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Research Article

On the Numerical Solution of Fractional Parabolic Partial Differential Equations with the Dirichlet Condition

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The first and second order of accuracy stable difference schemes for the numerical solution of the mixed problem for the fractional parabolic equation are presented. Stability and almost coercive stability estimates for the solution of these difference schemes are obtained. A procedure of modified Gauss elimination method is used for solving these difference schemes in the case of one-dimensional fractional parabolic partial differential equations.

1. Introduction

It is known that various problems in fluid mechanics (dynamics, elasticity) and other areas of physics lead to fractional partial differential equations. Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see, e.g., [1–28] and the references therein).

The role played by stability inequalities (well posedness) in the study of boundary value problems for parabolic partial differential equations is well known (see, e.g., [29–34]). In the present paper, the mixed boundary value problem for the fractional parabolic equation

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} + D_t^{1/2}u(t,x) - \sum_{p=1}^{m}\left(a_p(x)u_{x_p}\right)_{x_p} = f(t,x), \\
x = (x_1,\ldots,x_m) \in \Omega, \quad 0 < t < T, \\
u(t,x) = 0, \quad x \in S, \\
u(0,x) = 0, \quad x \in \overline{\Omega}
\end{gathered} \tag{1.1}$$

is considered. Here $D_t^{1/2} = D_{0+}^{1/2}$ is the standard Riemann-Liouville's derivative of order $1/2$ and Ω is the open cube in the m-dimensional Euclidean space

$$\mathbb{R}^m : \{x \in \Omega : x = (x_1, \ldots, x_m);\ 0 < x_j < 1,\ 1 \le j \le m\} \tag{1.2}$$

with boundary $S, \overline{\Omega} = \Omega \cup S$, $a_p(x)(x \in \Omega)$ and $f(t,x)(t \in (0,T), x \in \Omega)$ are given smooth functions and $a_p(x) \ge a > 0$.

The first and second order of accuracy in t and second orders of accuracy in space variables difference schemes for the approximate solution of problem (1.1) are presented. The stability and almost coercive stability estimates for the solution of these difference schemes are established. A procedure of modified Gauss elimination method is used for solving these difference schemes in the case of one-dimensional fractional parabolic partial differential equations.

2. Difference Schemes and Stability Estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, let us define the grid space

$$\begin{gathered}\overline{\Omega}_h = \{x = x_p = (h_1 p_1, \ldots, h_m p_m),\ p = (p_1, \ldots, p_m),\\ 0 \le p_j \le M_j,\ h_j M_j = 1,\ j = 1, \ldots, m\},\\ \Omega_h = \overline{\Omega}_h \cap \Omega, \qquad S_h = \overline{\Omega}_h \cap S.\end{gathered} \tag{2.1}$$

We introduce the Hilbert space $L_{2h} = L_2(\overline{\Omega}_h)$ of the grid function $\varphi^h(x) = \{\varphi(h_1 j_1, \ldots, h_m j_m)\}$ defined on $\overline{\Omega}$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2(\overline{\Omega}_h)} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m\right)^{1/2}. \tag{2.2}$$

To the differential operator A^x generated by problem (1.1), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{p=1}^{m} \left(a_p(x) u_{\overline{x}_p}^h\right)_{x_p, j_p} \tag{2.3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. Here,

$$\begin{aligned}\varphi_{x_p, j_p} &= \frac{1}{h_p}\left(\varphi(h_1 j_1, \ldots, h_j(j_j+1), \ldots, h_m j_m) - \varphi(h_1 j_1, \ldots, h_j j_j, \ldots, h_m j_m)\right),\\ \varphi_{\overline{x}_p, j_p} &= \frac{1}{h_p}\left(\varphi(h_1 j_1, \ldots, h_j j_j, \ldots, h_m j_m) - \varphi(h_1 j_1, \ldots, h_j(j_j-1), \ldots, h_m j_m)\right).\end{aligned} \tag{2.4}$$

With the help of A_h^x, we arrive at the initial boundary value problem

$$\frac{dv^h(t,x)}{dt} + D_t^{1/2} v^h(t,x) + A_h^x v^h(t,x) = f^h(t,x), \quad 0 < t < T,\ x \in \Omega_h,$$
$$v^h(0,x) = 0, \quad x \in \overline{\Omega} \tag{2.5}$$

for a finite system of ordinary fractional differential equations.

In the second step, applying the first order of approximation formula

$$D_{t_k}^{1/2} u_k = \frac{1}{\sqrt{\pi}} \sum_{r=1}^{k} \frac{\Gamma(k-r+1/2)}{(k-r)!} \left(\frac{u_r - u_{r-1}}{\tau^{1/2}} \right) \tag{2.6}$$

for

$$D_{t_k}^{1/2} u(t_k) = \frac{1}{\Gamma(1/2)} \int_0^{t_k} (t_k - s)^{-1/2} u'(s) ds \tag{2.7}$$

(see [35]) and using the first order of accuracy stable difference scheme for parabolic equations, one can present the first order of accuracy difference scheme with respect to t

$$\frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + D_{t_k}^{1/2} u_k^h(x) + A_h^x u_k^h(x) = f_k^h(x), \quad x \in \overline{\Omega}_h,$$
$$f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \tag{2.8}$$
$$u_0^h(x) = 0, \quad x \in \overline{\Omega}_h$$

for the approximate solution of problem (2.5). Here

$$\Gamma\left(k - r + \frac{1}{2}\right) = \int_0^{\infty} t^{k-r+1/2} e^{-t} dt. \tag{2.9}$$

Moreover, applying the second order of approximation formula

$$D_{t_{k-\tau/2}}^{1/2} u_k = \begin{cases} -\dfrac{2\sqrt{2}}{3\sqrt{\pi}\sqrt{\tau}} u_0 + \dfrac{2\sqrt{2}}{3\sqrt{\pi}\sqrt{\tau}} u_1 + \dfrac{\sqrt{2}\sqrt{\tau}}{3\sqrt{\pi}} u'(0), & k = 1, \\ \dfrac{\sqrt{6}}{\sqrt{\pi}\sqrt{\tau}} \left\{ \dfrac{4}{5} u_0 + \dfrac{2}{5} u_1 + \dfrac{2}{5} u_2 \right\} - \dfrac{\sqrt{6}\sqrt{\tau}}{5\sqrt{\pi}} u'(0), & k = 2, \\ d \displaystyle\sum_{m=2}^{k-1} \{ [(k-m) b_1(k-m) + b_2(k-m)] u_{m-2} & \\ \qquad + [(2m - 2k - 1) b_1(k-m) - 2 b_2(k-m)] u_{m-1} & \\ \qquad + [(k-m+1) b_1(k-m) + b_2(k-m)] u_m \} & \\ \quad + c[-u_{k-2} - 4u_{k-1} + 5u_k], & 3 \le k \le N \end{cases} \tag{2.10}$$

for

$$D^{1/2}_{t_{k-\tau/2}} u\left(t_k - \frac{\tau}{2}\right) = \frac{1}{\Gamma(1/2)} \int_0^{t_k-\tau/2} \left(t_k - \frac{\tau}{2} - s\right)^{-1/2} u'(s)ds \tag{2.11}$$

(see [27]) and the Crank-Nicholson difference scheme for parabolic equations, one can present the second order of accuracy difference scheme with respect to t and to x and

$$\begin{gathered}\frac{u^h_k(x) - u^h_{k-1}(x)}{\tau} + D^{1/2}_{t_k} u^h_k(x) + \frac{1}{2} A^x_h \left(u^h_k(x) + u^h_{k-1}(x)\right) = f^h_k(x), \quad x \in \overline{\Omega}_h, \\ f^h_k(x) = f\left(t_k - \frac{\tau}{2}, x\right), \quad t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \\ u^h_0(x) = 0, \quad x \in \overline{\Omega}_h\end{gathered} \tag{2.12}$$

for the approximate solution of problem (2.5). Here and in the future

$$\begin{gathered}d = \frac{2}{\sqrt{\pi}\sqrt{\tau}}, \qquad c = \frac{\sqrt{2}}{6\sqrt{\pi}\sqrt{\tau}}, \qquad b_1(r) = \sqrt{r + \frac{1}{2}} - \sqrt{r - \frac{1}{2}}, \\ b_2(r) = -\frac{1}{3}\left(\left(r + \frac{1}{2}\right)^{3/2} - \left(r - \frac{1}{2}\right)^{3/2}\right).\end{gathered} \tag{2.13}$$

Theorem 2.1. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, the solutions of difference scheme* (2.8) *and* (2.12) *satisfy the following stability estimate:*

$$\max_{1\le k\le N} \left\| u^h_k \right\|_{L_{2h}} \le C_1 \max_{1\le k\le N} \left\| f^h_k \right\|_{L_{2h}}, \tag{2.14}$$

where C_1 does not depend on τ, h and f^h_k, $1 \le k \le N$.

Proof. We consider the difference scheme (2.8). We have that

$$u^h_k(x) = \sum_{s=1}^{k} R^{k-s+1} F^h_s(x)\tau, \quad 1 \le k \le N, \tag{2.15}$$

where

$$\begin{gathered}R = \left(I + \tau A^x_h\right)^{-1}, \qquad F^h_k(x) = f^h_k(x) - D^{1/2}_{t_k} u^h_k(x), \\ D^{1/2}_{t_k} u^h_k(x) = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!} \tau^{-1/2} \left[-D^{1/2}_{t_m} u^h_m(x) + f^h_m(x)\right].\end{gathered} \tag{2.16}$$

Using formula (2.15), we can write

$$
\begin{aligned}
u_k^h(x) &= \sum_{s=1}^{k} R^{k-s+1}\left[-D_{t_s}^{1/2} u_s^h(x) + f_s^h(x)\right]\tau \\
&= -\sum_{s=1}^{k} R^{k-s+1} D_{t_s}^{1/2} u_s^h(x)\tau + \sum_{s=1}^{k} R^{k-s+1} f_s^h(x)\tau, \quad 1 \le k \le N.
\end{aligned}
\tag{2.17}
$$

First, we will prove that

$$
\max_{1\le k\le N}\left\|D_{t_k}^{1/2} u_k^h\right\|_{L_{2h}} \le M \max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}.
\tag{2.18}
$$

Using formula (2.17), we get

$$
\begin{aligned}
\frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} &= -D_{t_k}^{1/2} u_k^h(x) + f_k^h(x) - A_h^x u_k^h(x) \\
&= -D_{t_k}^{1/2} u_k^h(x) + f_k^h(x) + \sum_{s=1}^{k} A_h^x R^{k-s+1} D_{t_s}^{1/2} u_s^h(x)\tau - \sum_{s=1}^{k} A_h^x R^{k-s+1} f_s^h(x)\tau.
\end{aligned}
\tag{2.19}
$$

Using formulas (2.16) and (2.19), we obtain

$$
\begin{aligned}
D_{t_k}^{1/2} u_k^h(x) &= \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!}\left(\frac{u_m^h(x) - u_{m-1}^h(x)}{\tau^{1/2}}\right) \\
&= \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!}\tau^{1/2}\left[-D_{t_m}^{1/2} u_m^h(x) + f_m^h(x)\right] \\
&\quad + \frac{1}{\sqrt{\pi}} \sum_{s=1}^{k}\sum_{m=s}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!}\tau^{3/2} A_h^x R^{m-s+1} D_{t_s}^{1/2} u_s^h(x) \\
&\quad - \frac{1}{\sqrt{\pi}} \sum_{s=1}^{k}\sum_{m=s}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!}\tau^{3/2} A_h^x R^{m-s+1} f_s^h(x).
\end{aligned}
\tag{2.20}
$$

Now, let us estimate $z_k = \|D_{t_k}^{1/2} u_k^h\|_{L_{2h}}, 1 \le k \le N$. Applying the triangle inequality and the estimate [34]

$$
\left\|A_h^x R^k\right\|_{L_{2h}\to L_{2h}} \le \frac{M}{k\tau}, \quad \left\|R^k\right\|_{L_{2h}\to L_{2h}} \le M, \quad 1 \le k \le N,
\tag{2.21}
$$

we get

$$
\begin{aligned}
z_k \leq{}& \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!} \tau^{1/2} \left[z_m + \left\| f_m^h \right\|_{L_{2h}} \right] \\
&+ \frac{1}{\sqrt{\pi}} \sum_{s=1}^{k} \left\| \sum_{m=s}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!} A_h^x R^{m-s+1} \right\|_{L_{2h} \to L_{2h}} z_s \tau^{3/2} \\
&+ \frac{1}{\sqrt{\pi}} \sum_{s=1}^{k} \left\| \sum_{m=s}^{k} \frac{\Gamma(k-m+1/2)}{(k-m)!} A_h^x R^{m-s+1} \right\|_{L_{2h} \to L_{2h}} \left\| f_s^h \right\|_{L_{2h}} \tau^{3/2} \\
\leq{}& M_3 \sum_{s=1}^{k-1} \frac{1}{\sqrt{(k-s)\tau}} \tau \left[z_s + \left\| f_s^h \right\|_{L_{2h}} \right] + M_4 \left[z_s + \left\| f_s^h \right\|_{L_{2h}} \right] \tau^{1/2}
\end{aligned}
\tag{2.22}
$$

for any $k = 1, \ldots, N$. Then, using the difference analogy of integral inequality, we get (2.18).

Second, applying formula (2.17), estimates (2.18) and (2.21), we obtain

$$
\begin{aligned}
\left\| u_k^h \right\|_{L_{2h}} ={}& \sum_{s=1}^{k} \left\| R^{k-s+1} \right\|_{L_{2h} \to L_{2h}} \left\| D_{t_s}^{1/2} u_s^h \right\|_{L_{2h}} \tau \\
&+ \sum_{s=1}^{k} \left\| R^{k-s+1} \right\|_{L_{2h} \to L_{2h}} \left\| f_s^h \right\|_{L_{2h}} \tau \leq C_1 \max_{1 \leq k \leq N} \left\| f_k^h \right\|_{L_{2h}}.
\end{aligned}
\tag{2.23}
$$

Estimate (2.14) for the solution of (2.8) is proved. The proof of estimate (2.14) for the solution of (2.12) follows the scheme of the proof of estimate (2.14) for the solution of (2.8) and rely on the estimate

$$
\left\| A_h^x B^k C^2 \right\|_{L_{2h} \to L_{2h}} \leq \frac{1}{k\tau}, \quad \left\| B^k \right\|_{L_{2h} \to L_{2h}} \leq 1, \quad 1 \leq k \leq N. \tag{2.24}
$$

Here,

$$
B = \left(I - \frac{\tau}{2} A_h^x \right) \left(I + \frac{\tau}{2} A_h^x \right)^{-1}, \qquad C = \left(I + \frac{\tau}{2} A_h^x \right)^{-1}. \tag{2.25}
$$

Theorem 2.1 is proved. □

Theorem 2.2. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, the solutions of difference scheme* (2.8) *satisfy the following almost coercive stability estimate:*

$$
\max_{1 \leq k \leq N} \left\| \frac{u_k^h - u_{k-1}^h}{\tau} \right\|_{L_{2h}} + \max_{1 \leq k \leq N} \sum_{p=1}^{m} \left\| \left(u_k^h \right)_{\overline{x}_p x_p, j_p} \right\|_{L_{2h}} \leq C_2 \ln \frac{1}{\tau + |h|} \max_{1 \leq k \leq N} \left\| f_k^h \right\|_{L_{2h}}, \tag{2.26}
$$

where C_2 is independent of τ, h and f_k^h, $1 \leq k \leq N$.

Proof. We will prove the estimate

$$\max_{1\le k\le N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}\le M\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|A_h^x\right\|_{L_{2h}\to L_{2h}}\right|\right\}\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}. \tag{2.27}$$

Using formula (2.19) and estimate (2.21), we obtain

$$\begin{gathered}\max_{1\le k\le N}\sum_{s=1}^{m}\left\|A_h^xR^{k-s+1}f_s^h\tau\right\|_{L_{2h}}\le M\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|A_h^x\right\|_{L_{2h}\to L_{2h}}\right|\right\}\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}},\\ \max_{1\le k\le N}\sum_{s=1}^{m}\left\|A_h^xR^{k-s+1}D_{t_s}^{1/2}u_s^h\tau\right\|_{L_{2h}}\le M\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|A_h^x\right\|_{L_{2h}\to L_{2h}}\right|\right\}\max_{1\le k\le N}\left\|D_{t_k}^{1/2}u_k^h\tau\right\|_{L_{2h}}\end{gathered} \tag{2.28}$$

and estimate (2.18), the triangle inequality and equation (2.8), we get (2.27). From that it follows:

$$\max_{1\le k\le N}\left\|A_h^xu_k^h\right\|_{L_{2h}}\le M_1\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|A_h^x\right\|_{L_{2h}\to L_{2h}}\right|\right\}\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}. \tag{2.29}$$

Then, the proof of estimate (2.26) is based on estimates (2.27), (2.29), and the following theorem on coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 2.3. *For the solutions of the elliptic difference problem*

$$\begin{gathered}A_h^xu^h(x)=w^h(x),\quad x\in\Omega_h,\\ u^h(x)=0,\quad x\in S_h\end{gathered} \tag{2.30}$$

the following coercivity inequality holds (see [14, 36])

$$\sum_{p=1}^{m}\left\|u_{x_p\bar{x}_p,j_p}^h\right\|_{L_{2h}}\le C\left\|w^h\right\|_{L_{2h}}, \tag{2.31}$$

where C does not depend on h and w^h.

Theorem 2.2 is proved. □

Theorem 2.4. *Let τ and $|h|=\sqrt{h_1^2+\cdots+h_m^2}$ be sufficiently small positive numbers. Then, the solutions of difference scheme* (2.12) *satisfy the following almost coercive stability estimate:*

$$\max_{1\le k\le N}\left\|\frac{u_k^h-u_{k-1}^h}{\tau}\right\|_{L_{2h}}+\max_{1\le k\le N}\frac{1}{2}\sum_{p=1}^{m}\left\|\left(u_k^h+u_{k-1}^h\right)_{\bar{x}_px_p,j_p}\right\|_{L_{2h}}\le C_3\ln\frac{1}{\tau+|h|}\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{2h}}, \tag{2.32}$$

where C_3 does not depend on τ, h and f_k^h, $1\le k\le N$.

The proof of Theorem 2.4 follows the proof of Theorem 2.2 and on the estimate (2.24) and the self-adjointness and positive definiteness of operator A_h^x in L_{2h} and Theorem 2.3.

Remark 2.5. The stability estimates of Theorems 2.1, 2.2, and 2.4 are satisfied in the case of operator

$$Au = -\sum_{k=1}^{n} a_k(x)\frac{\partial^2 u}{\partial x_k^2} + \sum_{k=1}^{n} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u \tag{2.33}$$

with Dirichlet condition $u = 0$ in S. In this case, A is not self-adjoint operator in H. Nevertheless, $Au = A_0u + Bu$ and A_0 is a self-adjoint positive definite operator in H and BA_0^{-1} is bounded in H. The proof of this statement is based on the abstract results of [14] and difference analogy of integral inequality.

The method of proofs of Theorems 2.1, 2.2, and 2.4 enables us to obtain the estimate of convergence of difference schemes of the first and second order of accuracy for approximate solutions of the initial-boundary value problem

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} - \sum_{p=1}^{n} a_p(x)u_{x_p x_p} + \sum_{p=1}^{n} b_p(x)u_{x_p} + D_t^{\alpha}u(t,x) \\
= f(t,x;u(t,x),u_{x_1}(t,x),\ldots,u_{x_n}(t,x)), \\
x = (x_1,\ldots,x_n) \in \Omega, \quad 0 < t < T, \\
u(0,x) = 0, \quad x \in \overline{\Omega}, \\
u(t,x) = 0, \quad x \in S
\end{gathered} \tag{2.34}$$

for semilinear fractional parabolic partial differential equations.

Note that, one has not been able to obtain a sharp estimate for the constant figuring in the stability estimates of Theorems 2.1, 2.2, and 2.4. Therefore, our interest in the present paper is studying the difference schemes (2.8) and (2.12) by numerical experiments. Applying these difference schemes, the numerical methods are proposed in the following section for solving the one-dimensional fractional parabolic partial differential equation. The method is illustrated by numerical experiments.

3. Numerical Results

For the numerical result, the mixed problem

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} + D_t^{1/2}u(t,x) - \frac{\partial}{\partial x}\left((1+x)\frac{\partial u(t,x)}{\partial x}\right) = f(t,x), \\
f(t,x) = \left(3 + \frac{16\sqrt{t}}{5\sqrt{\pi}} + \pi^2 t(1+x)\right)t^2 \sin \pi x - \pi t^3 \cos \pi x, \quad 0 < t < 1,\ 0 < x < 1, \\
u(t,0) = u(t,1) = 0, \quad 0 \le t \le 1, \\
u(0,x) = 0, \quad 0 \le x \le 1
\end{gathered} \tag{3.1}$$

for the one-dimensional fractional parabolic partial differential equation is considered. The exact solution of problem (3.1) is

$$u(t,x) = t^3 \sin \pi x. \tag{3.2}$$

First, applying difference scheme (2.8), we obtain

$$\begin{gathered}
\frac{u_n^k - u_n^{k-1}}{\tau} + \frac{1}{\sqrt{\pi}} \sum_{r=1}^{k} \frac{\Gamma(k-r+1/2)}{(k-r)!} \left(\frac{u_n^r - u_n^{r-1}}{\tau^{1/2}} \right) \\
- \frac{1}{h} \left[(1+x_{n+1}) \frac{u_{n+1}^k - u_n^k}{h} - (1+x_n) \frac{u_n^k - u_{n-1}^k}{h} \right] = \varphi_n^k, \\
\varphi_n^k = f(t_k, x_n), \quad t_k = k\tau,\ x_n = nh,\ 1 \le k \le N,\ 1 \le n \le M-1, \\
u_0^k = u_M^k = 0, \quad 0 \le k \le N, \\
u_n^0 = 0, \quad 0 \le n \le M.
\end{gathered} \tag{3.3}$$

We can rewrite it in the system of equations with matrix coefficients

$$\begin{gathered}
AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, \quad 1 \le n \le M-1, \\
U_0 = \widetilde{0}, \qquad U_M = \widetilde{0}.
\end{gathered} \tag{3.4}$$

Here and in the future $\widetilde{0}$ is the $(N+1) \times 1$ zero matrix and $A = a_n D, C = c_n D$,

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{N,1} & b_{N,2} & b_{N,3} & \cdots & b_{N,N} & 0 \\ b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad U_q = \begin{bmatrix} u_q^0 \\ u_q^1 \\ u_q^2 \\ \vdots \\ u_q^{N-1} \\ u_q^N \end{bmatrix}_{(N+1)\times 1}, \quad q = n \pm 1, n,$$

$$a_n = -\frac{1 + x_{n+1}}{h^2}, \qquad c_n = -\frac{1 + x_n}{h^2},$$

$$b_{11} = 1, \qquad b_{21} = -\frac{1}{\sqrt{\tau}} - \frac{1}{\tau}, \qquad b_{22} = \frac{1}{\sqrt{\tau}} + \frac{1}{\tau} + \frac{2 + x_{n+1} + x_n}{h^2},$$

$$b_{31} = -\frac{\Gamma(1 + 1/2)}{\sqrt{\pi\tau}}, \qquad b_{32} = \frac{\Gamma(1 + 1/2) - \Gamma(1/2)}{\sqrt{\pi\tau}} - \frac{1}{\tau}, \qquad b_{33} = \frac{1}{\sqrt{\tau}} + \frac{1}{\tau} + \frac{2 + x_{n+1} + x_n}{h^2},$$

$$b_{ij} = \begin{cases} -\dfrac{\Gamma(i - 2 + 1/2)}{\sqrt{\pi\tau}(i-2)!}, & j = 1, \\[2ex] \dfrac{\Gamma(i - j + 1/2)}{\sqrt{\pi\tau}(i-j)!} - \dfrac{\Gamma(i - j - 1 + 1/2)}{\sqrt{\pi\tau}(i-j-1)!}, & 2 \le j \le i - 2, \\[2ex] \dfrac{\Gamma(1 + 1/2) - \Gamma(1/2)}{\sqrt{\pi\tau}} - \dfrac{1}{\tau}, & j = i - 1, \\[2ex] \dfrac{1}{\sqrt{\tau}} + \dfrac{1}{\tau} + \dfrac{2 + x_{n+1} + x_n}{h^2}, & j = i, \\ 0, & i < j \le N + 1 \end{cases} \tag{3.5}$$

for $i = 4, 5, \ldots, N + 1$ and

$$\varphi_n^k = \left[3 + \frac{16\sqrt{k\tau}}{5\sqrt{\pi}} + \pi^2(k\tau)(1 + nh)\right](k\tau)^2 \sin(\pi nh) - \pi(k\tau)^3 \cos(\pi nh). \tag{3.6}$$

So, we have the second-order difference equation with respect to n matrix coefficients. This type system was developed by Samarskii and Nikolaev [37]. To solve this difference equation we have applied a procedure for difference equation with respect to k matrix coefficients. Hence, we seek a solution of the matrix equation in the following form:

$$U_j = \alpha_{j+1} U_{j+1} + \beta_{j+1}, \quad U_M = 0, \quad j = M - 1, \ldots, 2, 1, \tag{3.7}$$

where $\alpha_j (j = 1, 2, \ldots, M)$ are $(N + 1) \times (N + 1)$ square matrices and $\beta_j (j = 1, 2, \ldots, M)$ are $(N + 1) \times 1$ column matrices defined by

$$\alpha_{j+1} = -\left(B + C\alpha_j\right)^{-1} A, \tag{3.8}$$

$$\beta_{j+1} = \left(B + C\alpha_j\right)^{-1} \left(D\varphi_j - C\beta_j\right), \quad j = 1, 2, \ldots, M - 1, \tag{3.9}$$

where $j = 1, 2, \ldots, M - 1$, α_1 is the $(N + 1) \times (N + 1)$ zero matrix and β_1 is the $(N + 1) \times 1$ zero matrix.

Second, applying difference scheme (2.12), we obtain

$$
\begin{gathered}
\frac{u_n^k-u_n^{k-1}}{\tau}+D_{t_k-\tau/2}^{1/2}u_n^k-\frac{1}{2}\left[(1+x_n)\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+\frac{u_{n+1}^k-u_{n-1}^k}{2h}\right.\\
\left.+(1+x_n)\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\frac{u_{n+1}^{k-1}-u_{n-1}^{k-1}}{2h}\right]=\varphi_n^k,\\
\varphi_n^k=f\left(t_k-\frac{\tau}{2},x_n\right),\quad t_k=k\tau,\ x_n=nh,\ 1\le k\le N,\ 1\le n\le M-1,\\
u_0^k=u_M^k=0,\quad 0\le k\le N,\\
u_n^0=0,\quad 0\le n\le M,
\end{gathered}
\tag{3.10}
$$

where

$$
D_{t_k-\tau/2}^{1/2}u_n^k=\begin{cases}
-\dfrac{2\sqrt{2}}{3\sqrt{\pi}\sqrt{\tau}}u_n^0+\dfrac{2\sqrt{2}}{3\sqrt{\pi}\sqrt{\tau}}u_n^1+\dfrac{\sqrt{2}\sqrt{\tau}}{3\sqrt{\pi}}u'(0,x_n), & k=1,\\[2ex]
\dfrac{\sqrt{6}}{\sqrt{\pi}\sqrt{\tau}}\left\{\dfrac{4}{5}u_n^0+\dfrac{2}{5}u_n^1+\dfrac{2}{5}u_n^2\right\}-\dfrac{\sqrt{6}\sqrt{\tau}}{5\sqrt{\pi}}u'(0,x_n), & k=2,\\[2ex]
d\displaystyle\sum_{m=2}^{k-1}\Big\{[(k-m)b_1(k-m)+b_2(k-m)]u_n^{m-2} & \\
\qquad+[(2m-2k-1)b_1(k-m)-2b_2(k-m)]u_n^{m-1} & \\
\qquad+[(k-m+1)b_1(k-m)+b_2(k-m)]u_n^{m}\Big\} & \\
\quad+c\left[-u_n^{k-2}-4u_n^{k-1}+5u_n^k\right], & 3\le k\le N
\end{cases}
\tag{3.11}
$$

for any n, $1\le n\le M-1$. We get the system of equations in the matrix form

$$
\begin{gathered}
AU_{n+1}+BU_n+CU_{n-1}=D\varphi_n,\quad 1\le n\le M-1,\\
U_0=\widetilde{0},\qquad U_M=\widetilde{0},
\end{gathered}
\tag{3.12}
$$

where $A = a_n F, C = c_n F$,

$$F = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{N,1} & b_{N,2} & b_{N,3} & \cdots & b_{N,N} & 0 \\ b_{N+1,1} & b_{N+1,2} & b_{N+1,3} & \cdots & b_{N+1,N} & b_{N+1,N+1} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \varphi_n^2 \\ \vdots \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}, \quad U_q = \begin{bmatrix} U_q^0 \\ U_q^1 \\ U_q^2 \\ \vdots \\ U_q^{N-1} \\ U_q^N \end{bmatrix}_{(N+1)\times 1}, \quad q = n \pm 1, n,$$

$$a_n = -\frac{1}{2}\left(\frac{1+x_n}{h^2} + \frac{1}{2h}\right), \qquad c_n = -\frac{1}{2}\left(\frac{1+x_n}{h^2} - \frac{1}{2h}\right),$$

$$b_{11} = 1, \qquad b_{21} = -\frac{2\sqrt{2}}{3\sqrt{\pi\tau}} - \frac{1}{\tau} + \frac{1+x_n}{h^2}, \qquad b_{22} = \frac{2\sqrt{2}}{3\sqrt{\pi\tau}} + \frac{1}{\tau} + \frac{1+x_n}{h^2},$$

$$b_{31} = \frac{4\sqrt{6}}{5\sqrt{\pi\tau}}, \qquad b_{32} = \frac{2\sqrt{6}}{5\sqrt{\pi\tau}} - \frac{1}{\tau} + \frac{1+x_n}{h^2}, \qquad b_{33} = \frac{2\sqrt{6}}{5\sqrt{\pi\tau}} + \frac{1}{\tau} + \frac{1+x_n}{h^2},$$

$$b_{41} = d[1b_1(1) + b_2(1)], \qquad b_{42} = d[-3b_1(1) - 2b_2(1)] - c,$$

$$b_{43} = d[2b_1(1) + b_2(1)] - 4c - \frac{1}{\tau} + \frac{1+x_n}{h^2}, \qquad b_{44} = 5c + \frac{1}{\tau} + \frac{1+x_n}{h^2},$$

$$b_{51} = d[2b_1(2) + b_2(2)], \qquad b_{52} = d[-5b_1(2) - 2b_2(2) + 1b_1(1) + b_2(1)],$$

$$b_{53} = d[3b_1(2) + b_2(2) - 3b_1(1) - 2b_2(1)] - c,$$

$$b_{54} = d[2b_1(1) + \quad b_2(1)] - 4c - \frac{1}{\tau} + \frac{1+x_n}{h^2}, \qquad b_{55} = 5c + \frac{1}{\tau} + \frac{1+x_n}{h^2},$$

$$b_{ij} = \begin{cases} d\left[(i-3)b_1(i-3)+b_2(i-3)\right], & j=1, \\ d\left[(5-2i)b_1(i-3)-2b_2(i-3)+(i-4)b_1(i-4)+b_2(i-4)\right], & j=2, \\ d\left[(i-j+1)b_1(i-j)+b_2(i-j)+(2j-2i+1)b_1(i-j-1)\right. & \\ \quad \left. -2b_2(i-j-1)+(i-j-2)b_1(i-j-2)+b_2(i-j-2)\right], & 3 \le j \le i-3, \\ d\left[3b_1(2)+b_2(2)-3b_1(1)-2b_2(1)\right]- \quad c, & j=i-2, \\ d\left[2b_1(1)+b_2(1)\right]-4c-\dfrac{1}{\tau}+\dfrac{1+x_n}{h^2}, & j=i-1, \\ 5c+\dfrac{1}{\tau}+\dfrac{1+x_n}{h^2}, & j=i, \\ 0, & i<j\le N+1 \end{cases} \tag{3.13}$$

for $i = 6, 7, \ldots, N+1$ and

$$\varphi_n^k = \left[3+\frac{16\sqrt{k\tau}}{5\sqrt{\pi}}+\pi^2(k\tau)(1+nh)\right](k\tau)^2\sin(\pi nh)-\pi(k\tau)^3\cos(\pi nh). \tag{3.14}$$

So, we have again the second-order difference equation with respect to n matrix coefficients. Therefore, applying the same procedure of modified Gauss elimination method (3.7) and (3.8) difference equation (3.12).

Finally, we give the results of the numerical analysis. The numerical solutions are recorded for different values of N and M and u_n^k represents the numerical solutions of these difference schemes at (t_k, x_n). The error is computed by the following formula:

$$E_M^N = \max_{1\le k\le N,\, 1\le n\le M-1}\left|u(t_k, x_n)-u_n^k\right|. \tag{3.15}$$

Table 1 is constructed for $N = M = 20$, 40, and 80, respectively.

Thus, by using the Crank-Nicholson difference scheme, the accuracy of solution increases faster than the first order of accuracy difference scheme.

4. Conclusion

In this study, the first and second order of accuracy stable difference schemes for the numerical solution of the mixed problem for the fractional parabolic equation are investigated. We have obtained stability and almost coercive stability estimates for the solution of these difference schemes. The theoretical statements for the solution of these difference schemes for one-dimensional parabolic equations are supported by numerical example in computer.

Table 1: Error analysis.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference scheme (2.8)	0.0040	0.0020	0.0010
Difference scheme (2.12)	0.0006726	0.0001678	0.00004187

We showed that the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

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