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Research Article

Global Attractors in $H^1(\mathbb{R}^N)$ for Nonclassical Diffusion Equations

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We study the existence of global attractors for nonclassical diffusion equations in $H^1(\mathbb{R}^N)$. The nonlinearity satisfies the arbitrary order polynomial growth conditions.

1. Introduction

In this paper, we investigate the long-time behavior of the solutions for the following nonclassical diffusion equations:

$$u_t - \Delta u_t - \Delta u + f(x,u) = g(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

with the initial data

$$u(x,0) = u_0, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $g(x) \in L^2(\mathbb{R}^N)$, and the nonlinearity $f(x,u) = f_1(u) + a(x)f_2(u)$ satisfies

(F_1) $\alpha_1|u|^p - \beta_1|u|^2 \le f_1(u)(u) \le \gamma_1|u|^p + \delta_1|u|^2$, $f_1(u)u \ge 0$, $p \ge 2$, and $f_1'(u) \ge -c$,

(F_2) $\alpha_2|u|^p - \beta_2 \le f_2(u)(u) \le \gamma_2|u|^p + \delta_2$, $p \ge 2$, and $f_2'(u) \ge -c$,

and

(A) $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a(x) > 0$,

where $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1,2$, and c are all positive constants. Moreover, without loss of generality, we also assume $f_1(0) = f_2(0) = 0$.

In 1980, Aifantis in [1–3] pointed out that the classical reaction-diffusion equation

$$u_t - \Delta u = f(u) + g(x) \tag{1.3}$$

does not contain each aspect of the reaction-diffusion problem, and it neglects viscidity, elasticity, and pressure of medium in the process of solid diffusion and so forth. Furthermore, Aifantis found out that the energy constitutional equation revealing the diffusion process is different along with the different property of the diffusion solid. For example, the energy constitutional equation is different, when conductive medium has pressure and viscoelasticity or not. He constructed the mathematical model by some concrete examples, which contains viscidity, elasticity, and pressure of medium, that is the following nonclassical diffusion equation:

$$u_t - \Delta u_t - \Delta u = f(u) + g(x). \tag{1.4}$$

This equation is a special form of the nonclassical diffusion equation used in fluid mechanics, solid mechanics, and heat conduction theory (see [1–4]). Recently, Aifantis presented a new model about this problem and scrutinized the concrete process of constructing model; the reader can refer to [5] for details.

The longtime behavior of (1.1) acting on a bounded domain Ω has been extensively studied by several authors in [6–13] and references therein. In [12] the existence of a global attractor for the autonomous case has been shown provided that the nonlinearity is critical and $g(x) \in H^{-1}(\Omega)$. Furthermore, for the non-autonomous, the existence of a uniform attractor and exponential attractors has been scrutinized when the time-dependent forcing term $g(x,t)$ only satisfies the translation bounded domain instead of translation compact, namely, $g(x,t) \in L_b^2(\mathbb{R}, L^2(\Omega))$. A similar problem was discussed in [13] by virtue of the standard method based on the so-called squeezing property. To our best knowledge, the dynamics of (1.1) acting on an unbounded domain $\mathbb{R}^N$ has not been considered by predecessors.

As we know, if we want to prove the existence of global attractors, the key point is to obtain the compactness of the semigroup in some sense. For bounded domains, the compactness is obtained by a priori estimates and compactness of Sobolev embeddings. This method does not apply to unbounded domains since the embeddings are no longer compact. To overcome the difficulty of the noncompact embedding, in [14], using the idea of Ball [15], the author proved that the solutions are uniformly small for large space and time variables and then showed that the weak asymptotic compactness is equivalent to the strong asymptotic compactness in certain circumstances. In [16], the authors provided new a priori estimates for the existence of global attractors in unbounded domains and then applied this approach to a nonlinear reaction-diffusion equation with a nonlinearity having a polynomial growth for arbitrary order $p-1$ ($p \geq 2$) and with distribution derivatives in homogeneous term. More recently, in [17] the authors achieved the existence of global attractors for reaction-diffusion equations in $L^2(\mathbb{R}^n)$, by using the methods presented in [18]. Our purpose in this paper is to study the existence of global attractors of (1.1) on the unbounded domains $\mathbb{R}^n$, and we borrow the idea of [17, 18]. Our main result is Theorem 4.6.

This paper is organized as follows. In Section 2, we recall some basic definitions and related theorems that will be used later. In Section 3, we prove the existence of weak solution for nonclassical diffusion equations in $H^1(\mathbb{R}^N)$. The main result is stated and proved in Section 4.

2. Preliminaries

In this section, we iterate some notations and abstract results.

Definition 2.1 (see [18]). Let M be a metric space, and let A be bounded subsets of M. The Kuratowski measure of noncompactness $\gamma(A)$ of A defined by

$$\gamma(A) = \inf\{\delta > 0 \mid A \text{ admits a finite cover by sets whose diameter } \leq \delta\}. \tag{2.1}$$

Definition 2.2 (see [18]). Let X be a Banach space, and let $\{S(t)\}_{t\geq 0}$ be a family of operators on X. We say that $\{S(t)\}_{t\geq 0}$ is a continuous semigroup (C_0 semigroup) (or norm-to-weak continuous semigroup) on X, if $\{S(t)\}_{t\geq 0}$ satisfies

(i) $S(0) = \mathrm{Id}$ (the identity),

(ii) $S(t)S(s) = S(t+s)$, for all $t,\ s \geq 0$,

(iii) $S(t_n)x_n \to S(t)x$, if $t_n \to t,\ x_n \to x$ in X (or (iii) $S(t_n)x_n \rightharpoonup S(t)x$, if $t_n \to t,\ x_n \to x$ in X).

Definition 2.3 (see [18]). A C_0 semigroup (or norm-to-weak continuous semigroup) $\{S(t)\}_{t\geq 0}$ in a complete metric space M is called ω-limit compact if for every bounded subset B of M and for every $\varepsilon > 0$, there is a $t(B) > 0$, such that

$$\gamma\left(\bigcup_{t\geq t(B)} S(t)B\right) \leq \varepsilon. \tag{2.2}$$

Condition C (see [18]). For any bounded set B of a Banach space X, there exists a $t(B) > 0$ and a finite dimensional subspace X_1 of X such that $\{\|P_m S(t)B\|\}$ is bounded and

$$\|(I - P_m)S(t)x\| < \varepsilon \quad \text{for } t \geq t(B),\ x \in B, \tag{2.3}$$

where $P_m : X \to X_1$ is a bounded projector.

Lemma 2.4 (see [18]). *Let X be a Banach space, and let $\{S(t)\}_{t\geq 0}$ be a C_0 semigroup (or norm-to-weak continuous semigroup) in X.*

(1) *If Condition C holds, the $\{S(t)\}_{t\geq 0}$ is ω-limit compact.*

(2) *Let X be a uniformly convex Banach space. Then $\{S(t)\}_{t\geq 0}$ is ω-limit compact if and only if Condition C holds.*

Lemma 2.5 (see [18]). *Let X be a Banach space, and let $\{S(t)\}_{t\geq 0}$ be a C_0 semigroup (or norm-to-weak continuous semigroup) in X.*

(1) *If Condition C holds, the $\{S(t)\}_{t\geq 0}$ is ω-limit compact;*

(2) *Let X be a uniformly convex Banach space. Then $\{S(t)\}_{t\geq 0}$ is ω-limit compact if and only if Condition C holds.*

Theorem 2.6 (see [18]). *Let X be a Banach space. Then the C_0 semigroup (or norm-to-weak continuous semigroup) $\{S(t)\}_{t\geq 0}$ has a global attractor in X if and only if*

(1) *there is a bounded absorbing set $B \subset X$.*

(2) *$\{S(t)\}_{t\geq 0}$ is ω-limit compact.*

Lemma 2.7 (see [19]). *Let Φ be an absolutely continuous positive function on $\mathbb{R}^+$, which satisfies for some $\varepsilon > 0$ the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) + 2\varepsilon\Phi(t) \leq g(t)\Phi(t) + h(t), \tag{2.4}$$

for almost every $t \in \mathbb{R}^+$, where g and h are functions on $\mathbb{R}^+$ such that

$$\int_{\tau}^{t} |g(y)| \mathrm{d}y \leq m_1\left(1 + (t-\tau)^{\mu}\right), \quad \forall t \geq \tau \geq 0, \tag{2.5}$$

for some $m_1 \geq 0$ and $\mu \in [0, 1)$, and

$$\sup_{t\geq 0} \int_{t}^{t+1} |h(y)| \mathrm{d}y \leq m_2, \tag{2.6}$$

for some $m_2 \geq 0$. Then

$$\Phi(t) \leq \beta\Phi(0)e^{-\varepsilon t} + \rho, \quad \forall t \in R^+, \tag{2.7}$$

for some $\beta = \beta(m_1, \mu) \geq 1$ and

$$\rho = \frac{\beta m_2 e^{\varepsilon}}{1 - e^{-\varepsilon}}. \tag{2.8}$$

Lemma 2.8 (see [20]). *Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0, T; X)$, and $\mathrm{d}u_n/\mathrm{d}t$ is uniformly bounded in $L^p(0, T; Y)$, for some $p > 1$. Then there is a subsequence that converges strongly in $L^2(0, T; H)$.*

3. Unique Weak Solution

Theorem 3.1. *Assume (F_1), (F_2), and (A) are satisfied. Then for any $T > 0$ and $u_0 \in H^1(R^N)$, there is a unique solution u of* (1.1)-(1.2) *such that*

$$u \in C^1\left([0, T]; H^1\left(\mathbb{R}^N\right)\right) \cap L^p\left(0, T; L^p\left(\mathbb{R}^N\right)\right). \tag{3.1}$$

Moreover, the solution continuously depends on the initial data.

Proof. We decompose our proof into three steps for clarity.

Step 1. For any $n \in N$, we consider the existence of the weak solution for the following problem in $B(0,n) \triangleq B_n \subset R^N$:

$$\begin{gathered} u_t - \Delta u_t - \Delta u + f(x,u) = g(x), \quad x \in B_n, \\ u(x,0) = u_0 \in H^1(B_n), \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{3.2}$$

Choose a smooth function $\chi_n(x)$ with

$$\chi_n(x) = \begin{cases} 1, & x \in B_{n-1}, \\ 0, & x \notin B_n. \end{cases} \tag{3.3}$$

Since B_n is a bounded domain, so the existence and uniqueness of solutions can be obtained by the standard Faedo-Galerkin methods; see [6, 8, 11, 16]; we have the unique weak solution

$$u_n \in C^1\Big([0,T]; H^1(B_n)\Big) \cap L^p(0,T; L^p(B_n)), \quad u_n(x,0) = \chi_n(x)u_0(x). \tag{3.4}$$

Step 2. According to Step 1, we denote $(\mathrm{d}/\mathrm{d}t)u_n = u_{nt}$; then u_n satisfies

$$u_{nt} - \Delta u_{nt} - \Delta u_n + f(x,u_n) = g(x), \quad x \in B_n, \tag{3.5}$$

$$u_n(x,0) = \chi_n(x)u_0(x), \tag{3.6}$$

$$u_n|_{\partial B_n} = 0. \tag{3.7}$$

For the mathematical setting of the problem, we denote $\|\cdot\|_{L^2(B_n)} \triangleq \|\cdot\|_{B_n}$, $\|\cdot\|_{L^1(R^N)} \triangleq \|\cdot\|_1$, $\|\cdot\|_{L^2(R^N)} \triangleq \|\cdot\|$, $\|\cdot\|_{L^\infty(R^N)} \triangleq \|\cdot\|_\infty$.

Multiplying (3.5) by u_n in B_n, using $f_1(u)u \geq 0$, (F_2) and (A), we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla u_n\|_{B_n}^2 + \|u_n\|_{B_n}^2\Big) + \|\nabla u_n\|_{B_n}^2 &\leq \int_{B_n} a(x)(\beta_2 - \alpha_2|u|^p)\mathrm{d}x + \int_{B_n} g u_n \mathrm{d}x \\ &\leq \beta_2\|a(x)\|_1 - \int_{B_n} \alpha_2 a(x)|u|^p \mathrm{d}x + \frac{\|g\|^2}{2\lambda} + \frac{\lambda}{2}\|u_n\|_{B_n}^2. \end{aligned} \tag{3.8}$$

By the Poincaré inequality, for some $\nu > 0$, we conclude that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla u_n\|_{B_n}^2 + \|u_n\|_{B_n}^2\Big) + \nu\Big(\|\nabla u_n\|_{B_n}^2 + \|u_n\|_{B_n}^2\Big) + \int_{B_n} \alpha_2 a(x)|u|^p \mathrm{d}x \\ &\quad \leq \beta_2\|a(x)\|_1 + \frac{\|g\|^2}{2\lambda}. \end{aligned} \tag{3.9}$$

Hence, it follows that

$$\begin{aligned}
&\|\nabla u_n(T)\|_{B_n}^2+\|u_n(T)\|_{B_n}^2+2\nu\int_0^T\left(\|\nabla u_n(T)\|_{B_n}^2+\|u_n(T)\|_{B_n}^2\right)+2\int_0^T\int_{B_n}\alpha_2 a(x)|u|^p\mathrm{d}x\\
&\quad\le\left(2\beta_2\|a(x)\|_1+\frac{\|g\|^2}{\lambda}\right)T.
\end{aligned}\tag{3.10}$$

We get the following estimate:

$$\begin{gathered}
\sup_{t\in[0,T]}\|\nabla u_n(t)\|_{B_n}^2+\|u_n(t)\|_{B_n}^2\le C,\\
\int_0^T\left(\|\nabla u_n(t)\|_{B_n}^2+\|u_n(t)\|_{B_n}^2\right)\le C,\\
\int_0^T\int_{B_n}\alpha_2 a(x)|u(t)|^p\mathrm{d}x\le C.
\end{gathered}\tag{3.11}$$

Similar to (3.9), using (F_1), (F_2), and (A), we get

$$\int_0^T\int_{B_n}|u(t)|^p\mathrm{d}x\le C,\tag{3.12}$$

where C is independent of n.

(F_1) and (F_2) yield

$$\begin{gathered}
\left|f_1(u_n)\right|\le C\left(|u_n|^{p-1}+|u_n|\right),\\
\left|f_2(u_n)\right|\le C\left(|u_n|^{p-1}+1\right).
\end{gathered}\tag{3.13}$$

Choose q such that $(1/p)+(1/q)=1$; then $(p-1)q=p$. Noting that $p\ge 2$, then $1<q\le 2$, and we have the embedding $L^p(B_n)\hookrightarrow L^q(B_n)$. According to (3.12) and (3.13), we get

$$\begin{aligned}
\int_0^T\int_{B_n}\left|f_1(u)\right|^q&\le C\int_0^T\int_{B_n}\left(|u_n|^{p-1}+|u_n|\right)^q\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^T\int_{B_n}|u_n|^{(p-1)q}\mathrm{d}x\,\mathrm{d}t+C\int_0^T\int_{B_n}|u_n|^q\mathrm{d}x\,\mathrm{d}t\\
&\le C\int_0^T\int_{B_n}|u_n|^p+C\int_0^T\int_{B_n}|u_n|^p\mathrm{d}x\,\mathrm{d}t\\
&\le C,
\end{aligned}$$

$$\begin{aligned}\int_0^T\int_{B_n}\left|f_2(u)\right|^q &\le C\int_0^T\int_{B_n}|a(x)|^q\left(|u_n|^{p-1}+1\right)^q\mathrm{d}x\,\mathrm{d}t\\ &\le C|a(x)|_\infty^{q-1}\int_0^T\int_{B_n}a(x)\left(|u_n|^{(p-1)q}+1\right)\mathrm{d}x\,\mathrm{d}t\\ &\le C|a(x)|_\infty^{q-1}\left(C|a(x)|_1+\int_0^T\int_{B_n}a(x)|u_n|^p\mathrm{d}x\,\mathrm{d}t\right)\\ &\le C,\end{aligned} \tag{3.14}$$

where C is independent of n.

Thanks to (3.14), $f_1(u_n)$ is bounded in $L^p(0,T;L^q(B_n))$, and $af_2(u_n)$ is bounded in $L^p(0,T;L^q(B_n))$.

For $\forall v\in L^2(0,T;H_0^1(B_n))$,

$$\begin{aligned}\int_0^T\int_{B_n}-\Delta u_n v &= \int_0^T\int_{B_n}\nabla u_n\nabla v\\ &\le\left(\int_0^T\|\nabla u_n\|_{B_n}^2\right)^{1/2}\left(\int_0^T\|\nabla v\|_{B_n}^2\right)^{1/2}\\ &\le\left(\int_0^T\|\nabla u_n\|^2\right)^{1/2}\left(\int_0^T\|\nabla v\|_{B_n}^2\right)^{1/2}\\ &\le C\|\nabla v\|_{H_0^1(B_n)},\end{aligned} \tag{3.15}$$

where C is independent of n. We can obtain that $-\Delta u_n$ is bounded in $L^2(0,T;H^{-1}(B_n))$.

Since $g(x)\in L^2(\mathbb{R}^N)$,

$$g(x)\in L^2\left(0,T;\mathbb{R}^N\right). \tag{3.16}$$

Therefore, there exists $s>0$, such that $L^2(0,T;H^{-1}(B_n))$, $L^2(0,T;H_0^1(B_n))$, $L^q(0,T;L^q(B_n))$, and $L^2(0,T;L^2(B_n))$ are continuous embedding to $L^q(0,T;H^{-s}(B_n))$.

According to (3.5) and (3.14)–(3.16), we obtain

$$u_{nt}-\Delta u_{nt}\in L^q\left(0,T;H^{-s}(B_n)\right). \tag{3.17}$$

So u_n has a subsequent (we also denote u_n) weak* convergence to u in $L^2(0,T;H^{-1}(B_n))$ and $L^2(0,T;L^2(B_n))$; $u_{nt}-\Delta u_{nt}$ has a subsequent (we also denote $u_{nt}-\Delta u_{nt}$) weak* convergence to $u_t-\Delta u_t$. Let $u_n=0$ outside of B_n; we can extend u_n to $\mathbb{R}^N$.

As introduced in [6, 20], $C_c^\infty(\mathbb{R}^N)$ is dense in the dual space of $H^{-1}(B_n)$, $L^2(B_n)$, $L^q(B_n)$, and $H^{-s}(B_n)$, so we can choose for all $\phi \in L^2(0,T;C_c^\infty(\mathbb{R}^N)) \cap L^q(0,T;C_c^\infty(\mathbb{R}^N))$ as a test function such that

$$\begin{gathered} \langle \Delta u_n, \phi \rangle \longrightarrow \langle \Delta u, \phi \rangle, \\ \langle u_{nt} - \Delta u_{nt}, \phi \rangle \longrightarrow \langle u_t - \Delta u_t, \phi \rangle. \end{gathered} \tag{3.18}$$

Since for all $\phi \in C_c^\infty(\mathbb{R}^N)$, there exists bounded domain $\Omega \subset \mathbb{R}^N$ such that $\phi = 0$, $x \notin \Omega$. It follows that u_n is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$, and $u_{nt} - \Delta u_{nt} \in L^q(0,T;H^{-s}(\Omega))$. Since $H_0^1(\Omega) \subset\subset L^2(\Omega) \subset H^{-s}(\Omega)$, according to Lemma 2.8, there is a subsequence u_n (we also denote u_n) that converges strongly to u in $L^2(0,T;L^2(\Omega))$.

Using the continuity of f_1 and f_2, we have

$$\begin{gathered} \langle f_1(u_n), \phi \rangle \longrightarrow \langle f_1(u), \phi \rangle, \\ \langle a(x) f_2(u_n), \phi \rangle \longrightarrow \langle a(x) f_2(u), \phi \rangle. \end{gathered} \tag{3.19}$$

In line with (3.18) and (3.19), and let $n \to \infty$, we geting for all $\phi \in L^2(0,T;C_c^\infty(\mathbb{R}^N)) \cap L^q(0,T;C_c^\infty(\mathbb{R}^N))$:

$$\langle u_t - \Delta u_t - \Delta u + f_1(u) + a(x) f_2(u), \phi \rangle = \langle g(x), \phi \rangle. \tag{3.20}$$

Thus, u is the weak solution of (3.2) and satisfies

$$u \in C^1\left([0,T]; H^1\left(\mathbb{R}^N\right)\right) \cap L^p\left(0,T; L^p\left(\mathbb{R}^N\right)\right). \tag{3.21}$$

Step 3 (uniqueness and continuous dependence). Let u_0, v_0 be in $H^1(\mathbb{R}^N)$, and setting $w(t) = u(t) - v(t)$, we see that $w(t)$ satisfies

$$w_t - \Delta w_t - \Delta w + f_1(u) - f_1(v) + a(x)\big(f_2(u) - f_2(v)\big) = 0, \quad x \in \mathbb{R}^N. \tag{3.22}$$

Taking the inner product with w of (3.22), using (F_1), (F_2), and (A), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla w\|^2 + \|w\|^2\right) + \|\nabla w\|^2 \\ &\quad \le \left| \int (f_1(u) - f_1(v)) w \,\mathrm{d}x \right| \\ &\qquad + \left| \int a(x)(f_2(u) - f_2(v)) w \,\mathrm{d}x \right| \\ &\quad \le C(1 + \|a\|_\infty)\|w\|^2. \end{aligned} \tag{3.23}$$

By the Gronwall Lemma, we get

$$\|\nabla w(t)\|^2 + \|w(t)\|^2 \le e^{Ct}\left(\|\nabla w(0)\|^2 + \|w(0)\|^2\right). \tag{3.24}$$

This is uniqueness and is continuous dependence on initial conditions.

Thanks to Theorem 3.1, and leting $S(t)u_0 = u(t)$, $S(t) : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is a C^0 semigroup. □

4. Global Attractor in $\mathbb{R}^N$

Lemma 4.1. *Assume (F_1), (F_2), and (A) are satisfied. There is a positive constant ρ_1 such that for any bounded subset $B \subset H^1(\mathbb{R}^N)$, there exists $T_1 = T_1(B)$ such that*

$$\|\nabla u(t)\| \le \rho_1, \quad \forall t \ge T_1,\ u_0 \in B. \tag{4.1}$$

Proof. Multiplying (1.1) by u in $\mathbb{R}^N$, using $f_1(u)u \ge 0$, (F_2) and (A), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \|u\|^2\right) + \|\nabla u\|^2 &\le \int_{\mathbb{R}^N} a(x)\left(\beta_2 - \alpha_2|u|^p\right)\mathrm{d}x + \int_{\mathbb{R}^N} gu\,\mathrm{d}x \\ &\le \beta_2\|a(x)\|_1 - \int_{\mathbb{R}^N}\alpha_2 a(x)|u|^p\mathrm{d}x + \frac{\|g\|^2}{2\lambda} + \frac{\lambda}{2}\|u\|^2_{B_n}.\end{aligned} \tag{4.2}$$

By virtue of the Poincaré inequality, for some $\nu > 0$, there holds

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \|u\|^2\right) + \nu\left(\|\nabla u\|^2 + \|u\|^2\right) + \int_{\mathbb{R}^N}\alpha_2 a(x)|u|^p\mathrm{d}x \\ &\quad\le \beta_2\|a(x)\|_1 + \frac{\|g\|^2}{2\lambda}.\end{aligned} \tag{4.3}$$

Furthermore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \|u\|^2\right) + \nu\left(\|\nabla u\|^2 + \|u\|^2\right) \le \beta_2\|a(x)\|_1 + \frac{\|g\|^2}{2\lambda}. \tag{4.4}$$

By the Gronwall Lemma, we get

$$\|\nabla u(t)\|^2 + \|u(t)\|^2 \le e^{-\nu t}\left(\|\nabla u(0)\|^2 + \|u(0)\|^2\right) + 2\beta_2\|a(x)\|_1 + \frac{\|g\|^2}{\lambda}. \tag{4.5}$$

We completed the proof. □

According to Lemma 4.1, we know that

$$\mathcal{B}_0 = \left\{u \in H^1\left(\mathbb{R}^N\right) : \|\nabla u\| \leq \rho\right\} \tag{4.6}$$

is a compact absorbing set of a semigroup of operators $\{S(t)\}_{t\geq 0}$ generalized by (1.1)-(1.2), $(F1)$, $(F2)$, and (A).

Lemma 4.2. *Assume (F_1), (F_2), and (A) hold. Then for any $u_0 \in H^1(\mathbb{R}^N)$ and $\varepsilon > 0$, there are some $T(\varepsilon)$ and $k(\varepsilon)$ such that*

$$\int_{|x|\geq 2k} |\nabla u(t)|^2 \mathrm{d}t \leq C\varepsilon, \tag{4.7}$$

whenever $k \geq T(\varepsilon)$ and $t \geq t(\varepsilon)$.

Proof. Choose a smooth function $\theta(x)$ with

$$\theta(x) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2, \end{cases} \tag{4.8}$$

where $0 \leq \theta(s) \leq 1$, $1 \leq s \leq 2$, and there is a constant c such that $|\theta'(s)| \leq c$.

Multiplying (1.1) with $\theta^2(|x|^2/k^2)u$ and integrating on $\mathbb{R}^N$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u|^2 + |u|^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)u\Delta u\,\mathrm{d}x \\
= -\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)f_1(u)u\mathrm{d}x - \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)a(x)f_2(u)u\,\mathrm{d}x \\
+ \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)ug\,\mathrm{d}x \\
\leq -\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)f_1(u)u\,\mathrm{d}x - \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)a(x)f_2(u)u\,\mathrm{d}x \\
+ \frac{\lambda}{2}\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2\mathrm{d}x + \frac{1}{2\lambda}\int_{\mathbb{R}^N} |g|^2\mathrm{d}x.
\end{aligned} \tag{4.9}$$

Next we deal with the right hand side of (4.9) one by one:

$$\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)u\Delta u\,\mathrm{d}x = -\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x - \int_{\mathbb{R}^N} \frac{4x}{k^2}\theta\left(\frac{|x|^2}{k^2}\right)\theta'\left(\frac{|x|^2}{k^2}\right)u\nabla u\,\mathrm{d}x. \tag{4.10}$$

According to the condition $|\theta'(s)| \le c$ and the bounded absorbing set in $H^1(\mathbb{R}^N)$ for $t \ge t_*$, it follows that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \frac{4x}{k^2}\theta\left(\frac{|x|^2}{k^2}\right)\theta'\left(\frac{|x|^2}{k^2}\right)u\nabla u\,\mathrm{d}x\right| &= \left|\int_{k\le|x|\le\sqrt{2}k} \frac{4x}{k^2}\theta\left(\frac{|x|^2}{k^2}\right)\theta'\left(\frac{|x|^2}{k^2}\right)u\nabla u\,\mathrm{d}x\right| \\ &\le \frac{4\sqrt{2}}{k}\int_{k\le|x|\le\sqrt{2}k}\theta^2\left(\frac{|x|^2}{k^2}\right)|u||\nabla u|\mathrm{d}x \\ &\le \frac{2\sqrt{2}}{k}\left(\int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^N}|u|^2\mathrm{d}x\right) \\ &\le \frac{C}{k}\int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x + \frac{C}{k},\end{aligned} \tag{4.11}$$

where C is independent of k. For any $0 < \varepsilon < 1$ given, let

$$k_1(\varepsilon) = \frac{C}{\varepsilon}. \tag{4.12}$$

Hence, combining (4.10) with (4.11), when $k \ge k_1(\varepsilon)$, we conclude that

$$\int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)u\Delta u\,\mathrm{d}x \le -\frac{1}{2}\int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x + \varepsilon. \tag{4.13}$$

Using $f_1(u)u \ge 0$ and (F_2), it yields

$$\begin{aligned}&-\int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)f_1(u)u\,\mathrm{d}x - \int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)a(x)f_2(u)u\,\mathrm{d}x \\ &\quad\le \int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)a(x)(\beta_2 - \alpha_2|u|^p)\mathrm{d}x \\ &\quad\le \beta_2\int_{\mathbb{R}^N}\theta^2\left(\frac{|x|^2}{k^2}\right)a(x)\mathrm{d}x \\ &\quad\le \beta_2\int_{|x|\ge k}a(x)\mathrm{d}x.\end{aligned} \tag{4.14}$$

Since $a \in L^1(\mathbb{R}^N)$, there exist $k_2(\varepsilon) > k_1(\varepsilon)$, such that

$$\int_{|x|\ge k}a(x)\mathrm{d}x \le \frac{\varepsilon}{2\beta_2}. \tag{4.15}$$

Then

$$-\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right) f_1(u)u\,\mathrm{d}x - \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right) a(x) f_2(u)u\,\mathrm{d}x \le \frac{\varepsilon}{2}. \tag{4.16}$$

From the assumption $g(x) \in L^2(\mathbb{R}^N)$, provide $k \ge k(\varepsilon) \ge k_2(\varepsilon)$, such that

$$\int_{|x|\ge k} |g|^2 \mathrm{d}x \le \varepsilon\lambda. \tag{4.17}$$

Thus combining (4.9), (4.13), (4.16), and (4.17), we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u|^2 + |u|^2\right)\mathrm{d}x + \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2 \mathrm{d}x \le 4\varepsilon. \tag{4.18}$$

Furthermore, there holds

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u|^2 + |u|^2\right)\mathrm{d}x + \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u|^2 + |u|^2\right)\mathrm{d}x \\
&\quad\le 2\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u|^2 + |u|^2\right)\mathrm{d}x + 4\varepsilon.
\end{aligned} \tag{4.19}$$

According to Lemma 2.7, we obtain

$$\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u(t)|^2 + |u(t)|^2\right) \le \beta\int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u(0)|^2 + |u(0)|^2\right)e^{-t/2} + \frac{\beta e^{1/2}}{1-e^{-1/2}}\varepsilon. \tag{4.20}$$

Thus, we get

$$\int_{|x|\ge 2k} |\nabla u(t)|^2 \mathrm{d}t \le \int_{\mathbb{R}^N} \theta^2\left(\frac{|x|^2}{k^2}\right)\left(|\nabla u(t)|^2 + |u(t)|^2\right) \le C\varepsilon, \tag{4.21}$$

provided $T \ge T(\varepsilon)$ and $k \ge \widetilde{k}(\varepsilon)$, we complete the proof. □

Lemma 4.3. *Assume* (F_1), (F_2), *and* (A) *hold. There is a positive constant* ρ_2 *such that for any bounded subset* $B \subset H^2(\mathbb{R}^N)$, *there exists* $T_2 = T_2(B)$ *such that*

$$\|\Delta u(t)\| \le \rho_2, \quad \forall t \ge T_2,\ u_0 \in B. \tag{4.22}$$

Proof. Multiplying (1.1) by $-\Delta u$ in $\mathbb{R}^N$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2+\|\Delta u\|^2\right)+\|\Delta u\|^2 = \int_{\mathbb{R}^N} f_1(u)\Delta u\,\mathrm{d}x+\int_{\mathbb{R}^N} a(x)f_2(u)\Delta u\,\mathrm{d}x-\int_{\mathbb{R}^N} g\Delta u\,\mathrm{d}x. \tag{4.23}$$

Using (F_1), (F_2), and (A), we have the following estimates:

$$\begin{gathered}\int_{\mathbb{R}^N} f_1(u)\Delta u\,\mathrm{d}x \le \int_{\mathbb{R}^N} f_1'(u)|\nabla u|^2\mathrm{d}x \le c\|\nabla u\|^2,\\ \int_{\mathbb{R}^N} a(x)f_2(u)\Delta u\,\mathrm{d}x \le \int_{\mathbb{R}^N} a(x)f_2'(u)|\nabla u|^2\mathrm{d}x \le c\|\nabla u\|^2,\\ \left|\int_{\mathbb{R}^N} g\Delta u\,\mathrm{d}x\right| \le c\|g(x)\|^2+\frac{1}{2}\|\Delta u\|^2.\end{gathered} \tag{4.24}$$

Together with (4.6) and (4.19)–(4.21), by the Poincaré inequality, for some $\mu > 0$, this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2+\|\Delta u\|^2\right)+\mu\left(\|\nabla u\|^2+\|\Delta u\|^2\right) \le C\|g(x)\|^2+C. \tag{4.25}$$

By the Gronwall Lemma, we get

$$\|\nabla u(t)\|^2+\|\Delta u(t)\|^2 \le e^{-\mu t}\left(\|\nabla u(0)\|^2+\|\Delta u(0)\|^2\right)+C. \tag{4.26}$$

We complete the proof. □

Remark 4.4. There is a constant $C > 0$, such that for any bounded subset $B \subset B(0,\rho_2) \subset H^1(\mathbb{R}^N)$, when $t > t_*$, there holds

$$\int_t^{t+1}\left(\|\nabla u\|^2+\|\Delta u\|^2\right) \le C. \tag{4.27}$$

Lemma 4.5. *Assume (F_1), (F_2), and (A) are satisfied. Then the semigroup $\{S(t)\}_{t\ge 0}$ associated with the initial value problems* (1.1) *and* (1.2) *is ω-limit compact.*

Proof. Denote $B_R = B(0;R)\cap\mathbb{R}^N$, and we split $u(t)$ as

$$u(t) = \chi(x)u(t)+(1-\chi(x))u(t) = u_1(t)+u_2(t), \tag{4.28}$$

where $\theta(x)$ is a smooth function:

$$\chi(x) = \begin{cases}1, & x\in B_R,\\ 0, & x\notin B_{R+1},\end{cases} \tag{4.29}$$

with $0 \leq \chi(x) \leq 1$, and there is a positive constant c such that $|\chi'(x)| \leq c$. Then

$$u_1(t) = \begin{cases} u(t), & x \in B_R, \\ 0, & x \notin B_{R+1}, \\ \chi(x)u(t), & \text{others}, \end{cases}$$
$$u_2(t) = \begin{cases} 0, & x \in B_R, \\ u(t), & x \notin B_{R+1}, \\ (1-\chi(x))u(t), & \text{others}. \end{cases} \tag{4.30}$$

From Lemma 4.1, we know that $u_1(t) \in H^1(B_R)$ as $t \geq T_1$.

For any $\varepsilon > 0$ given, we can choose R large enough; by Remark 4.4, we can assume

$$\int_{|x|\geq R} |\nabla u|^2 \mathrm{d}x \leq \frac{\varepsilon}{2}. \tag{4.31}$$

So we conclude that

$$\|\nabla u_2\|^2 \leq \frac{\varepsilon}{2}. \tag{4.32}$$

For any bounded set $B \subset H^1(\mathbb{R}^N)$, $\{S(t)B\}_{t\geq 0} = \{S(t)u_0 \mid u_0 \in B\}_{t\geq 0}$ can be split as

$$S(t)B = \chi(x)s(t)B + (1-\chi(x))s(t)B. \tag{4.33}$$

Then in line with the property of noncompact measure, it follows that

$$\gamma(S(t)B) = \gamma(\chi(x)s(t)B) + \gamma((1-\chi(x))s(t)B). \tag{4.34}$$

On the other hand,

$$\gamma(\chi(x)s(t)B) = \{\chi(x)s(t)u_0 = u_1(t) \mid u_0 \in B\}. \tag{4.35}$$

From Lemma 4.3, we get

$$\|u_1\|_{H^2(B_{R+1})} \leq C, \quad \forall t > t_* + 1. \tag{4.36}$$

Recall that

$$(1-\chi(x))s(t)B = \{(1-\chi(x))s(t)u_0 = u_2 \mid u_2 \in B\}. \tag{4.37}$$

On account of Remark 4.4, it yields

$$\gamma((1-\chi(x))s(t)B) \leq \varepsilon, \quad \forall t > t_* + 1. \tag{4.38}$$

Therefore, we have

$$\gamma(S(t)B)B \leq \varepsilon, \quad \forall t > t_* + 1. \tag{4.39}$$

That is, $\{S(t)\}_{t\geq 0}$ is ω-limit compact in $H^1(\mathbb{R}^N)$. □

Theorem 4.6. *Assume* (F_1), (F_2), *and* (A) *hold. Then the semigroup* $\{S(t)\}_{t\geq 0}$ *associated with the initial value problems* (1.1) *and* (1.2) *has a global attractor* $\mathcal{A}$ *in* $H^1(\mathbb{R}^N)$.

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