Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 676871, 12 pages
doi:10.1155/2012/676871

Research Article

The Optimal Control and MLE of Parameters of a Stochastic Single-Species System

Huili Xiang and Zhijun Liu

Department of Mathematics, Hubei University for Nationalities, Hubei, Enshi 445000, China

Correspondence should be addressed to Zhijun Liu, zhijun_liu47@hotmail.com

Received 5 May 2012; Revised 4 September 2012; Accepted 20 September 2012

Academic Editor: Garyfalos Papaschinopoulos

This paper investigates the optimal control and MLE (maximum likelihood estimation) for a single-species system subject to random perturbation. With the help of the techniques of stochastic analysis and mathematical statistics, sufficient conditions for the optimal control threshold value, the optimal control moment, and the maximum likelihood estimation of parameters are established, respectively. An example is presented to illustrate the feasibility of our theoretical results.

1. Introduction

The Malthus model is usually expressed as

$$\frac{dx(t)}{dt} = rx(t), \tag{1.1}$$

where $x(0) = x_0 > 0$, $x(t)$ stands for the density of species x at t moment, and r is the intrinsic growth rate. As everyone knows, model (1.1) has epoch-making significance in mathematics and ecology and later, many deterministic mathematical models have been widely studied (see [1–5]). In fact, a population system is inevitably affected by the environmental noise in the real world. As a consequence, it is reasonable to study a corresponding stochastic model. Notice that some recent results, especially on optimal control, for the following stochastic model

$$\frac{dx(t)}{x(t)} = rdt + \sigma dw(t), \tag{1.2}$$

have been obtained (see [6–9]), where $w(t)$ stands for the standard Brownian motion. However, for some pest populations, their generations are nonoverlapping (e.g., poplar and willow weevil, osier weevil and paranthrene tabaniformis) and the discrete models are more appropriate than the continuous ones. Compared with the continuous ones, the study on discrete mathematical models is more challenging. Inspired by [1–12], in this paper we will consider the following discrete model of system (1.2)

$$x(n) = x(n-1)\exp(r + \sigma\varepsilon_n), \tag{1.3}$$

where $x(0) > 0$, $\varepsilon_i \sim N(0,1)$, $i = 1,2,\ldots,n$, and any two of them are independent. σ stands for the noise intensity. We will focus on the optimal control threshold value, the optimal control moment, and the maximum likelihood estimation of parameters. To the best of our knowledge, no work has been done for system (1.3).

The rest of this paper is organized as follows. In Section 2, some preliminaries are introduced. In Section 3, we give three results of this paper. As applications of our main results; an example is presented to illustrate the feasibility of our theoretical results in Section 4.

2. Preliminaries

In this section, we summarize several definitions, assumptions, and lemmas which are useful for the later sections.

Definition 2.1. Only when the quantity of pest population reaches U one starts to control the pest population, and the real number U is called to be a control threshold value.

Definition 2.2. Until the N_0th generation, the total quantity of pest population first reaches the control threshold value, then one says that N_0 is the first reaching time.

Two main goals of this paper are to seek the optimal control threshold value and the optimal control moment from the point of view of the lowest control cost. Considering that the practical control to some pest population must be in the limited time range, we give the first assumption:

(H_1) $n \leq n_0$, where n is the number of generation of pest population in a control period and n_0 is a positive integer.

Denote $M = \max\{x(0), x(1), \ldots, x(n_0)\}$. Usually, at the beginning, the number of pest population is very small, so we give the second assumption:

(H_2) The first reaching time $N_0 > 0$.

Let the life period of pest population x be τ, we should annihilate pest at $N_0\tau$ moment from the point of view of the lowest control cost. We further give the third assumption.

(H_3) The number of pest population x will not reach the extent which can cause damage again after being annihilated.

By (H_3), we have

$$P\{M < U\} = P\{N_0 > n_0\} \quad \text{or} \quad P\{M \geq U\} = P\{N_0 \leq n_0\}. \tag{2.1}$$

So we can give the expression of the total loss caused by pest and expending for annihilating pest, respectively. It is obvious that the loss caused by pest population comes from the quantity of population and damaging time. We need to the fourth assumption

(H_4) The generation of pest population is nonoverlapping.

On one hand, the loss caused by pest can be expressed as

$$S(N_0) = k_1 \sum_{n=0}^{N_0} E[x(n)]\tau, \tag{2.2}$$

where k_1 stands for the loss caused by unit number pest in one generation, $E[x(n)]$ is the mean function of $x(n)$. On the other hand, the expending for annihilating pest can be expressed as

$$k_2 H(M-U), \tag{2.3}$$

where $H(x)$ is defined by

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{2.4}$$

that is,

$$H(M-U) = \begin{cases} 1, & M > U, \\ 0, & M \le U, \end{cases} \tag{2.5}$$

where k_2 stands for the expending for annihilating pest once. Since $S(N_0)$ is dependent on random variable N_0 and $k_2 H(M-U)$ is dependent on random variable M and threshold value U, the total cost is a random variable, which can be expressed as

$$J(U) = E[S(N_0)] + E[k_2 H(M-U)] = E[S(N_0)] + k_2 P\{M > U\}. \tag{2.6}$$

Thus, we need to search for U^* such that $J(U^*)$ is minimum and consequently, we can give the optimal control moment.

Next, we will give some lemmas which are very important to the proofs of three theorems in the following section.

Lemma 2.3. *The solution of system* (1.3) *can be expressed as*

$$x(n) = x(0) \exp\left(nr + \sigma \sum_{i=1}^{n} \varepsilon_i \right). \tag{2.7}$$

Proof. By (1.3), we have

$$\frac{x(n)}{x(n-1)} = \exp(r + \sigma\varepsilon_n), \quad (n = 1, 2 \ldots). \tag{2.8}$$

Thus, one has

$$\begin{aligned} \frac{x(1)}{x(0)} &= \exp(r + \sigma\varepsilon_1), \\ \frac{x(2)}{x(1)} &= \exp(r + \sigma\varepsilon_2), \\ &\vdots \\ \frac{x(n)}{x(n-1)} &= \exp(r + \sigma\varepsilon_n). \end{aligned} \tag{2.9}$$

By a simplification, we obtain

$$\frac{x(n)}{x(0)} = \exp\left(nr + \sigma\sum_{i=1}^{n}\varepsilon_i\right), \tag{2.10}$$

that is,

$$x(n) = x(0)\exp\left(nr + \sigma\sum_{i=1}^{n}\varepsilon_i\right). \tag{2.11}$$

□

Lemma 2.4. *If $E(x(n))$ is the mean-value function of the solution of system* (1.3)*, then one has*

$$E[x(n)] = x(0)\exp\left[n\left(r + \frac{\sigma^2}{2}\right)\right]. \tag{2.12}$$

Proof. One has

$$\begin{aligned} E[x(n)] &= E\left[x(0)\exp\left(nr + \sum_{i=1}^{n}\varepsilon_i\right)\right] \\ &= E\left[x(0)\exp(nr)\exp\left(\sigma\sum_{i=1}^{n}\varepsilon_i\right)\right] \\ &= x(0)\exp(nr)E\left[\exp\left(\sigma\sum_{i=1}^{n}\varepsilon_i\right)\right]. \end{aligned} \tag{2.13}$$

Let $Y = \sum_{i=1}^{n} \varepsilon_i$. Since $\varepsilon_i \sim N(0,1),\ i = 1,2,\ldots,n$, we have

$$Y \sim N(0,n), \tag{2.14}$$

and the probability density function of random variable Y is

$$f(y) = \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-y^2}{2n}\right). \tag{2.15}$$

It follows from (2.13) and (2.15) that we have

$$\begin{aligned}
E[x(n)] &= x(0)\exp(nr)E[\exp(\sigma Y)] \\
&= x(0)\exp(nr)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-y^2}{2n}\right) dy \\
&= x(0)\exp(nr)\frac{1}{\sqrt{2\pi n}}\int_{-\infty}^{+\infty} \exp\left(\sigma y - \frac{-y^2}{2n}\right) dy \\
&= x(0)\exp(nr)\frac{1}{\sqrt{2\pi n}}\int_{-\infty}^{+\infty} \exp\left[\frac{(y-\sigma n)^2 - (\sigma n)^2}{2n}\right] dy \\
&= x(0)\exp(nr)\frac{\exp(n\sigma^2/2)}{\sqrt{2\pi n}}\int_{-\infty}^{+\infty} \exp\left[\frac{(y-\sigma n)^2}{2n}\right] dy \\
&= x(0)\exp(nr)\frac{\exp(n\sigma^2/2)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp\left[\frac{((y-\sigma n)/\sqrt{n})^2}{2}\right] d\left(\frac{y-\sigma n}{\sqrt{n}}\right) \\
&= x(0)\exp\left[n\left(r + \frac{\sigma^2}{2}\right)\right].
\end{aligned} \tag{2.16}$$

□

Lemma 2.5. *Let the life period of pest population be τ, let k_1 be the loss caused by unit number pest in one generation, and let k_2 be the expending for annihilating pests once time. The loss caused by pest can be expressed as*

$$S(N_0) = \frac{k_1 x(0)}{1 - \exp(r + (\sigma^2/2))}\left[1 - \exp\left(N_0\left(r + \frac{\sigma^2}{2}\right)\right)\right]. \tag{2.17}$$

Proof. Consider

$$
\begin{aligned}
S(N_0) &= k_1\tau\sum_{n=0}^{N_0} x(n) \\
&= k_1\tau\left\{x(0) + x(0)\exp\left(r+\frac{\sigma^2}{2}\right) + x(0)\exp 2\left(r+\frac{\sigma^2}{2}\right)\right. \\
&\qquad \left. + \cdots + x(0)\exp N_0\left(r+\frac{\sigma^2}{2}\right)\right\} \\
&= k_1\tau x(0)\frac{1-\exp\left(N_0\left(r+\left(\sigma^2/2\right)\right)\right)}{1-\exp\left(r+\left(\sigma^2/2\right)\right)} \\
&= \frac{k_1\tau x(0)}{1-\exp\left(r+\left(\sigma^2/2\right)\right)}\left[1-\exp\left(N_0\left(r+\frac{\sigma^2}{2}\right)\right)\right].
\end{aligned}
\tag{2.18}
$$

□

Lemma 2.6. *Let $P\{N_0 = k\} = P_k$, One has*

$$
P_k = \left[1-\Phi\left(\frac{\ln(U/x(0)) - kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0)) - nr}{\sigma\sqrt{n}}\right), \quad k = 1,2,\ldots,n_0,
\tag{2.19}
$$

where

$$
\Phi(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt.
\tag{2.20}
$$

Proof. By the definition of N_0, we have

$$
x(N_0) \geq U, \qquad x(n) < U, \qquad n = 0,1,\ldots,N_0-1.
\tag{2.21}
$$

Then

$$
\begin{aligned}
P\{N_0 = k\} &= P\{x(k) \geq U, x(n) < U, (n = 0,1,\ldots,k-1)\} \\
&= P\{x(k) \geq U\}\prod_{n=1}^{k-1}P\{x(n) < U\}.
\end{aligned}
\tag{2.22}
$$

By (H_2), we have

$$
P\{x(0) < U\} = 1.
\tag{2.23}
$$

Furthermore,

$$
P\{x(k) \geq U\} = 1 - P\{x(k) < U\}.
\tag{2.24}
$$

By (2.24), we have

$$P\{N_0 = k\} = [1 - P\{x(k) < U\}]\prod_{n=1}^{k-1} P\{x(n) < U\}. \tag{2.25}$$

Moreover, one has

$$\begin{aligned}
P\{x(n) < U\} &= P\left\{ x(0)\exp\left(nr + \sigma\sum_{i=1}^{n}\varepsilon_i \right) < U \right\} \\
&= P\left\{ \exp\left(nr + \sigma\sum_{i=1}^{n}\varepsilon_i \right) < \frac{U}{x(0)} \right\} \\
&= P\left\{ \left(nr + \sigma\sum_{i=1}^{n}\varepsilon_i \right) < \ln\frac{U}{x(0)} \right\} \\
&= P\left\{ \sum_{i=1}^{n}\varepsilon_i < \frac{\ln(U/x(0)) - nr}{\sigma} \right\} \\
&= P\left\{ \frac{\sum_{i=1}^{n}\varepsilon_i}{\sqrt{n}} < \frac{\ln(U/x(0)) - nr}{\sigma\sqrt{n}} \right\} \\
&= \Phi\left(\frac{\ln(U/x(0)) - nr}{\sigma\sqrt{n}} \right),
\end{aligned} \tag{2.26}$$

and then we obtain

$$P_k = \left[1 - \Phi\left(\frac{\ln(U/x(0)) - kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0)) - nr}{\sigma\sqrt{n}}\right), \quad k = 1, 2, \dots, n_0. \tag{2.27}$$

□

Lemma 2.7. *The mean-value function of the loss caused by pest population is*

$$\begin{aligned}
E[S(N_0)] = \sum_{k=1}^{n_0}\Bigg\{ &\left[1 - \exp\left(k\left(r + \frac{\sigma^2}{2}\right)\right)\right]\left[1 - \Phi\left(\frac{\ln(U/x(0)) - kr}{\sigma\sqrt{k}}\right)\right] \\
&\times \prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0)) - nr}{\sigma\sqrt{n}}\right)\Bigg\}.
\end{aligned} \tag{2.28}$$

Proof. By the definition of mean value function, we have

$$E[S(N_0)] = \sum_{k=1}^{n_0}\left\{\left[1 - \exp\left(k\left(r + \frac{\sigma^2}{2}\right)\right)\right]P\{N_0 = k\}\right\}, \tag{2.29}$$

then by Lemma 2.6, we obtain

$$E[S(N_0)] = \sum_{k=1}^{n_0}\left\{\left[1-\exp\left(k\left(r+\frac{\sigma^2}{2}\right)\right)\right]\left[1-\Phi\left(\frac{\ln(U/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\right.$$
$$\left.\times\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0))-nr}{\sigma\sqrt{n}}\right)\right\}. \tag{2.30}$$

□

Lemma 2.8. *The following equality holds*

$$P\{M>U\} = \sum_{k=1}^{n_0-1}\left\{\left[1-\Phi\left(\frac{\ln(U/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0))-nr}{\sigma\sqrt{n}}\right)\right\}. \tag{2.31}$$

Proof. By the definitions of M and U, we have

$$\begin{aligned} P\{M>U\} &= P\{N_0<n_0\} = \sum_{k=1}^{n_0-1}P\{N_0=k\} \\ &= \sum_{k=1}^{n_0-1}\left\{\left[1-\Phi\left(\frac{\ln(U/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0))-nr}{\sigma\sqrt{n}}\right)\right\}. \end{aligned} \tag{2.32}$$

□

3. Main Results

In this section, we give three main results. We first give the optimal control threshold value.

Theorem 3.1. *If the assumptions* (H_1)–(H_4) *are satisfied, then the optimal control threshold value of system* (1.3) *is the minimal nonnegative solution of the following equation about* U

$$\frac{k_1x(0)\sum_{k=1}^{n_0}p + k_2\left(1-\exp\left(r+(1/2)\sigma^2\right)\right)\sum_{k=1}^{n_0-1}p}{\exp\left(r+(1/2)\sigma^2\right)-1} = 0, \tag{3.1}$$

where

$$\begin{aligned} p_1 &= 2^{-k}\prod_{n=1}^{k-1}\left[1+f\left(\frac{\sqrt{2}}{2\sqrt{n}\sigma}\ln\left(\frac{U-nrx(0)}{x(0)}\right)\right)\right], \\ p_2 &= \exp\left(-\frac{1}{2k\sigma^2}\right)\left(\ln\left(\frac{U-krx(0)}{x(0)}\right)\right)^2, \\ p_3 &= \sqrt{k\pi}\sigma(krx(0)-U), \\ p &= -\frac{\sqrt{2}p_1\,p_2}{p_3}, \\ f(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \end{aligned} \tag{3.2}$$

Proof. By Lemmas 2.3–2.8, we obtain that the total loss can be expressed as

$$\begin{aligned}
J(U) &= E[S(N_0)] + k_2 P(M > U) \\
&= \sum_{k=1}^{n_0}\left\{\left[1-\exp\left(k\left(r+\frac{\sigma^2}{2}\right)\right)\right]\left[1-\Phi\left(\frac{\ln(U/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0))-nr}{\sigma\sqrt{n}}\right)\right\} \\
&\quad + k_2\sum_{k=1}^{n_0-1}\left\{\left[1-\Phi\left(\frac{\ln(U/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U/x(0))-nr}{\sigma\sqrt{n}}\right)\right\}.
\end{aligned} \tag{3.3}$$

A calculation leads to

$$J'(U) = \frac{k_1 x(0)\sum_{k=1}^{n_0} p + k_2\left(1-\exp\left(r+(1/2)\sigma^2\right)\right)\sum_{k=1}^{n_0-1} p}{\exp\left(r+(1/2)\sigma^2\right)-1}. \tag{3.4}$$

Denote U^* is the minimal nonnegative solution of the above equation, it follows from (3.4) that $J'(U^*) = 0$ and U^* is the optimal control threshold value of system (1.3). The proof of Theorem 3.1 is complete. □

In the following, we give the optimal control moment.

Theorem 3.2. *If the assumptions* (H_1)*–*(H_4) *hold, then the optimal control moment of system* (1.3) *can be expressed as*

$$T_0 = \tau\sum_{k=1}^{n_0}\left\{k\left[1-\Phi\left(\frac{\ln(U^*/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U^*/x(0))-nr}{\sigma\sqrt{n}}\right)\right\}. \tag{3.5}$$

where U^* *is defined in Theorem 3.1 and* τ *is the life period of the pest population.*

Proof. By the definition of N_0, we have $T_0 = E[N_0\tau]$. Furthermore, it follows from Lemma 2.6 that

$$\begin{aligned}
T_0 &= \tau\sum_{k=1}^{n_0} kP\{N_0 = k\} \\
&= \tau\sum_{k=1}^{n_0}\left\{k\left[1-\Phi\left(\frac{\ln(U^*/x(0))-kr}{\sigma\sqrt{k}}\right)\right]\prod_{n=1}^{k-1}\Phi\left(\frac{\ln(U^*/x(0))-nr}{\sigma\sqrt{n}}\right)\right\}.
\end{aligned} \tag{3.6}$$

The proof of Theorem 3.2 is complete. □

Finally, we give the estimate of the maximum likelihood estimations of the parameters r and σ of system (1.3).

Theorem 3.3. *Let $\hat{r}$ and $\hat{\sigma}$ be the maximum likelihood estimations of the parameters r and σ, one has*

$$\hat{r} = \frac{y(n) - y(0)}{n},$$
$$\hat{\sigma} = \left[\frac{\sum_{i=1}^{n}\left[y(i) - y(i-1) - \hat{r}\right]^2}{n}\right]^{1/2}, \tag{3.7}$$

where $y(n) = \ln x(n)$.

Proof. From system (1.3), we have

$$\ln x(n) - \ln x(n-1) = r + \sigma\varepsilon_n, \tag{3.8}$$

let $y(n) = \ln x(n)$, then we obtain

$$y(n) - y(n-1) = r + \sigma\varepsilon_n. \tag{3.9}$$

Since ε_n i.i.d $N(0,1)$, we have $[y(n) - y(n-1)]$ i.i.d $N(r,\sigma^2)$. Let $x(n)$ be the quantity of the nth generation pest population, we can obtain corresponding values $y(0), y(1), \ldots, y(n)$, then the likelihood function of parameters r and σ is

$$L(r,\sigma) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left[y(i) - y(i-1) - r\right]^2}{2\sigma^2}\right)$$
$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\frac{1}{2\sigma^2}\right)\sum_{i=1}^{n}\left[y(i) - y(i-1) - r\right]^2. \tag{3.10}$$

Further, we have

$$\ln L(r,\sigma) = -n\ln\sqrt{2\pi}\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[y(i) - y(i-1) - r\right]^2. \tag{3.11}$$

From (3.11), we obtain the following likelihood equation

$$\frac{\partial \ln L(r,\sigma)}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}\left[y(i) - y(i-1) - r\right]^2}{\sigma^3} = 0,$$
$$\frac{\partial \ln L(r,\sigma)}{\partial r} = \frac{y(n) - y(0) - nr}{\sigma^2} = 0, \tag{3.12}$$

Table 1: The average value and absolute error of MLE of parameters with different number of sample.

True	Size	Aver		AE	
(r, σ)	n	r-MLE	σ-MLE	r	σ
	500	0.3049901	0.1000706	0.0049901	0.0000706
$(0.3, 0.1)$	1000	0.3048290	0.0999396	0.0048290	0.0000604
	2000	0.3040423	0.1000054	0.0040423	0.0000054

and the maximum likelihood estimations of r and σ are

$$\hat{r} = \frac{y(n) - y(0)}{n},$$
$$\hat{\sigma} = \left[\frac{\sum_{i=1}^{n}\left[y(i) - y(i-1) - \hat{r}\right]^2}{n}\right]^{1/2}. \tag{3.13}$$

The proof of Theorem 3.3 is complete. □

4. An Example

In this section, to illustrate the feasibility of our theoretical results, we will give the following example.

Example 4.1. Consider the following system

$$x(n) = x(n-1)\exp(0.3 + 0.08\varepsilon_n). \tag{4.1}$$

The choose the loss caused by the unit number pest $k_1 = 0.8$, the expending for annihilating pest once $k_2 = 0.2$, and initial value $x(0) = 0.01$, $n_0 = 5$. By Theorems 3.1 and 3.2, we can obtain the approximates of the optimal control threshold value $U^* = 0.1168326$ and the optimal control moment $T_0 = 3.3217$.

Next, we give the MLE of the parameters r and σ to compare the true value with estimation. In Table 1, for the given true value of parameters $r = 0.3$ and $\sigma = 0.1$, the number of the sample “size n” increases from 500 to 2000, the data of the columns r-MLE and σ-MLE are obtained by the average of 10 MLEs from the data coming from system (1.3). The columns of AE shows the absolute error of MLE. Table 1 shows that, with the augment of the number of the sample, the absolute error of MLE of r and σ will decrease, which implies that it is reasonable to estimate the parameters of system (1.3) by MLE.

Acknowledgments

The authors would like to thank anonymous reviewers for their helpful comments which improved the presentation of their work. The work is supported by the National Natural Science Foundation of China (no. 11261017), the Key Project of Chinese Ministry of Education (no. 210134, 212111) and the Innovation Term of Hubei University for Nationalities (no. MY2011T007).

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