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Research Article

Multiple Periodic Solutions of a Ratio-Dependent Predator-Prey Discrete Model

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A delayed ratio-dependent predator-prey discrete-time model with nonmonotone functional response is investigated in this paper. By using the continuation theorem of Mawhins coincidence degree theory, some new sufficient conditions are obtained for the existence of multiple positive periodic solutions of the discrete model. An example is given to illustrate the feasibility of the obtained result.

1. Introduction

It is known that one of important factors impacted on a predator-prey system is the functional response. Holling proposed three types of functional response functions, namely, Holling I, Holling II, and Holling III, which are all monotonously nondescending [1]. But for some predator-prey systems, when the prey density reaches a high level, the growth of predator may be inhibited; that is, to say, the predator's functional response is not monotonously increasing. In order to describe such kind of biological phenomena, Andrews proposed the so-called Holling IV functional response function [2]

$$g(x) = \frac{cx}{m^2 + nx + x^2}, \tag{1.1}$$

which is humped and declines at high prey densities x. Recently, many authors have explored the dynamics of predator-prey systems with Holling IV type functional responses [3–11]. For example, Ruan and Xiao considered the following predator-prey model [5]:

$$\begin{aligned}\frac{dx}{dt} &= x(t)\left[a - bx(t) - \frac{cy(t)}{m^2 + x^2(t)}\right],\\ \frac{dy}{dt} &= y(t)\left[-d + \frac{hx(t-\tau)}{m^2 + x^2(t-\tau)}\right],\end{aligned} \tag{1.2}$$

where $x(t)$ and $y(t)$ represent predator and prey densities, respectively. In (1.2), the functional response function $g_{\mathrm{IV}}(x) = cx/(m^2 + x^2)$ is a special case of Holling IV functional response.

The functional response functions mentioned previously only depend on the prey x. But some biologists have argued that the functional response should be ratio dependent or semi-ratio dependent in many situations. Based on biological and physiological evidences, Arditi and Ginzburg first proposed the ratio-dependent predator-prey model [12]

$$\begin{aligned}\frac{dx}{dt} &= x(t)\left[a - bx(t) - \frac{cy(t)}{my(t) + x(t)}\right],\\ \frac{dy}{dt} &= y(t)\left[-d + \frac{hx(t)}{my(t) + x(t)}\right],\end{aligned} \tag{1.3}$$

where the functional response function $g_r(x, y) = (cx/y)/(m + x/y)$ is ratio dependent. Many researchers have putted up a great lot of works on the ratio-dependent or semi-ratio-dependent predator-prey system [13–19].

Recently, some researchers incorporated the ratio-dependent theory and the inhibitory effect on the specific growth rate into the predator-prey model [3, 7, 11, 15]. Ding et al. considered a semi-ratio-dependent predator-prey system with nonmonotonic functional response and time delay [11]; they obtained some sufficient conditions for the existence and global stability of a positive periodic solution to this system. Hu and Xia considered a functional response function [7, 15]:

$$g_{\mathrm{IV}}\left(\frac{x}{y}\right) = \frac{cxy}{m^2y^2 + x^2}. \tag{1.4}$$

With the functional response function, Xia and Han proposed the following periodic ratio-dependent model with nonmonotone functional response [15]:

$$\begin{aligned}\frac{dx(t)}{dt} &= x(t)\left[a(t) - b(t)\int_{-\infty}^{t} K(t-s)x(s)ds - \frac{c(t)y^2(t)}{m^2y^2(t) + x^2(t)}\right],\\ \frac{dy(t)}{dt} &= y(t)\left[-d(t) + \frac{h(t)x(t-\tau(t))y(t-\tau(t))}{m^2y^2(t-\tau(t)) + x^2(t-\tau(t))}\right],\end{aligned} \tag{1.5}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $h(t)$ are all positive periodic continuous functions with period $\omega > 0$, m is a positive real constant, and $K(s) : \mathbb{R}^+ \to \mathbb{R}^+$ is a delay kernel function. Based on Mawhins coincidence degree, they obtained some sufficient conditions for the existence of two positive periodic solutions of the ratio-dependent model (1.5).

It is well known that discrete population models are more appropriate than the continuous models when the populations do not overlap among generations. Therefore, many scholars have studied some discrete population models [3, 4, 14, 16–19]. For example, Lu and Wang considered the following discrete semi-ratio-dependent predator-prey system with Holling type IV functional response and time delay [3]:

$$\begin{aligned} x(n+1) &= x(n)\exp\left[r_1(n) - a_{11}(n)x(n-\tau) - \frac{a_{12}(n)y(n)}{m^2 + x^2(n)}\right],\\ y(n+1) &= y(n)\exp\left[r_2(n) - a_{21}(n)\frac{y(n)}{x(n)}\right]. \end{aligned} \tag{1.6}$$

They proved that the system (1.6) is permanent and globally attractive under some appropriate conditions. Furthermore, they also obtained some sufficient conditions which guarantee the existence and global attractivity of positive periodic solution.

Motivated by the mentioned previously, this paper is to investigate the existence of multiple periodic solutions of the following discrete ratio-dependent model with nonmonotone functional response:

$$\begin{aligned} x(n+1) &= x(n)\exp\left[a(n) - b(n)\sum_{l=0}^{+\infty}K(l)x(n-l) - \frac{c(n)y^2(n)}{m^2y^2(n) + x^2(n)}\right],\\ y(n+1) &= y(n)\exp\left[-d(n) + \frac{h(n)x(n-\tau(n))y(n-\tau(n))}{m^2y^2(n-\tau(n)) + x^2(n-\tau(n))}\right], \end{aligned} \tag{1.7}$$

for $n \in \mathbb{Z}_0^+$, where $a, d : \mathbb{Z}_0^+ \to \mathbb{R}$, $b, c, h : \mathbb{Z}_0^+ \to \mathbb{R}^+$, and $\tau : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+$ are all ω-periodic sequences, ω is a positive integer, m is a positive real constant, and $K : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+$ satisfies $\sum_{l=0}^{+\infty} K(l) = 1$, where $\mathbb{Z}$, $\mathbb{Z}_0^+$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}_0^+$, and $\mathbb{R}^+$ denote the sets of all integers, nonnegative integers, positive integers, real numbers, nonnegative real numbers, and positive real numbers, respectively. The model (1.7) is created from the continuous-time system (1.5) by employing the semidiscretization technique.

The initial conditions associated with (1.7) are of the form

$$x(n) = \phi(n), \quad y(n) = \varphi(n), \quad n \in \mathbb{Z} - \mathbb{Z}^+, \tag{1.8}$$

where $\phi(n) \ge 0$, $\varphi(n) \ge 0$ for $n \in \mathbb{Z} - \mathbb{Z}_0^+$ and $\phi(0) > 0$, $\varphi(0) > 0$.

2. Preliminaries

For convenience, we will use the following notations in the discussion:

$$I_\omega = \{0, 1, \ldots, \omega - 1\}, \qquad \overline{f} := \frac{1}{\omega}\sum_{k=0}^{\omega-1} f(k), \qquad \Delta u(n) = u(n+1) - u(n), \tag{2.1}$$

where f is a ω-periodic sequence of real numbers defined for $k \in \mathbb{Z}$.

In the system (1.7), the time delay kernel sequence $K(l)$ satisfies $\sum_{l=0}^{+\infty} K(l) = 1$. Therefore, if we define

$$G(l) = \sum_{k=0}^{+\infty} K(l + k\omega), \quad l \in I_\omega, \tag{2.2}$$

then $G(l)$ is uniformly convergent with respect to $l \in I_\omega$, and it satisfies $\sum_{l=0}^{\omega-1} G(l) = 1$.

Lemma 2.1. *$(x^*(n), y^*(n))$ is a positive ω-periodic solution of system* (1.7) *if and only if $(u_1^*(n), u_2^*(n)) = (\ln(x^*(n)/y^*(n)), \ln y^*(n))$ is a ω-periodic solution of the following system* (2.3)*:*

$$\begin{aligned} \Delta u_1(n) &= a(n) + d(n) - b(n)\sum_{l=0}^{\omega-1} G(l)\exp[u_1(n-l) + u_2(n-l)] \\ &\quad - \frac{c(n)}{m^2 + \exp[2u_1(n)]} - \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2 + \exp[2u_1(n-\tau(n))]}, \\ \Delta u_2(n) &= -d(n) + \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2 + \exp[2u_1(n-\tau(n))]}, \end{aligned} \tag{2.3}$$

where $a(n)$, $b(n)$, $c(n)$, $d(n)$, $h(n)$, and $\tau(n)$ are the same as those in model (1.7).

Proof. Let $(u_1(n), u_2(n)) = (\ln(x(n)/y(n)), \ln y(n))$; then the system (1.7) can be rewritten as

$$\begin{aligned} \Delta u_1(n) &= a(n) + d(n) - b(n)\sum_{l=0}^{+\infty} K(l)\exp[u_1(n-l) + u_2(n-l)] \\ &\quad - \frac{c(n)}{m^2 + \exp[2u_1(n)]} - \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2 + \exp[2u_1(n-\tau(n))]}, \\ \Delta u_2(n) &= -d(n) + \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2 + \exp[2u_1(n-\tau(n))]}. \end{aligned} \tag{2.4}$$

Therefore, $(x^*(n), y^*(n))$ is a positive ω-periodic solution of system (1.7) if and only if $(u_1^*(n), u_2^*(n)) = (\ln(x^*(n)/y^*(n)), \ln y^*(n))$ is a ω-periodic solution of the system (2.4).

Notice that

$$\begin{aligned} &\sum_{l=0}^{+\infty} K(l)\exp[u_1(n-l) + u_2(n-l)] \\ &\quad = \sum_{k=0}^{+\infty} \sum_{l=k\omega}^{(k+1)\omega-1} K(l)\exp[u_1(n-l) + u_2(n-l)] \\ &\quad = \sum_{k=0}^{+\infty} \sum_{s=0}^{\omega-1} K(s + k\omega)\exp[u_1(n-s-k\omega) + u_2(n-s-k\omega)]. \end{aligned} \tag{2.5}$$

If $(u_1(n), u_2(n))$ is ω-periodic, then we have

$$\begin{aligned}&\sum_{l=0}^{+\infty} K(l)\exp[u_1(n-l)+u_2(n-l)]\\&\qquad=\sum_{k=0}^{+\infty}\sum_{s=0}^{\omega-1} K(s+k\omega)\exp[u_1(n-s)+u_2(n-s)].\end{aligned}\tag{2.6}$$

Because $G(l)=\sum_{k=0}^{+\infty} K(l+k\omega)$ is uniformly convergent with respect to $l\in I_\omega$, so we have

$$\begin{aligned}&\sum_{l=0}^{+\infty} K(l)\exp[u_1(n-l)+u_2(n-l)]\\&\qquad=\sum_{s=0}^{\omega-1}\sum_{k=0}^{+\infty} K(s+k\omega)\exp[u_1(n-s)+u_2(n-s)]\\&\qquad=\sum_{s=0}^{\omega-1} G(s)\exp[u_1(n-s)+u_2(n-s)].\end{aligned}\tag{2.7}$$

Therefore, $(u_1^*(n), u_2^*(n))$ is a ω-periodic solution of the system (2.3) if and only if it is a ω-periodic solution of the system (2.4). This completes the proof. □

From (1.8), the initial conditions associated with (2.3) are of the form

$$x(n)=\phi(n),\quad y(n)=\psi(n),\quad n\in\{0,-1,-2,\ldots,\tau_0\},\tag{2.8}$$

where $\tau_0=\max_{n\in\mathbb{Z}-\mathbb{Z}^+}\{\omega-1,\tau(n)\}$, $\phi(n)\ge 0$, $\psi(n)\ge 0$ for $n\in\mathbb{Z}-\mathbb{Z}_0^+$, and $\phi(0)>0$, $\psi(0)>0$.

Throughout this paper, we assume that

(H1) $\overline{d}>0$, $\overline{h}>2m\overline{d}\exp[(\overline{|a|}+\overline{|d|}+\overline{a}+\overline{d})\omega]$;

(H2) $m^2\overline{a}>\overline{c}$.

Under the assumption (H1), there exist the following six positive numbers:

$$\begin{aligned}l_\pm&=\frac{\overline{h}\exp\left[\left(\overline{|a|}+\overline{|d|}+\overline{a}+\overline{d}\right)\omega\right]\pm\sqrt{\overline{h}^2\exp\left[4\left(\overline{a}+\overline{d}\right)\omega\right]-4m^2\overline{d}^2}}{2\overline{d}},\\v_\pm&=\frac{\overline{h}\exp\left[-\left(\overline{|a|}+\overline{|d|}+\overline{a}+\overline{d}\right)\omega\right]\pm\sqrt{\overline{h}^2\exp\left[-4\left(\overline{a}+\overline{d}\right)\omega\right]-4m^2\overline{d}^2}}{2\overline{d}},\\u_\pm&=\frac{\overline{h}\pm\sqrt{\overline{h}^2-4m^2\overline{d}^2}}{2\overline{d}}.\end{aligned}\tag{2.9}$$

Obviously,

$$l_- < u_- < v_- < v_+ < u_+ < l_+. \tag{2.10}$$

In this paper, we adopt coincidence degree theory to prove the existence of multiple positive periodic solutions of (1.7). We first summarize some concepts and results from the book by Gaines and Mawhin [20]. Let X and Y be normed vector spaces. Define an abstract equation in X,

$$Lx = \lambda Nx, \tag{2.11}$$

where $L : \mathrm{Dom}\, L \subset X \to Y$ is a linear mapping, and $N : X \to Y$ is a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \ker L = \mathrm{codim}\ \mathrm{Im}\ L < +\infty$ and $\mathrm{Im}\, L$ is closed in Y. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}\, P = \ker L$ and $\mathrm{Im}\ L = \ker Q = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}\, L \cap \ker P} : (I - P)X \to \mathrm{Im}\ L$ is invertible, and its inverse is denoted by K_p. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Because $\mathrm{Im}\, Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \mathrm{Im}\, Q \to \ker L$.

In our proof of the existence, we also need the following two lemmas.

Lemma 2.2 (continuation theorem [20]). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose that*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \mathrm{Dom}\, L$, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega \cap \ker L, QNx \neq 0$;*

(c) $\deg(JQN, \Omega \cap \ker L, 0) \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\mathrm{Dom}\, L \cap \overline{\Omega}$.

Lemma 2.3 (see [14]). *If $u : \mathbb{Z} \to \mathbb{R}$ is a ω-periodic sequence, then for any fixed n_1,$n_2 \in I_\omega$, one has*

$$u(n) \le u(n_1) + \sum_{k=0}^{\omega-1} |\Delta u(k)|, \qquad u(n) \ge u(n_2) - \sum_{k=0}^{\omega-1} |\Delta u(k)|. \tag{2.12}$$

3. Existence of Two Positive Periodic Solutions

We are ready to state and prove our main theorem.

Theorem 3.1. *Suppose that (H1) and (H2) hold. Then model* (1.7) *has at least two positive ω-periodic solutions.*

Proof. It is easy to see that if the system (2.3) has a ω-periodic solution $(u_1^*(n), u_2^*(n))$, then $(x^*(n), y^*(n)) = (\exp(u_1^*(n) - u_2^*(n)), \exp(u_2^*(n)))$ is a positive ω-periodic solution to the system (1.7). Therefore, to complete the proof, it suffices to show that the system (2.3) has at least two ω-periodic solutions.

We take

$$X = Y = \{(u_1(n), u_2(n)) \mid u_i(n+\omega) = u_i(n),\ i = 1,2,\ n \in \mathbb{Z}\} \tag{3.1}$$

and define the norm of X and Y

$$\|u\| = \max_{n\in I_\omega}|u_1(n)| + \max_{n\in I_\omega}|u_2(n)|, \tag{3.2}$$

for $u = (u_1, u_2) \in X$ or Y. Then X and Y are Banach spaces when they are endowed with the previous norm $\|\cdot\|$.

For any $u = (u_1, u_2) \in X$, because of its periodicity, it is easy to verify that

$$\begin{aligned} \Lambda_1(u,n) &= a(n) + d(n) - b(n)\sum_{l=0}^{\omega-1} G(l)\exp[u_1(n-l) + u_2(n-l)] \\ &\quad - \frac{c(n)}{m^2 + \exp[2u_1(n)]} - \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2 + \exp[2u_1(n-\tau(n))]}, \\ \Lambda_2(u,n) &= -d(n) + \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2 + \exp[2u_1(n-\tau(n))]} \end{aligned} \tag{3.3}$$

are ω-periodic with respect to n.

Set

$$\begin{aligned} L &: \operatorname{Dom} L \cap X \longrightarrow Y, \qquad (Lu)(n) = (L(u_1,u_2))(n) = (\Delta u_1(n), \Delta u_2(n)), \\ N &: X \longrightarrow Y, \qquad (Nu)(n) = (N(u_1,u_2))(n) = (\Lambda_1(u,n), \Lambda_2(u,n)). \end{aligned} \tag{3.4}$$

Obviously, $\ker L = \mathbb{R}^2$, $\operatorname{Im} L = \{(u_1, u_2) \in Y : \sum_{n=0}^{\omega-1} u_i(n) = 0,\ i = 1,2\}$ is closed in Y, and $\dim\ker L = \operatorname{codim} \operatorname{Im} L = 2$. Therefore, L is a Fredholm mapping of index zero.

Define two mappings P and Q as

$$\begin{aligned} P &: X \longrightarrow X, \qquad Pu = \left(\frac{1}{\omega}\sum_{n=0}^{\omega-1} u_1(n), \frac{1}{\omega}\sum_{n=0}^{\omega-1} u_2(n)\right), \quad u = (u_1,u_2) \in X, \\ Q &: Y \longrightarrow Y, \qquad Qv = \left(\frac{1}{\omega}\sum_{n=0}^{\omega-1} v_1(n), \frac{1}{\omega}\sum_{n=0}^{\omega-1} v_2(n)\right), \quad v = (v_1,v_2) \in Y. \end{aligned} \tag{3.5}$$

It is easy to prove that P and Q are two projectors such that $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$. Furthermore, by a simple computation, we find that the inverse K_p of $L_p : \operatorname{Im} L \to \operatorname{Dom} L \cap \ker P$ has the form

$$K_p(u_1,u_2) = \left(\sum_{k=0}^{n-1} u_1(k) - \frac{1}{\omega}\sum_{k=0}^{\omega-1}(\omega-k)u_1(k), \sum_{k=0}^{n-1} u_2(k) - \frac{1}{\omega}\sum_{k=0}^{\omega-1}(\omega-k)u_2(k)\right). \tag{3.6}$$

Evidently,

$$QN(u_1,u_2) = \left(\frac{1}{\omega}\sum_{n=0}^{\omega-1}\Lambda_1(u,n), \frac{1}{\omega}\sum_{n=0}^{\omega-1}\Lambda_2(u,n)\right) \tag{3.7}$$

and $K_p(I-Q)N$ are continuous by the Lebesgues convergence theorem. Moreover, by Arzela Ascolis theorem, $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for the open bounded set $\Omega \subset X$. Therefore, N is L-compact on $\overline{\Omega}$ for the open bounded set $\Omega \subset X$.

Corresponding to the operator equation (2.11), we get the following system:

$$\begin{aligned} \Delta u_1(n) &= \lambda\Lambda_1(u,n), \\ \Delta u_2(n) &= \lambda\Lambda_2(u,n), \end{aligned} \tag{3.8}$$

where $\lambda \in (0,1)$. Suppose that $(u_1(n), u_2(n)) \in \mathrm{X}$ is an arbitrary solution of system (3.8) for a constant $\lambda \in (0,1)$. Summing (3.8) over I_ω, we obtain

$$\overline{a}\omega = \sum_{n=0}^{\omega-1}\left\{b(n)\sum_{l=0}^{\omega-1}G(l)\exp[u_1(n-l)+u_2(n-l)] + \frac{c(n)}{m^2+\exp[2u_1(n)]}\right\}, \tag{3.9}$$

$$\overline{d}\omega = \sum_{n=0}^{\omega-1}\frac{h(n)\exp[u_1(n-\tau(n))]}{m^2+\exp[2u_1(n-\tau(n))]}. \tag{3.10}$$

From system (3.8), we have

$$\begin{aligned} \sum_{n=0}^{\omega-1}|\Delta u_1(n)| < {}& \sum_{n=0}^{\omega-1}(|a(n)|+|d(n)|) \\ & + \sum_{n=0}^{\omega-1}\left\{b(n)\sum_{l=0}^{\omega-1}G(l)\exp[u_1(n-l)+u_2(n-l)]\right. \\ & \left. + \frac{c(n)}{m^2+\exp[2u_1(n)]} + \frac{h(n)\exp[u_1(n-\tau(n))]}{m^2+\exp[2u_1(n-\tau(n))]}\right\}, \\ \sum_{n=0}^{\omega-1}|\Delta u_2(n)| < {}& \sum_{n=0}^{\omega-1}|d(n)| + \sum_{n=0}^{\omega-1}\frac{h(n)\exp[u_1(n-\tau(n))]}{m^2+\exp[2u_1(n-\tau(n))]}. \end{aligned} \tag{3.11}$$

By using (3.9) and (3.10), we obtain

$$\sum_{n=0}^{\omega-1}|\Delta u_1(n)| < \left(\overline{|a|}+\overline{|d|}+\overline{a}+\overline{d}\right)\omega, \tag{3.12}$$

$$\sum_{n=0}^{\omega-1}|\Delta u_2(n)| < \left(\overline{|d|}+\overline{d}\right)\omega. \tag{3.13}$$

Obviously, there exist $\xi_i, \eta_i \in I_\omega$, such that

$$u_i(\xi_i) = \min_{n \in I_\omega} u_i(n), \quad u_i(\eta_i) = \max_{n \in I_\omega} u_i(n), \quad i = 1, 2. \tag{3.14}$$

From (3.10), it follows that

$$\overline{d}\omega \le \overline{h}\omega \frac{\exp[u_1(\eta_1)]}{m^2 + \exp[2u_1(\xi_1)]}, \tag{3.15}$$

therefore

$$u_1(\eta_1) \ge \ln\left[\frac{\overline{d}}{\overline{h}}\left(m^2 + \exp[2u_1(\xi_1)]\right)\right]. \tag{3.16}$$

By using Lemma 2.3 and (3.12), we obtain

$$u_1(n) \ge u_1(\eta_1) - \sum_{s=0}^{\omega-1} |\Delta u_1(s)| > \ln\left[\frac{\overline{d}}{\overline{h}}\left(m^2 + \exp[2u_1(\xi_1)]\right)\right] - \left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega. \tag{3.17}$$

In particular, we have

$$u_1(\xi_1) > \ln\left[\frac{\overline{d}}{\overline{h}}\left(m^2 + \exp[2u_1(\xi_1)]\right)\right] - \left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega, \tag{3.18}$$

or

$$\overline{d}\exp[2u_1(\xi_1)] - \overline{h}\exp\left[\left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega\right]\exp[u_1(\xi_1)] + m^2\overline{d} < 0. \tag{3.19}$$

The assumption (H1) implies that $\overline{h}\exp[(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d})\omega] > 2m\overline{d}$. So we have

$$\ln l_- < u_1(\xi_1) < \ln l_+. \tag{3.20}$$

From (3.10), we also have

$$\overline{d}\omega \ge \overline{h}\omega \frac{\exp[u_1(\xi_1)]}{m^2 + \exp[2u_1(\eta_1)]}, \tag{3.21}$$

it follows that

$$u_1(\xi_1) \le \ln\left[\frac{\overline{d}}{\overline{h}}\left(m^2 + \exp[2u_1(\eta_1)]\right)\right]. \tag{3.22}$$

By using Lemma 2.3 and (3.12) again, we have

$$u_1(n) \le u_1(\xi_1) + \sum_{s=0}^{\omega-1}|\Delta u_1(s)| < \ln\left[\frac{\overline{d}}{\overline{h}}\left(m^2 + \exp[2u_1(\eta_1)]\right)\right] + \left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega. \quad (3.23)$$

In particular, we have

$$u_1(\eta_1) < \ln\left[\frac{\overline{d}}{\overline{h}}\left(m^2 + \exp[2u_1(\eta_1)]\right)\right] + \left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega, \quad (3.24)$$

or

$$\overline{d}\exp[2u_1(\eta_1)] - \overline{h}\exp\left[-\left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega\right]\exp[u_1(\eta_1)] + m^2\overline{d} > 0. \quad (3.25)$$

Therefore,

$$u_1(\eta_1) < \ln v_- \quad \text{or} \quad u_1(\eta_1) > \ln v_+. \quad (3.26)$$

From (3.12) and (3.20), we have

$$u_1(n) \le u_1(\xi_1) + \sum_{s=0}^{\omega-1}|\Delta u_1(s)| < \ln l_+ + \left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega := B_{11}. \quad (3.27)$$

Similarly, from (3.12) and (3.26), we have

$$u_1(n) \ge u_1(\eta_1) - \sum_{s=0}^{\omega-1}|\Delta u_1(s)| > \ln v_+ - \left(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d}\right)\omega := B_{12}. \quad (3.28)$$

By using (3.14), (3.27), and (3.28), it follows from (3.9) that

$$\overline{a}\omega \ge \overline{b}\omega\exp[u_2(\xi_2) + B_{12}], \quad (3.29)$$

$$\overline{a}\omega \le \overline{b}\omega\exp[u_2(\eta_2) + B_{11}] + \frac{\overline{c}\omega}{m^2}. \quad (3.30)$$

From (3.29), we have

$$u_2(\xi_2) \le \ln\frac{\overline{a}}{\overline{b}} - B_{12}. \quad (3.31)$$

In view of (3.12), we obtain

$$u_2(n) \le u_2(\xi_2) + \sum_{s=0}^{\omega-1}|\Delta u_2(s)| < \ln\frac{\overline{a}}{\overline{b}} - B_{12} + \left(\overline{|d|} + \overline{d}\right)\omega := B_{21}. \quad (3.32)$$

Under the assumption (H2), it follows from (3.30) that

$$u_2(\eta_2) \geq \ln \frac{\overline{a} - (\overline{c}/m^2)}{\overline{b}} - B_{11}. \tag{3.33}$$

By using (3.12), we obtain again

$$u_2(n) \geq u_2(\eta_2) - \sum_{s=0}^{\omega-1} |\Delta u_2(s)| > \ln \frac{\overline{a} - (\overline{c}/m^2)}{\overline{b}} - B_{11} - \left(\overline{|d|} + \overline{d}\right)\omega := B_{22}. \tag{3.34}$$

It follows from (3.32) and (3.34) that

$$\max_{n \in I_\omega} |u_2(n)| < \max\{|B_{21}|, |B_{22}|\} := B_2. \tag{3.35}$$

Notice that

$$QN(u_1, u_2) = \left[\overline{a} + \overline{d} - \overline{b}\exp(u_1 + u_2) - \frac{\overline{c} + \overline{h}\exp(u_1)}{m^2 + \exp(2u_1)}, -\overline{d} + \frac{\overline{h}\exp(u_1)}{m^2 + \exp(2u_1)}\right] \tag{3.36}$$

for $u = (u_1, u_2) \in \mathbb{R}^2$. Under the conditions (H1) and (H2), we can obtain two distinct solutions of $QN(u_1, u_2) = 0$

$$\begin{gathered} u^- = (u_1^-, u_2^-) = \left(\ln u_-, \ln \frac{\overline{a}(m^2 + u_-^2) - \overline{c}}{\overline{b}u_-(m^2 + u_-^2)}\right), \\ u^+ = (u_1^+, u_2^+) = \left(\ln u_+, \ln \frac{\overline{a}(m^2 + u_+^2) - \overline{c}}{\overline{b}u_+(m^2 + u_+^2)}\right). \end{gathered} \tag{3.37}$$

After choosing a constant $C > 0$ such that

$$C > \max\left\{\left|\ln \frac{\overline{a}(m^2 + u_-^2) - \overline{c}}{\overline{b}u_-(m^2 + u_-^2)}\right|, \left|\ln \frac{\overline{a}(m^2 + u_+^2) - \overline{c}}{\overline{b}u_+(m^2 + u_+^2)}\right|\right\}, \tag{3.38}$$

we can define two bounded open subsets of X as follows:

$$\begin{gathered} \Omega_1 = \left\{u = (u_1, u_2) \in X \mid u_1 \in (\ln l_-, \ln v_-),\ \max_{n \in I_\omega} |u_2| < B_2 + C\right\}, \\ \Omega_2 = \left\{u = (u_1, u_2) \in X \mid \min_{n \in I_\omega} u_1 \in (\ln l_-, \ln l_+),\ \max_{n \in I_\omega} u_1 \in (\ln v_+, B_{11}),\ \max_{n \in I_\omega} |u_2| < B_2 + C, \right\}. \end{gathered} \tag{3.39}$$

It follows from (2.10) and (3.38) that $u^- \in \Omega_1$ and $u^+ \in \Omega_2$. Because of $\ln v_- < \ln v_+$, it is easy to see that $\Omega_1 \cap \Omega_2$ is empty, and Ω_i satisfies the condition (a) in Lemma 2.2 for $i = 1, 2$. Moreover,

$QNu \neq 0$ for $u \in \partial\Omega_i \bigcap \ker L = \partial\Omega_i \bigcap \mathbb{R}^2$. This shows that the condition (b) in Lemma 2.2 is satisfied.

Because $\operatorname{Im} Q = \ker L$, we can take the isomorphic J as the identity mapping, then we have

$$\deg(JQN(u_1,u_2),\Omega_i \cap \ker L,(0,0)) = \deg(QN(u_1,u_2),\Omega_i \cap \ker L,(0,0)). \tag{3.40}$$

From (3.37), $QN(u_1,u_2) = 0$ has two solutions $u^- = (u_1^-,u_2^-) \in \Omega_1 \cap \operatorname{Ker} L$ and $u^+ = (u_1^+,u_2^+) \in \Omega_2 \cap \operatorname{Ker} L$. Therefore we have

$$\begin{aligned}
&\deg(QN(u_1,u_2),\Omega_1 \cap \ker L,(0,0)) \\
&= \operatorname{sign}\begin{vmatrix} -\bar{b}\exp(u_1^- + u_2^-) - \dfrac{\bar{h}\exp(u_1^-)(m^2 - \exp(2u_1^-))}{(m^2+\exp(2u_1^-))^2} & -\bar{b}\exp(u_1^- + u_2^-) \\ \dfrac{\bar{h}\exp(u_1^-)(m^2 - \exp(2u_1^-))}{(m^2+\exp(2u_1^-))^2} & 0 \end{vmatrix} \\
&= \operatorname{sign}\left(\frac{\bar{b}\,\bar{h}\exp(2u_1^- + u_2^-)(m^2 - \exp(2u_1^-))}{(m^2+\exp(2u_1^-))^2}\right) = \operatorname{sign}(m - \exp(u_1^-)) \\
&= \operatorname{sign}\left(\frac{\sqrt{\bar{e} - 2m\bar{d}}\left(\sqrt{\bar{e} + 2m\bar{d}} - \sqrt{\bar{e} - 2m\bar{d}}\right)}{2\bar{d}}\right) \\
&= 1 \neq 0.
\end{aligned} \tag{3.41}$$

Similarly, we can obtain that

$$\begin{aligned}
&\deg(QN(u_1,u_2),\Omega_2 \cap \ker L,(0,0)) \\
&= \operatorname{sign}(m - \exp(u_1^+)) = \operatorname{sign}\left(-\frac{\sqrt{\bar{e} - 2m\bar{d}}\left(\sqrt{\bar{e} + 2m\bar{d}} + \sqrt{\bar{e} - 2m\bar{d}}\right)}{2\bar{d}}\right) = -1 \neq 0.
\end{aligned} \tag{3.42}$$

So the condition (c) in Lemma 2.2 is also satisfied.

By now we know that Ω_i $(i = 1,2)$ satisfies all the requirements of Lemma 2.2. Hence the system (2.3) has at least two ω-periodic solutions. This completes the proof. □

4. An Example

In the system (1.7), let $a(n) = 0.5 + 0.25\cos((2/3)\pi n)$, let $b(n) = 1.1 + \cos((2/3)\pi n)$, let $c(n) = 0.11 + 0.1\cos((2/3)\pi n)$, let $d(n) = 0.011 + 0.01\sin((2/3)\pi n)$, let $h(n) = 1 + 0.5\cos((2/3)\pi n)$, and let $\tau(n) = 2$. Obviously, they are positive periodic sequences with period $\omega = 3$. The time delay kernel sequence $K(n) = (1 - \exp(-1))\exp(-n)$, which satisfies $\sum_{n=0}^{+\infty} K(n) = 1$. It is easy to obtain that $\overline{d} = 0.011 > 0$, $\overline{h} - 2m\overline{d}\exp[(\overline{|a|} + \overline{|d|} + \overline{a} + \overline{d})\omega] \approx 0.0559 > 0$, $m^2\overline{a} - \overline{c} = 1.89 > 0$. Therefore, the conditions (H1) and (H2) are satisfied. From Theorem 3.1, the system (1.7) has at least two 3-periodic solutions.

5. Conclusion

In [3], Lu and Wang investigated a discrete time semi-ratio-dependent predator-prey system (1.6) with Holling type IV functional response and time delay. They established sufficient conditions which guarantee the existence and global attractivity of a positive periodic solution of the system. In this paper, a ratio-dependent predator-prey discrete-time model with discrete distributed delays and nonmonotone functional response is investigated. By using the continuation theorem of Mawhins coincidence degree theory, we prove that the system (1.7) has at least two positive periodic solutions under conditions (H1) and (H2). As [3], we would like to know the local stability of the two positive periodic solutions of system (1.7), which is our future work.

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