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Research Article

On the Nonhomogeneous Fourth-Order p-Laplacian Generalized Sturm-Liouville Nonlocal Boundary Value Problems

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We study the nonlinear nonhomogeneous n-point generalized Sturm-Liouville fourth-order p-Laplacian boundary value problem by using Leray-Schauder nonlinear alternative and Leggett-Williams fixed-point theorem.

1. Introduction

In this paper, we prove the existence of one and multiple positive solutions of the following differential equations:

$$\begin{gathered}
\left(\phi_p\left(u''(t)\right)\right)'' - k^2\phi_p\left(u''(t)\right) = g(t)f(t,u(t)), \quad t \in [0,1], \\
u''(0) = \phi_q(a), \qquad u''(1) = \phi_q(b), \\
\alpha u(0) - \beta u'(0) = \sum_{i=1}^{n-2} a_i u(\xi_i), \\
\gamma u(1) + \delta u'(1) = \sum_{i=1}^{n-2} b_i u(\xi_i),
\end{gathered} \tag{1.1}$$

where ϕ_p is p-Laplacian operator, that is, $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p+1/q = 1m$, $k \neq 0$, α, β, γ, $\delta \geqslant 0$, $\xi_i \in (0,1)$, a, b, a_i, $b_i \in (0,\infty)$ $(i = 1,2,\ldots,n-2)$, $f \in C([0,1] \times [0,\infty), [0,\infty))$, $f(t,0) \not\equiv 0$, $g(t) \in C([0,1],[0,\infty))$.

Recently, much attention has been paid to the existence of positive solutions for nonlocal nonlinear boundary value problems (BVPs for short), see [1–4] and references therein. Such problems have potential applications in physics, biology, chemistry, and so forth. For example, a second-order three-point is used as a model for the membrane response of a spherical cap in nonlinear diffusion generated by nonlinear sources and in chemical reactor theory.

At the same time, the boundary value problems with p-Laplacian operator have been discussed extensively, for example, see [1–3, 5–7].

In [1], Feng et al. researched the boundary value problem

$$\begin{gathered} (\phi_p(u'))'(t) = q(t)f(t,u(t)), \quad 0 \leqslant t \leqslant 1, \\ u(0) = \sum_{i=1}^{m} a_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} b_i u(\xi_i); \end{gathered} \tag{1.2}$$

they obtained at least one or two positive solutions under some assumptions imposed on the nonlinearity of f by applying Krasnoselskii fixed-point theorem.

Zhou and Ma studied the existence and iteration of positive solutions for the following third-order generalized right-focal boundary value problem with p-Laplacian operator in [3]:

$$\begin{gathered} (\phi_p(u''))'(t) = q(t)f(t,u(t)), \quad 0 \leqslant t \leqslant 1, \\ u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u'(\eta) = 0, \qquad u''(1) = \sum_{i=1}^{n} \beta_i u''(\theta_i); \end{gathered} \tag{1.3}$$

they established a corresponding iterative scheme for (1.4) by employing the monotone iterative technique.

We would also like to mention the work of Zhang and Liu in [7], in which they considered the existence of positive solutions for

$$\begin{gathered} (\phi_p(u''(t)))'' = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{n-2} a_i u(\xi_i), \qquad u(1) = 0, \qquad u''(0) = \sum_{i=1}^{n-2} b_i u''(\xi_i), \qquad u''(1) = 0, \end{gathered} \tag{1.4}$$

by virtue of monotone iterative techniques, and they established a necessary and sufficient condition of positive solutions for their problem.

However, to the best of our knowledge, there are not many results concerning about the existence and multiple solutions of fourth-order p-Laplacian generalized Sturm-Liouville n-point boundary value problems. In this paper, motivated by the study of [4, 8], we committed to consider the fourth-order p-Laplacian generalized Sturm-Liouville nonlocal boundary value problem without assuming any monotonicity condition on the nonlinearity f.

The rest of the paper is arranged as follows. We state some definitions and several preliminary results in Section 2 that we will use in the sequel. Then in Section 3 we present the existence of one positive solution of BVP (1.1) by Leray-Schauder nonlinear alternative. In Section 4 we get three solutions by Leggett-Williams fixed-point theorem.

2. Preliminaries and Some Lemmas

The basic space used in this paper is $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$.

Denote

$$\varphi(t) = \beta + \alpha t, \quad \psi(t) = \gamma + \delta - \gamma t, \quad t \in [0,1], \qquad \rho = \alpha\gamma + \beta\gamma + \alpha\delta,$$
$$\Delta = \begin{vmatrix} -\sum_{i=1}^{n-2} a_i\varphi(\xi_i) & \rho - \sum_{i=1}^{n-2} a_i\psi(\xi_i) \\ \rho - \sum_{i=1}^{n-2} b_i\varphi(\xi_i) & -\sum_{i=1}^{n-2} b_i\psi(\xi_i) \end{vmatrix}. \tag{2.1}$$

Definition 2.1. A function u is said to be a solution of the boundary value problem (1.1) if $u \in C^2[0,1]$ satisfies (1.1) and $\phi_p(u) \in C^2[0,1]$. In addition, u is said to be a positive solution if $u(t) > 0$ for $t \in (0,1)$, and u is a solution of BVP (1.1).

Throughout the paper, we assume the following condition is satisfied:

(H0) $\rho > 0$, $\rho - \sum_{i=1}^{n-2} a_i\psi(\xi_i) > 0$, $\rho - \sum_{i=1}^{n-2} b_i\varphi(\xi_i) > 0$, $\Delta < 0$, $f(t,u(t)) \geqslant k^2(a+b)/\min_{t\in[0,1]} g(t)$.

Let $y(t) = -\phi_p(u''(t))$, then BVP (1.1) is divided into the following two parts:

$$\begin{gathered} -y'' + k^2 y = g(t)f(t,u(t)), \quad t \in (0,1), \\ y(0) = a, \qquad y(1) = b, \end{gathered} \tag{2.2}$$

$$\begin{gathered} u'' + \phi_q(y) = 0, \quad t \in (0,1), \\ \alpha u(0) - \beta u'(0) = \sum_{i=1}^{n-2} a_i u(\xi_i), \qquad \gamma u(1) + \delta u'(1) = \sum_{i=1}^{n-2} b_i u(\xi_i). \end{gathered} \tag{2.3}$$

It is not difficult that we can transform (2.2) into the following differential equations:

$$\begin{gathered} -y'' + k^2 y = g(t)f(t,u(t)) - k^2 a(1-t) - k^2 bt, \quad t \in (0,1), \\ y(0) = 0, \qquad y(1) = 0. \end{gathered} \tag{2.4}$$

By routine calculations we can get the following three Lemmas.

Lemma 2.2. *The BVP* (2.4) *has a unique solution*

$$y(t) = \int_0^1 G_1(t,s)\left[g(s)f(s,u(s)) - k^2 a(1-s) - k^2 bs\right]ds, \tag{2.5}$$

where

$$G_1(t,s)=\frac{1}{\rho}\begin{cases}\dfrac{\sinh ks\cdot\sinh k(1-t)}{k\sinh k}, & 0\leqslant s\leqslant t\leqslant 1,\\ \dfrac{\sinh kt\cdot\sinh k(1-s)}{k\sinh k} & 0\leqslant t\leqslant s\leqslant 1.\end{cases}\tag{2.6}$$

Lemma 2.3. *The BVP* (2.3) *has a unique solution*

$$u(t)=\int_0^1 G_2(t,s)\phi_q(y(s))\,ds+A\varphi(t)+B\psi(t),\tag{2.7}$$

where

$$\begin{gathered}G_2(t,s)=\frac{1}{\rho}\begin{cases}\psi(t)\varphi(s), & 0\leqslant s\leqslant t\leqslant 1,\\ \psi(s)\varphi(t), & 0\leqslant t\leqslant s\leqslant 1,\end{cases}\\
A=\frac{1}{\Delta}\begin{vmatrix}\displaystyle\sum_{i=1}^{n-2}a_i\int_0^1 G_2(\xi_i,s)\phi_q(y(s))\,ds & \displaystyle\rho-\sum_{i=1}^{n-2}a_i\psi(\xi_i)\\ \displaystyle\sum_{i=1}^{n-2}b_i\int_0^1 G_2(\xi_i,s)\phi_q(y(s))\,ds & \displaystyle-\sum_{i=1}^{n-2}a_i\psi(\xi_i)\end{vmatrix},\\
B=\frac{1}{\Delta}\begin{vmatrix}\displaystyle-\sum_{i=1}^{n-2}a_i\varphi(\xi_i) & \displaystyle\sum_{i=1}^{n-2}a_i\int_0^1 G_2(\xi_i,s)\phi_q(y(s))\,ds\\ \displaystyle\rho-\sum_{i=1}^{n-2}a_i\varphi(\xi_i) & \displaystyle\sum_{i=1}^{n-2}b_i\int_0^1 G_2(\xi_i,s)\phi_q(y(s))\,ds\end{vmatrix}.\end{gathered}\tag{2.8}$$

The proof of Lemma 2.3 is similar to that of Lemma 5.5.1 in [8], so we omit it here. From Lemmas 2.2 and 2.3 we can get that $u(t)$ is a solution of BVP (1.1) if and only if

$$\begin{aligned}u(t)=&\int_0^1 G_2(t,s)\phi_q\left\{\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))-k^2a(1-\tau)-k^2b\tau\right]\right\}ds\\&+A(\phi_q(y))\varphi(t)+B(\phi_q(y))\psi(t),\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}&A(\phi_q(y))\\&=\frac{1}{\Delta}\begin{vmatrix}\displaystyle\sum_{i=1}^{n-2}a_i\int_0^1 G_2(\xi_i,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))-k^2a(1-\tau)-k^2b\tau\right]d\tau\right)ds & \displaystyle\rho-\sum_{i=1}^{n-2}a_i\psi(\xi_i)\\ \displaystyle\sum_{i=1}^{n-2}b_i\int_0^1 G_2(\xi_i,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))-k^2a(1-\tau)-k^2b\tau\right]d\tau\right)ds & \displaystyle-\sum_{i=1}^{n-2}a_i\psi(\xi_i)\end{vmatrix},\end{aligned}$$

$$B(\phi_q(y)) = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{n-2} a_i\varphi(\xi_i) & \sum_{i=1}^{n-2} a_i\int_0^1 G_2(\xi_i,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))-k^2a(1-\tau)-k^2b\tau\right]d\tau\right)ds \\ \rho-\sum_{i=1}^{n-2} a_i\varphi(\xi_i) & \sum_{i=1}^{n-2} b_i\int_0^1 G_2(\xi_i,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))-k^2a(1-\tau)-k^2b\tau\right]d\tau\right)ds \end{vmatrix}. \tag{2.10}$$

Lemma 2.4. *Consider,*

$$\begin{gathered} G_i(t,s) \leqslant G_i(s,s), \quad t,s\in[0,1],\ i=1,2, \\ G_1(t,s) \geqslant \Lambda_0 G_1(s,s), \quad t\in[\epsilon,1-\epsilon],\ s\in[0,1], \\ G_2(t,s) \geqslant \Lambda G_2(s,s), \quad t\in[\epsilon,1-\epsilon],\ s\in[0,1], \end{gathered} \tag{2.11}$$

where

$$\begin{gathered} \Lambda_0 = \frac{\sinh k\epsilon}{\sinh k}, \quad \epsilon\in\left(0,\frac{1}{2}\right), \\ \Lambda = \min\left\{\frac{\varphi(\epsilon)}{\varphi(1)}, \frac{\psi(1-\epsilon)}{\psi(0)}\right\}, \quad \epsilon\in\left(0,\frac{1}{2}\right). \end{gathered} \tag{2.12}$$

Denote

$$\Lambda_1 = \max\{\|\varphi\|,\|\psi\|\}, \quad \Lambda_2 = \min\left\{\min_{t\in[\epsilon,1-\epsilon]}\varphi(t), \min_{t\in[\epsilon,1-\epsilon]}\psi(t)\right\}, \quad \lambda = \min\left\{\Lambda,\frac{\Lambda_2}{\Lambda_1}\right\}. \tag{2.13}$$

Lemma 2.5. *Let*

$$\begin{aligned} K &= \left\{u \mid u\in C[0,1], u(t)\geqslant 0, \forall t\in[0,1], \min_{t\in[\epsilon,1-\epsilon]} u(t)\geqslant \lambda\|u\|\right\}, \\ Tu(t) &= \int_0^1 G_2(t,s)\phi_q\left\{\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))-k^2a(1-\tau)-k^2b\tau\right]\right\}ds \\ &\quad + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi(t), \end{aligned} \tag{2.14}$$

then

$$T(K)\subseteq K. \tag{2.15}$$

Proof. Firstly, we prove that $Tu(t)\geqslant 0$. For $f\in C([0,1]\times[0,\infty),[0,\infty)$, $g(t)\in C([0,1],[0,\infty))$, then we can get $\phi_q(y) = \phi_q(\int_0^1 G_1(s,\tau)[g(\tau)f(\tau,u(\tau)) - k^2a(1-\tau) - k^2b\tau]d\tau) > 0$, for all $u\in K$. Furthermore, condition (H0) leads to $A(\phi_q(y))\geqslant 0$, and $B(\phi_q(y))\geqslant 0$, thus, we get $Tu(t)\geqslant 0$.

Secondly, for $t \in [\epsilon, 1-\epsilon]$, we can get

$$\begin{aligned}
&\min_{t\in[\epsilon,1-\epsilon]} Tu(t) \\
&\quad = \min_{t\in[\epsilon,1-\epsilon]} \left\{ \int_0^1 G_2(t,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right] d\tau \right) ds \right. \\
&\qquad\qquad \left. + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi(t) \right\} \\
&\quad \geqslant \Lambda \int_0^1 G_2(s,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right] d\tau \right) ds \\
&\qquad + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi(t) \\
&\quad \geqslant \Lambda \int_0^1 G_2(s,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right] d\tau \right) ds \\
&\qquad + \frac{\Lambda_2}{\Lambda_1}\Lambda_1\left[A(\phi_q(y)) + B(\phi_q(y)]\right) \\
&\quad \geqslant \lambda \left\{ \int_0^1 G_2(s,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right] d\tau \right) ds \right. \\
&\qquad\qquad \left. + \Lambda_1\left[A(\phi_q(y)) + B(\phi_q(y))\right] \right\} = \lambda \|Tu\|.
\end{aligned} \tag{2.16}$$

Thus we can get that $\min_{t\in[\epsilon,1-\epsilon]} Tu(t) \geqslant \lambda\|Tu\|$, which means $T(K) \subseteq K$. □

We present here several definitions.

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp., convex) functional on K provided that $\alpha : K \to [0, +\infty)$ is continuous and

$$\begin{gathered}
\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y), \\
\left(\text{resp., } \alpha(tx + (1-t)y) \leqslant t\alpha(x) + (1-t)\alpha(y)\right),
\end{gathered} \tag{2.17}$$

for all, x, $y \in K$ and $t \in [0, 1]$.

Let $0 < a < b$ be given, and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{x \in K \mid \|x\| < r\}, \quad P(\alpha, a, b) = \{x \in K \mid a \leqslant \alpha(x), \|x\| \leqslant b\}. \tag{2.18}$$

For the convenience of the reader, we present here the Leggett-Williams fixed-point theorem and the Leray-Schauder nonlinear alternative theorem.

Lemma 2.6 (see [9], Leggett-Williams fixed-point theorem). *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator, and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leqslant \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \leqslant c$ such that*

(A1) *$\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$, and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$;*

(A2) *$\|Ax\| < a$ for $\|x\| \leqslant a$;*

(A3) *$\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points x_1, x_2, and x_3 and such that $\|x_1\| < a$, $b < \alpha(x_2)$ and $\|x_3\| > a$, with $\alpha(x_3) < b$.

Now we cite the Leray-Schauder nonlinear alternative.

Lemma 2.7 (see [10]). *Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3. Results of One Nontrivial Solution

In this section, we study the existence of one nontrivial solution of BVP (1.1) by Leray-Schauder nonlinear alternative.

Denote

$$
\begin{gathered}
H_1 = \phi_q\left(\int_0^1 G_1(\tau,\tau)g(\tau)p(\tau)d\tau\right), \\
H_2 = \phi_q\left(\int_0^1 G_1(\tau,\tau)g(\tau)r(\tau)d\tau\right), \\
\widehat{A} = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{n-2} a_i & \rho - \sum_{i=1}^{n-2} a_i\psi(\xi_i) \\ \sum_{i=1}^{n-2} b_i & -\sum_{i=1}^{n-2} a_i\psi(\xi_i) \end{vmatrix}, \\
\widehat{B} = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{n-2} a_i\varphi(\xi_i) & \sum_{i=1}^{n-2} a_i \\ \rho - \sum_{i=1}^{n-2} a_i\varphi(\xi_i) & \sum_{i=1}^{n-2} b_i \end{vmatrix}, \\
N = M \cdot 2^{q-1}\left(1 + \widehat{A}\Lambda_1 + \widehat{B}\Lambda_1\right), \\
l = \frac{NH_2}{1 - NH_1}, \qquad \Omega = \{u \in C[0,1], \|u\| < l\}.
\end{gathered}
\tag{3.1}
$$

Theorem 3.1. *Assume $NH_1 < 1$, $f(t,0) \not\equiv 0$, and there exist nonnegative functions p, $r \in L^1[0,1]$ such that $|f(t,u)| \leqslant p(t)|u|^{p-1} + r(t)$, a.e. $(t,u) \in [0,1] \times [0,+\infty)$, then BVP (1.1) has a nontrivial solution $u^* \in \overline{\Omega}$.*

Proof. If there exist two nonnegative functions p, $r \in L^1[0,1]$ such that $|f(t,u)| \leqslant p(t)|u|^{p-1} + r(t)$, a.e. $(t,u) \in [0,1] \times [0,+\infty)$, we can get that

$$\begin{aligned}
&\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right]d\tau\right)\\
&\quad\leqslant \phi_q\left(\int_0^1 G_1(s,\tau)g(\tau)f(\tau,u(\tau))d\tau\right)\\
&\quad\leqslant \phi_q\left[\int_0^1 G_1(\tau,\tau)g(\tau)\left(p(\tau)|u|^{p-1} + r(\tau)\right)d\tau\right]\\
&\quad= 2^{q-1}\left[\|u\|\phi_q\left(\int_0^1 G_1(\tau,\tau)g(\tau)p(\tau)d\tau + \phi_q\left(\int_0^1 G_1(\tau,\tau)a(\tau)r(\tau)\right)d\tau\right)\right]\\
&\quad= 2^{q-1}(\|u\|H_1 + H_2),
\end{aligned} \tag{3.2}$$

thus, we get

$$\begin{aligned}
&\int_0^1 G_2(\xi_i,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right]d\tau\right)ds\\
&\quad\leqslant \int_0^1 G_2(\xi_i,s)\phi_q\left(\int_0^1 G_1(s,\tau)g(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\quad\leqslant M\cdot 2^{q-1}(\|u\|H_1 + H_2).
\end{aligned} \tag{3.3}$$

In the same way, we obtain

$$\begin{aligned}
A(\phi_q(y)) &\leqslant M\cdot 2^{q-1}(\|u\|H_1 + H_2)\widehat{A},\\
B(\phi_q(y)) &\leqslant M\cdot 2^{q-1}(\|u\|H_1 + H_2)\widehat{B}.
\end{aligned} \tag{3.4}$$

Thus we have

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1}|Tu(t)|\\
&= \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G_2(t,s)\phi_q\left(\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2 a(1-\tau) - k^2 b\tau\right]\right)ds\right.\\
&\qquad\left. + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi(t)\right|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G_2(t,s)\phi_q\left(\int_0^1 G_1(s,\tau)g(\tau)f(\tau,u(\tau))d\tau\right)ds\right.\\
&\qquad\left.+\,A(\phi_q(y))\varphi(t)+B(\phi_q(y))\psi(t)\right|\\
&\leqslant M\cdot 2^{q-1}(\|u\|H_1+H_2)+M\cdot 2^{q-1}(\|u\|H_1+H_2)\left(\widehat{A}\Lambda_1+\widehat{B}\Lambda_1\right)\\
&= M\cdot 2^{q-1}(\|u\|H_1+H_2)\left(1+\widehat{A}\Lambda_1+\widehat{B}\Lambda_1\right)=N(\|u\|H_1+H_2).
\end{aligned} \tag{3.5}$$

Suppose that there exists $\mu > 1$ such that

$$Tu=\mu u,\quad u\in\partial\Omega. \tag{3.6}$$

Therefore,

$$\mu l=\mu\|u\|=\|Tu\|\leqslant l\left(NH_1+\frac{NH_2}{l}\right), \tag{3.7}$$

which leads to $\mu \leqslant NH_1 + NH_2/l = 1$, and this contradicts $\mu > 1$, then by Lemma 2.7, T has a fixed point $u^* \in \overline{\Omega}$; since $f(t,0)\not\equiv 0$, the BVP (1.1) has a nontrivial solution $u^* \in \overline{\Omega}$. This completes the proof of Theorem 3.1. □

4. Results of Multiple Positive Solutions

In the following parts, we will study the existence of multiple positive solutions of BVP (1.1) by using Leggett-Williams fixed-point theorem.

Denote

$$P_c=\{u\in K \mid \|u\|<c\}. \tag{4.1}$$

Define the nonnegative continuous concave functional on K by

$$\alpha(u)=\min_{\epsilon\leqslant t\leqslant 1-\epsilon}u(t). \tag{4.2}$$

It is obvious that for each $u \in K$, $\alpha(u) \leqslant \|u\|$.

Let $M = \max_{0\leqslant t\leqslant 1}\int_0^1 G_2(t,s)ds$, $m = \int_\epsilon^{1-\epsilon} G_2(s,s)ds$, $h = \phi_q(\int_\epsilon^{1-\epsilon} k^2(a+b)G_1(\tau,\tau)d\tau)$, and $\widehat{A}$, $\widehat{B}$, Λ, Λ_0, Λ_1 be defined in Sections 2 and 3.

We list the following three hypotheses:

(H1) $f(t,u) < \phi_p(c/M(1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B}))/\int_0^1 G_1(\tau,\tau)g(\tau)d\tau$, for all $t \in [0,1]$, $0 \leqslant u \leqslant c$;

(H2) $f(t,u) < \phi_p(a/M(1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B}))/\int_0^1 G_1(\tau,\tau)g(\tau)d\tau$, for all $t \in [0,1]$, $0 \leqslant u \leqslant a$;

(H3) $f(t,u) > \phi_p(2^{q-1}(b/m\Lambda(1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B})+h))/\Lambda_0\int_\epsilon^{1-\epsilon} G_1(\tau,\tau)g(\tau)d\tau$, for all $t \in [0,1]$, $b \leqslant u \leqslant b/\lambda$.

Theorem 4.1. *Assume* $(H1)$–$(H3)$ *hold, then BVP* (1.1) *has at least three positive solutions* u_1, u_2, *and* u_3, *such that* $\|u_1\| < a$, $b < \min_{[\epsilon,1-\epsilon]} u_2(t)$, *and* $\|u_3\| > a$, *with* $\min_{[\epsilon,1-\epsilon]} u_3(t) < b$.

Proof. Firstly, we prove that $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous.

From condition (H1), we can get

$$\begin{aligned} &\int_0^1 G_2(\xi_i, s)\phi_q\left\{\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2a(1-\tau) - k^2b\tau\right]d\tau\right\}ds \\ &\quad \leqslant M\phi_q\left(\int_0^1 G_1(\tau,\tau)g(\tau)f(\tau,u(\tau))d\tau\right) \\ &\quad \leqslant \frac{c}{1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B}}. \end{aligned} \tag{4.3}$$

Hence,

$$\begin{aligned} \|Tu\| &= \max_{0\leqslant t\leqslant 1}|Tu(t)| \\ &= \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G_2(t,s)\phi_q\left\{\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau)) - k^2a(1-\tau) - k^2b\tau\right]d\tau\right\}ds \right. \\ &\qquad \left. + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi(t)\right| \\ &\leqslant \max_{0\leqslant t\leqslant 1}\left|\int_0^1 G_2(t,s)\phi_q\left\{\int_0^1 G_1(s,\tau)\left[g(\tau)f(\tau,u(\tau))\right]d\tau\right\}ds \right. \\ &\qquad \left. + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi(t)\right| \\ &\leqslant \frac{c}{1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B}} + \frac{c}{1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B}}\Lambda_1\widehat{A} + \frac{c}{1+\Lambda_1\widehat{A}+\Lambda_1\widehat{B}}\Lambda_1\widehat{B} = c. \end{aligned} \tag{4.4}$$

Thus we get $\|Tu\| \leqslant c$; therefore, $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous by an application of Ascoli-Arzela theorem.

In the same way, condition (H2) implies that condition (A2) of Lemma 2.6 is satisfied. In the following, we show that condition (A1) of Lemma 2.6 is satisfied.

Let

$$u_0(t) = \frac{b}{\lambda}, \quad t \in [0,1], \tag{4.5}$$

then

$$u_0 \in P\left(\alpha, b, \frac{b}{\lambda}\right), \qquad \alpha(u_0) = \frac{b}{\lambda} > b, \tag{4.6}$$

thus, $\{u \in P(\alpha, b, b/\lambda) \mid \alpha(u) > b\} \neq \emptyset$.

If $u \in P(\alpha, b, b/\lambda)$, then $b \leqslant u(s) \leqslant b/\lambda$, $s \in [\epsilon, 1-\epsilon]$.

By condition (H3), we obtain

$$\begin{aligned}
&\int_0^1 G_2(\xi_i, s)\phi_q\left\{\int_0^1 G_1(s,\tau)\Big[g(\tau)f\Big(\tau, u(\tau) - k^2 a(1-\tau) - k^2 b\tau\Big)\Big]d\tau\right\}ds \\
&\quad \geqslant \Lambda\int_\epsilon^{1-\epsilon} G_2(s,s)\left[\frac{1}{2^{q-1}}\phi_q\left(\int_\epsilon^{1-\epsilon} G_1(s,\tau)g(\tau)f(\tau,u(\tau))\right)\right]d\tau\, ds \\
&\quad - \phi_q\left(\int_\epsilon^{1-\epsilon} G_1(s,\tau)\Big(k^2 a(1-\tau) + k^2 b\tau\Big)d\tau\right]ds \\
&\quad \geqslant \Lambda\int_\epsilon^{1-\epsilon} G_2(s,s)\left[\frac{1}{2^{q-1}}\phi_q\left(\int_\epsilon^{1-\epsilon} G_1(s,\tau)g(\tau)f(\tau,u(\tau))\right)\right]d\tau\, ds \\
&\quad - \phi_q\left(\int_\epsilon^{1-\epsilon} G_1(\tau,\tau)\Big(k^2 a(1-\tau) + k^2 b\tau\Big)d\tau\right]ds \\
&\quad \geqslant \Lambda\int_\epsilon^{1-\epsilon} G_2(s,s)\left[\frac{1}{2^{q-1}}\phi_q\left(\int_\epsilon^{1-\epsilon} G_1(s,\tau)g(\tau)f(\tau,u(\tau))\right)\right]d\tau \\
&\quad - \phi_q\left(\int_\epsilon^{1-\epsilon} G_1(\tau,\tau)\Big(k^2 a + k^2 b\Big)d\tau\right]ds \\
&\quad \geqslant \frac{b}{1 + \Lambda_1\widehat{A} + \Lambda_1\widehat{B}}.
\end{aligned} \tag{4.7}$$

Thus we get

$$\begin{aligned}
\alpha(Tu(t)) &= \min_{t\in[\epsilon,1-\epsilon]} Tu(t) \\
&\geqslant \min_{t\in[\epsilon,1-\epsilon]}\left(\int_\epsilon^{1-\epsilon} G_2(s,s)\phi_q\left\{\int_0^1 G_1(s,\tau)\Big[g(\tau)f\Big(\tau, u(\tau) - k^2 a(1-\tau) - k^2 b\tau\Big]\Big)d\tau\right\}\right)ds \\
&\quad + A(\phi_q(y))\varphi(t) + B(\phi_q(y))\psi((t)) \\
&\geqslant \frac{b}{1 + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}} + \frac{b}{1 + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}}\Lambda_2\widehat{A} + \frac{b}{1 + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}}\Lambda_2\widehat{B} = b.
\end{aligned} \tag{4.8}$$

Therefore, condition (A1) of Lemma 2.6 is satisfied.

Finally, we show that condition (A3) of Lemma 2.6 is satisfied.

If $u \in P(\alpha, b, c)$, and $\|Tu\| > b/\lambda$, then $\alpha(Tu(t)) = \min_{\epsilon\leqslant t\leqslant 1-\epsilon} Tu(t) \geqslant \lambda\|Tu\| > b$.

Therefore, condition (A3) of Lemma 2.6 is also satisfied. By Lemma 2.6, there exist three positive solutions u_1, u_2, and u_3 such that $\|u_1\| < a$, $b < \min_{t\in[\epsilon,1-\epsilon]} u_2(t)$, and $\|u_3\| > a$, with $\min_{t\in[\epsilon,1-\epsilon]} u_3(t) < b$. Thus we completed the proof. □

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