beta'(t)$ and $\alpha'(t)$ are at most linearly increasing, especially at infinity. Otherwise, $\sup_{t\in[0,+\infty)}\alpha'(t)$ and $\sup_{t\in[0,+\infty)}\beta'(t)$ may be equal to infinity.

Theorem 3.3. *Suppose $\phi \in L^1[0, +\infty)$ and the conditions (H_1)–(H_3) hold. Then BVP* (1.3) *has at least one solution $u \in C^2[0, +\infty) \cap C^3(0, +\infty)$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \alpha'(t) \leq u'(t) \leq \beta'(t), \quad t \in [0, +\infty). \tag{3.9}$$

Proof. Let $R > 0$ be the same definition in Lemma 3.1 and consider the boundary value problem

$$u'''(t) + \phi(t) f^*(t, u(t), u'(t), u''(t)) = 0, \quad t \in (0, +\infty),$$
$$u(0) = A, \qquad u'(0) - au''(0) = B, \qquad u''(+\infty) = C, \tag{3.10}$$

where

$$f^*(t,x,y,z) = \begin{cases} F_R(t,x,\alpha'(t),z) + \dfrac{y-\alpha'(t)}{1+|y-\alpha'(t)|}, & y < \alpha'(t), \\ F_R(t,x,y,z), & \alpha'(t) \le y \le \beta'(t), \\ F_R(t,x,\beta'(t),z) - \dfrac{y-\beta'(t)}{1+|y-\beta'(t)|}, & y > \beta'(t), \end{cases}$$
$$F_R(t,x,y,z) = \begin{cases} f_R(t,\alpha(t),y,z), & x < \alpha(t), \\ f_R(t,x,y,z), & \alpha(t) \le x \le \beta(t), \\ f_R(t,\beta(t),y,z), & x > \beta(t), \end{cases} \tag{3.11}$$
$$f_R(t,x,y,z) = \begin{cases} f(t,x,y,-R), & z < -R, \\ f(t,x,y,z), & -R \le z \le R, \\ f(t,x,y,R), & z > R. \end{cases}$$

Firstly we prove that BVP (3.10) has at least one solution u. To this end, define the operator $T : X \to X$ by

$$(Tu)(t) = p(t) + \int_0^{+\infty} G(t,s)\phi(s) f^*(s, u(s), u'(s), u''(s))\,ds. \tag{3.12}$$

By Lemma 2.3, we can see that the fixed points of T coincide with the solutions of BVP (3.10). So it is enough to prove that T has a fixed point.

We claim that $T : X \to X$ is completely continuous.

(1) $T : X \to X$ is well defined. For any $u \in X$, $\|u\| < +\infty$ and it holds

$$\int_0^{+\infty} \phi(s) f^*(s, u(s), u'(s), u''(s))\,ds \le \int_0^{+\infty} \phi(s)(H_0\varphi(s) + 1)\,ds < +\infty, \tag{3.13}$$

where $H_0 = \max_{0\leq s\leq \|u\|} h(s)$. By the Lebesgue-dominated convergence theorem, we have

$$\begin{aligned}
\lim_{t\to+\infty}\frac{(Tu)(t)}{v_0(t)} &= \lim_{t\to+\infty}\left(\frac{l(t)}{v_0(t)}+\int_0^{+\infty}\frac{G(t,s)}{v_0(t)}\phi(s)f^*(s,u(s),u'(s),u''(s))ds\right)\\
&= \frac{C}{2}+\frac{1}{2}\int_0^{+\infty}\phi(s)f^*(s,u(s),u'(s),u''(s))ds<+\infty,\\
\lim_{t\to+\infty}\frac{(Tu)'(t)}{v_1(t)} &= \lim_{t\to+\infty}\left(\frac{(aC+B)+Ct}{v_1(t)}+\int_0^{+\infty}\frac{g(t,s)}{v_1(t)}\phi(s)f^*(s,u(s),u'(s),u''(s))ds\right)\\
&= C+\int_0^{+\infty}\phi(s)f^*(s,u(s),u'(s),u''(s))ds<+\infty,\\
\lim_{t\to+\infty}(Tu)''(t) &= \lim_{t\to+\infty}\left(C+\int_t^{+\infty}\phi(s)f^*(s,u(s),u'(s),u''(s))ds\right)\\
&= C<+\infty,
\end{aligned}\tag{3.14}$$

so $Tu \in X$.

(2) $T : X \to X$ is continuous. For any convergent sequence $u_n \to u$ in X, there exists $r_1 > 0$ such that $\sup_{n\in N}\|u_n\| \leq r_1$. Similarly, we have

$$\begin{aligned}
\|Tu_n - Tu\| &= \max\left\{\|Tu_n-Tu\|_0, \left\|(Tu_n)'-(Tu)'\right\|_1, \left\|(Tu_n)''-(Tu)''\right\|_2\right\}\\
&\leq \int_0^{+\infty}\max\left\{\sup_{0\leq t<+\infty}\left|\frac{G(t,s)}{v_0(t)}\right|, \sup_{0\leq t<+\infty}\left|\frac{g(t,s)}{v_1(t)}\right|,1\right\}\\
&\quad\cdot\phi(s)\left|f^*(s,u_n(s),u'_n(s),u''_n(s))-f^*(s,u(s),u'(s),u''(s))\right|ds\\
&\leq \int_0^{+\infty}\phi(s)\left|f^*(s,u_n(s),u'_n(s),u''_n(s))-f^*(s,u(s),u'(s),u''(s))\right|ds\\
&\longrightarrow 0, \quad \text{as } n\longrightarrow+\infty,
\end{aligned}\tag{3.15}$$

so $T : X \to X$ is continuous.

(3) $T : X \to X$ is compact. Let B be any bounded subset of X, then there exists $r > 0$ such that $\|u\| \leq r$, for all $u \in B$. For any $u \in B$, one has

$$\begin{aligned}
\|Tu\| &= \max\left\{\|Tu\|_0, \left\|(Tu)'\right\|_1, \left\|(Tu)''\right\|_2\right\}\\
&\leq \int_0^{+\infty}\phi(s)\left|f^*(s,u(s),u'(s),u''(s))\right|ds\\
&\leq \int_0^{+\infty}\phi(s)\left(H_r\psi(s)+1\right)ds<+\infty,
\end{aligned}\tag{3.16}$$

where $H_r = \max_{0\le s\le r} h(s)$, so TB is uniformly bounded. Meanwhile, for any $T>0$, if $t_1, t_2 \in [0,T]$, we have

$$
\begin{aligned}
\left|\frac{Tu(t_1)}{v_0(t_1)} - \frac{Tu(t_2)}{v_0(t_2)}\right| &= \left|\int_0^{+\infty}\left(\frac{G(t_1,s)}{v_0(t_1)} - \frac{G(t_2,s)}{v_0(t_2)}\right)\phi(s) f^*(s,u(s),u'(s),u''(s))ds\right| \\
&\le \int_0^{+\infty}\left|\frac{G(t_1,s)}{v_0(t_1)} - \frac{G(t_2,s)}{v_0(t_2)}\right|\phi(s)\left(H_r\psi(s)+1\right)ds \\
&\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2, \\
\left|\frac{(Tu)'(t_1)}{v_1(t_1)} - \frac{(Tu)'(t_2)}{v_1(t_1)}\right| &= \left|\int_0^{+\infty}\left(\frac{g(t_1,s)}{v_1(t_1)} - \frac{g(t_2,s)}{v_1(t_2)}\right)\phi(s) f^*(s,u(s),u'(s),u''(s))ds\right| \\
&\le \int_0^{+\infty}\left|\frac{g(t_1,s)}{v_1(t_1)} - \frac{g(t_2,s)}{v_1(t_2)}\right|\phi(s)\left(H_r\psi(s)+1\right)ds \\
&\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2, \\
\left|\frac{(Tu)''(t_1)}{v_2(t_1)} - \frac{(Tu)''(t_2)}{v_2(t_1)}\right| &= \left|\int_{t_1}^{t_2}\phi(s) f^*(s,u(s),u'(s),u''(s))ds\right| \\
&\le \int_{t_1}^{t_2}\phi(s)\left(H_r\psi(s)+1\right)ds \\
&\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2;
\end{aligned}
\tag{3.17}
$$

that is, TB is equicontinuous. From Theorem 2.5, we can see that if TB is equiconvergent at infinity, then TB is relatively compact. In fact,

$$
\begin{aligned}
&\left|\frac{Tu(t)}{v_0(t)} - \lim_{t\to+\infty}\frac{Tu(t)}{v_0(t)}\right| \\
&\quad = \left|\frac{l(t)}{v_0(t)} - \frac{C}{2} + \int_0^{+\infty}\left(\frac{G(t,s)}{v_0(t)} - \frac{1}{2}\right)\phi(s) f^*(s,u(s),u'(s),u''(s))ds\right| \\
&\quad \le \left|\frac{l(t)}{v_0(t)} - \frac{C}{2}\right| + \int_0^{+\infty}\left|\frac{G(t,s)}{v_0(t)} - \frac{1}{2}\right|\phi(s)\left(H_r\psi(s)+1\right)ds \\
&\quad \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty,
\end{aligned}
$$

$$\begin{aligned}
&\left|\frac{(Tu)'(t)}{v_1(t)} - \lim_{t\to+\infty}\frac{(Tu)'(t)}{v_1(t)}\right| \\
&\quad = \left|\frac{(aC+B)+Ct}{v_1(t)} - C + \int_0^{+\infty}\left(\frac{g(t,s)}{v_1(t)} - 1\right)\phi(s)f^*(s,u(s),u'(s),u''(s))ds\right| \\
&\quad \le \left|\frac{(aC+B)+Ct}{v_1(t)} - C\right| + \int_0^{+\infty}\left|\frac{g(t,s)}{v_1(t)} - 1\right|\phi(s)\left(H_r\psi(s)+1\right)ds \\
&\quad \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \\
&\left|\frac{(Tu)''(t)}{v_2(t)}\right| \\
&\quad = \left|\int_t^{+\infty}\phi(s)f^*(s,u(s),u'(s),u''(s))ds\right| \\
&\quad \le \int_t^{+\infty}\phi(s)\left(H_r\psi(s)+1\right)ds \\
&\quad \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty.
\end{aligned} \tag{3.18}$$

Then we can obtain that $T : X \to X$ is completely continuous.

By the Schäuder fixed point theorem, T has at least one fixed point $u \in X$. Next we will prove u satisfying $\alpha'(t) \le u'(t) \le \beta'(t), t \in [0,+\infty)$. If $u'(t) \le \beta'(t), t \in [0,+\infty)$ does not hold, then,

$$\sup_{0\le t<+\infty}\left(u'(t) - \beta'(t)\right) > 0. \tag{3.19}$$

Because $u''(+\infty) - \beta''(+\infty) < 0$, so there are two cases. □

Case 1. Consider

$$\lim_{t\to 0^+}\left(u'(t) - \beta'(t)\right) = \sup_{0\le t<+\infty}\left(u'(t) - \beta'(t)\right) > 0. \tag{3.20}$$

Easily, $u''(0^+) - \beta''(0^+) \le 0$. While by the boundary condition, we have

$$a\left(u''(0) - \beta''(0)\right) \ge u'(0) - \beta'(0) > 0, \tag{3.21}$$

which is a contraction.

Case 2. There exists $t^* \in (0,+\infty)$ such that

$$\left(u'(t^*) - \beta'(t^*)\right) = \sup_{0\le t<+\infty}\left(u'(t) - \beta'(t)\right) > 0. \tag{3.22}$$

So, $u''(t^*) - \beta''(t^*) = 0, u'''(t^*) - \beta'''(t^*) \le 0$. Unfortunately,

$$\begin{aligned} u'''(t^*) - \beta'''(t^*) &\ge \phi(t^*)\big(f(t^*, \beta(t^*), \beta'(t^*), \beta''(t^*)) - f^*(t^*, u(t^*), u'(t^*), u''(t^*))\big) \\ &= \phi(t^*)\big(f(t^*, \beta(t^*), \beta'(t^*), \beta''(t^*)) - f^*(t^*, u(t^*), \beta'(t^*), \beta''(t^*))\big) \\ &\quad + \phi(t^*)\frac{u'(t^*) - \beta'(t^*)}{1 + |u'(t^*) - \beta'(t^*)|} \\ &\ge \phi(t^*)\frac{u'(t^*) - \beta'(t^*)}{1 + |u'(t^*) - \beta'(t^*)|} \\ &> 0. \end{aligned} \tag{3.23}$$

Consequently, $u'(t) \le \beta'(t)$ holds for all $t \in [0, +\infty)$. Similarly, we can show that $\alpha'(t) \le u'(t)$ for all $t \in [0, +\infty)$. Noticing that $\alpha(0) \le A \le \beta(0)$, from the inequality $\alpha'(t) \le u'(t) \le \beta'(t)$, we can obtain that $\alpha(t) \le u(t) \le \beta(t)$. Lemma 3.1 guarantee that $\|u''\|_\infty R$. So,

$$u'''(t) = -f^*(t, u(t), u'(t), u''(t)) = -f(t, u(t), u'(t), u''(t)); \tag{3.24}$$

that is, u is a solution of BVP (1.3).

Remark 3.4. For finite interval problem, it is sharp to define the lower and upper solutions satisfying $\alpha''(b) \le C$ and $\beta''(b) \ge C$; see [15].

If $f : [0, +\infty)^4 \to [0, +\infty)$, we can establish a criteria for the existence of positive solutions.

Theorem 3.5. *Let $f : [0, +\infty)^4 \to [0, +\infty)$ be continuous and $\phi \in L^1[0, +\infty)$. Suppose the condition (H_2) holds and the following conditions hold.*

(P_1) *BVP* (1.3) *has a pair of positive upper and lower solutions $\alpha, \beta \in X$ satisfying*

$$\alpha'(t) \le \beta'(t), \quad t \in [0, +\infty). \tag{3.25}$$

(P_2) *For any $r > 0$, there exists φ_r satisfying $\int_0^{+\infty} \phi(s)\varphi_r(s)ds < +\infty$ such that*

$$f(t, x, y, z) \le \varphi_r(t) \tag{3.26}$$

holds for all $t \in [0, +\infty), \alpha(t) \le x \le \beta(t), \alpha'(t) \le y \le \beta'(t)$, and $0 \le z \le r$.

Then BVP (1.3) *with $A, B, C \ge 0$ has at least one solution such that*

$$\alpha(t) \le u(t) \le \beta(t), \quad \alpha'(t) \le u'(t) \le \beta'(t), \quad t \in [0, +\infty). \tag{3.27}$$

Proof. Choose $R = (1/a)(B + \beta'(0))$ and consider the boundary value problem (3.10) except f_R substituting by

$$f_R(t,x,y,z) = \begin{cases} f(t,x,y,0), & z < 0, \\ f(t,x,y,z), & 0 \le z \le R, \\ f(t,x,y,R), & z > R. \end{cases} \tag{3.28}$$

Similarly, we can obtain that (3.10) has at least one solution u satisfying $\alpha(t) \le u(t) \le \beta(t)$ and $\alpha'(t) \le u'(t) \le \beta'(t), t \in [0,+\infty)$. Because

$$u'''(t) = -\phi(t) f^*(t, u(t), u'(t), u''(t)) \le 0 \tag{3.29}$$

and $u''(+\infty) = C \ge 0$, we have

$$0 \le u''(t) \le u''(0) = \frac{1}{a}(B + u'(0)) \le R. \tag{3.30}$$

Consequently, the solution u is a positive solution of (1.3). □

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