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Research Article

Warped Product Submanifolds of LP-Sasakian Manifolds

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We study of warped product submanifolds, especially warped product hemi-slant submanifolds of LP-Sasakian manifolds. We obtain the results on the nonexistance or existence of warped product hemi-slant submanifolds and give some examples of LP-Sasakian manifolds. The existence of warped product hemi-slant submanifolds of an LP-Sasakian manifold is also ensured by an interesting example.

1. Introduction

The notion of warped product manifolds was introduced by Bishop and O'Neill [1], and later it was studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. The existence or nonexistence of warped product manifolds plays some important role in differential geometry as well as in physics.

On the analogy of Sasakian manifolds, in 1989, Matsumoto [2] introduced the notion of LP-Sasakian manifolds. The same notion is also introduced by Mihai and Roşca [3] and obtained many interesting results. Later on, LP-Sasakian manifolds are also studied by several authors.

The notion of slant submanifolds in a complex manifold was introduced and studied by Chen [4], which is a natural generalization of both invariant and anti-invariant submanifolds. Chen [4] also found examples of slant submanifolds of complex Euclidean spaces C^2 and C^4. Then, Lotta [5] has defined and studied the slant immersions of a Riemannian manifold into an almost contact metric manifold and proved some properties of such immersions. Also, Cabrerizo et al. [6] studied slant immersions of K-contact and Sasakian manifolds.

In 1994, Papaghuic [7] introduced the notion of semi-slant submanifolds of almost Hermitian manifolds. Then, Cabrerizo et. al [8] defined and investigated semi-slant submanifolds of Sasakian manifolds. The idea of hemi-slant submanifolds was introduced by Carriazo as a particular class of bi-slant submanifolds and he called them anti-slant submanifolds [9]. Recently, these submanifolds were studied by Sahin for their warped products of Kähler manifolds [10]. Recently, Uddin [11] studied warped product CR-submanifolds of LP-Sasakian manifolds.

The purpose of the present paper is to study the warped product hemi-slant submanifolds of LP-Sasakian manifolds. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 deals with the study of warped and doubly warped product submanifolds of LP-Sasakian manifolds. In Section 4, we define hemi-slant submanifolds of LP-contact manifolds and investigate their warped products. Section 5 consists some examples of LP-Sasakian manifolds and their warped products.

2. Preliminaries

An n-dimensional smooth manifold M is said to be an LP-Sasakian manifold [3] if it admits a $(1,1)$ tensor field ϕ, a unit timelike contravariant vector field ξ, an 1-form η, and a Lorentzian metric g, which satisfy

$$\eta(\xi) = -1, \qquad g(X,\xi) = \eta(X), \qquad \phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \qquad \overline{\nabla}_X \xi = \phi X, \tag{2.2}$$

$$\left(\overline{\nabla}_X \phi\right)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. It can be easily seen that, in an LP-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \operatorname{rank}\phi = n-1. \tag{2.4}$$

Again, we put

$$\Omega(X,Y) = g(X, \phi Y) \tag{2.5}$$

for any vector fields X, Y tangent to M. The tensor field $\Omega(X,Y)$ is a symmetric (0,2) tensor field [2]. Also, since the vector field η is closed in an LP-Sasakian manifold, we have [2]

$$\left(\overline{\nabla}_X \eta\right)(Y) = \Omega(X,Y), \qquad \Omega(X,\xi) = 0, \tag{2.6}$$

for any vector fields X and Ytangent to M.

Let N be a submanifold of an LP-Sasakian manifold M with induced metric g and let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TN and the normal bundle $T^{\perp}N$ of N, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.7}$$

$$\overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{2.8}$$

for all $X, Y \in TN$ and $V \in T^{\perp}N$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of N into M. The second fundamental form h and the shape operator A_V are related by [12]

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.9}$$

for any $X, Y \in TN$ and $V \in T^{\perp}N$

For any $X \in TN$, we may write

$$\phi X = EX + FX, \tag{2.10}$$

where EX is the tangential component and FX is the normal component of ϕX.

Also, for any $V \in T^{\perp}N$, we have

$$\phi V = BV + CV, \tag{2.11}$$

where BV and CV are the tangential and normal components of ϕV, respectively. The covariant derivatives of the tensor fields E and F are defined as

$$\left(\overline{\nabla}_X E\right) Y = \nabla_X EY - E\nabla_X Y, \tag{2.12}$$

$$\left(\overline{\nabla}_X F\right) Y = \nabla_X^{\perp} FY - F\nabla_X Y \tag{2.13}$$

for any $X, Y \in TN$.

Throughout the paper, we consider ξ to be tangent to N. The submanifold N is said to be invariant if F is identically zero, that is, $\phi X \in TN$ for any $X \in TN$. On the other hand, N is said to anti-invariant if E is identically zero, that is, $\phi X \in T^{\perp}N$ for any $X \in TN$.

Furthermore, for a submanifold tangent to the structure vector field ξ, there is another class of submanifolds which is called a slant submanifold. For each nonzero vector X tangent to N at $x \in N$, the angle $\theta(X)$, $0 \leq \theta(X)(\pi/2)$ between ϕX and EX is called the *slant angle* or *wirtinger angle*. If the slant angle is constant then the submanifold is called *aslant submanifold*. Invariant and anti-invariant submanifolds are particular classes of slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold is said to be *proper* slant if the slant angle θ lies strictly between 0 and $\pi/2$, that is, $0 < \theta < \pi/2$ [6].

Theorem 2.1 (see [13]). *Let N be a submanifold of a Lorentzian almost paracontact manifold M such that ξ is tangent to N. Then, N is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$E^2 = \lambda(I + \eta \otimes \xi). \tag{2.14}$$

Furthermore, if θ is the slant angle of N, then $\lambda = \cos^2\theta$. Also from (2.14), we have

$$g(EX, EY) = \cos^2\theta[g(X,Y) + \eta(X)\eta(Y)], \tag{2.15}$$

$$g(FX, FY) = \sin^2\theta[g(X,Y) + \eta(X)\eta(Y)] \tag{2.16}$$

for any X, Y tangent to N.

The study of semi-slant submanifolds of almost Hermitian manifolds was introduced by Papaghuic [7], which was extended to almost contact manifold by Cabrerizo et al. [8]. The submanifold N is called semi-slant submanifold of M if there exist an orthogonal direct decomposition of TN as

$$TN = D_1 \oplus D_2 \oplus \{\xi\}, \tag{2.17}$$

where D_1 is an invariant distribution, that is, $\phi(D_1) = D_1$ and D_2 is slant with slant angle $\theta \neq 0$. The orthogonal complement of FD_2 in the normal bundle $T^{\perp}N$ is an invariant subbundle of $T^{\perp}N$ and is denoted by μ. Thus, we have for a semi-slant submanifold

$$T^{\perp}N = FD_2 \oplus \mu. \tag{2.18}$$

For an LP-contact manifold this study is extended by Yüksel et al. [13].

3. Warped and Doubly Warped Products

The notion of warped product manifolds was introduced by Bishop and O'Neill [1]. They defined the warped product manifolds as follows.

Definition 3.1. Let (N_1, g_1) and (N_2, g_2) be two semi-Riemannian manifolds and f be a positive differentiable function on N_1. Then, the warped product of N_1 and N_2 is a manifold, denoted by $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant.

More explicitely, if the vector fields X and Y are tangent to $N_1 \times_f N_2$ at (x, y), then

$$g(X,Y) = g_1(\pi_1 * X, \pi_1 * Y) + f^2(x) g_2(\pi_2 * X, \pi_2 * Y), \tag{3.2}$$

where π_i $(i = 1, 2)$ are the canonical projections of $N_1 \times N_2$ onto N_1 and N_2, respectively, and $*$ stands for the derivative map.

Let $N = N_1 \times_f N_2$ be a warped product manifold, which means that N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N, respectively.

For the warped product manifolds, we have the following result for later use [1].

Proposition 3.2. *Let $N = N_1 \times_f N_2$ be a warped product manifold. Then,*

(I) $\nabla_X Y \in TN_1$ *is the lift of* $\nabla_X Y$ *on* N_1*,*

(II) $\nabla_U X = \nabla_X U = (X \ln f)U$*,*

(III) $\nabla_U V = \nabla'_U V - g(U, V)\nabla \ln f$*,*

for any $X, Y \in TN_1$ and $U, V \in TN_2$, where ∇ and ∇' denote the Levi-Civita connections on N and N_2, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by Ünal [14]. A doubly warped product manifold of N_1 and N_2, denoted as ${}_{f_2}N_1 \times_{f_1} N_2$ is endowed with a metric g defined as

$$g = f_2^2 g_1 + f_1^2 g_2, \tag{3.3}$$

where f_1 and f_2 are positive differentiable functions on N_1 and N_2, respectively.

In this case formula (II) of Proposition 3.2 is generalized as

$$\nabla_X Z = (X \ln f_1) Z + (Z \ln f_2) X \tag{3.4}$$

for each X in TN_1 and Z in TN_2 [15].

One has the following theorem for doubly warped product submanifolds of an LP-Sasakian manifold [11].

Theorem 3.3. *Let $N = {}_{f_2}N_1 \times_{f_1} N_2$ be a doubly warped product submanifold of an LP-Sasakian manifold M where N_1 and N_2 are submanifolds of M. Then, f_2 is constant and N_2 is anti-invariant if the structure vector field ξ is tangent to N_1, and f_1 is constant and N_1 is anti-invariant if ξ is tangent to N_2.*

The following corollaries are immediate consequences of the above theorem.

Corollary 3.4. *There does not exist a proper doubly warped product submanifold in LP-Sasakian manifolds.*

Corollary 3.5. *There does not exist a warped product submanifold $N_1 \times_f N_2$ of an LP-Sasakian manifold M such that ξ is tangent to N_2.*

From the above theorem and Corollary 3.5, we have only the remaining case is to study the warped product submanifold $N_1 \times_f N_2$ with structure vector field ξ is tangent to N_1.

4. Warped Product Hemi-Slant Submanifolds

In this section, first we define hemi-slant submanifolds of an LP-contact manifold and then we will discuss their warped products.

Definition 4.1. A submanifold N of an LP-contact manifold M is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions D_1 and D_2 satisfying:

(i) $TN = D_1 \oplus D_2 \oplus \langle \xi \rangle$,

(ii) D_1 is a slant distribution with slant angle $\theta \neq \pi/2$,

(iii) D_2 is anti-invariant, that is, $\phi D_2 \subseteq T^{\perp}N$.

If μ is ϕ-invariant subspace of the normal bundle $T^{\perp}N$, then in case of hemi-slant submanifold, the normal bundle $T^{\perp}N$ can be decomposed as

$$T^{\perp}N = FD_1 \oplus FD_2 \oplus \mu. \tag{4.1}$$

Now, we discuss the warped product hemi-slant submanifolds of an LP-Sasakian manifold M. If $N = N_1 \times_f N_2$ be a warped product hemi-slant submanifold of an LP-Sasakian manifold M and N_θ and $N_\perp$ are slant and anti-invariant submanifolds of an LP-Sasakian manifold M, respectively then their warped product hemi-slant submanifolds may be given by one of the following forms:

(i) $N_\perp \times_f N_\theta$,

(ii) $N_\theta \times_f N_\perp$.

In the following theorem, we start with the case (i).

Theorem 4.2. *There does not exist a proper warped product hemi-slant submanifold $N = N_\perp \times_f N_\theta$ of an LP-Sasakian manifold M such that ξ is tangent to N_θ, where $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of M, respectively.*

Proof. Let $N = N_\perp \times_f N_\theta$ be a proper warped product hemi-slant submanifold of an LP-Sasakian manifold M such that ξ is tangent to N_θ. Then, for any $X \in TN_\theta$ and $U \in TN_\perp$, we have

$$\left(\overline{\nabla}_X \phi\right)U = \overline{\nabla}_X \phi U - \phi \overline{\nabla}_X U. \tag{4.2}$$

By virtue of (2.3) and (2.7)–(2.11), it follows from (4.2) that

$$\begin{aligned} \eta(U)X = & -A_{FU}X + \nabla^{\perp}_X FU - E\nabla_X U \\ & - F\nabla_X U - Bh(X,U) - Ch(X,U). \end{aligned} \tag{4.3}$$

Using Proposition 3.2(II) in (4.3) and then equating the tangential components, we get

$$\eta(U)X = A_{FU}X + (U \ln f)EX + Bh(X,U). \tag{4.4}$$

Taking the inner product with EX in (4.4) and using the fact that X and EX are mutually orthogonal vector fields, then we have

$$g(A_{FU}X, EX) + (U \ln f)g(EX, EX) + g(Bh(X, U), EX) = 0. \tag{4.5}$$

Using (2.9) and (2.15), we get

$$-(U \ln f)\cos^2\theta\|X\|^2 = g(h(X, EX), FU) - g(h(X, U), FEX). \tag{4.6}$$

Replacing X by EX in (4.6) and using (2.14), we obtain

$$-(U \ln f)\cos^2\theta\|X\|^2 = -g(h(X, EX), FU) + g(h(EX, U), EX). \tag{4.7}$$

Adding (4.6) and (4.7), we get

$$(U \ln f)\cos^2\theta\|X\|^2 = 0. \tag{4.8}$$

Since N_θ is proper slant and X is nonnull, (4.8) yields $U \ln f = 0$, which shows that f is constant and consequently the theorem is proved. □

The second case is dealt with the following theorem.

Theorem 4.3. *Let $N = N_\theta \times_f N_\perp$ be a warped product hemi-slant submanifold of an LP-Sasakian manifold M such that N_θ is a proper slant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of M. Then, $(\overline{\nabla}_X F)(U)$ lies in the invariant normal subbundle μ, for each $X \in TN_\theta$ and $U \in TN_\perp$.*

Proof. Consider $N = N_\theta \times_f N_\perp$ be a warped product hemi-slant submanifold of an LP-Sasakian manifold M such that N_θ is a proper slant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of M. Then, for any $X \in TN_\theta$ and $U \in TN_\perp$, we have

$$\overline{\nabla}_X \phi U = \phi \overline{\nabla}_X U. \tag{4.9}$$

Using (2.7) and (2.8), we obtain

$$-A_{FU}X + \nabla^\perp_X FU = \phi(\nabla_X U + h(X, U)). \tag{4.10}$$

By virtue of (2.10), (2.11) and Proposition 3.2(II), it follows from (4.10) that

$$\begin{aligned} -A_{FU}X + \nabla^\perp_X FU = {} & (X \ln f)EU + (X \ln f)FU \\ & + Bh(X, U) + Ch(X, U). \end{aligned} \tag{4.11}$$

Equating the normal components, we obtain

$$\nabla^\perp_X FU = (X \ln f)FU + Ch(X, U). \tag{4.12}$$

Taking the inner product of with FW_1, for any $W_1 \in TN_\perp$ in (4.13), we get

$$\begin{aligned} g\left(\nabla_X^\perp FU, FW_1\right) &= (X\ln f)g(FU, FW_1) + g(Ch(X,U), FW_1) \\ &= (X\ln f)g(\phi U, \phi W_1) + g(\phi h(X,U), \phi W_1) \\ &= (X\ln f)g(U, W_1). \end{aligned} \tag{4.13}$$

Also for any $X \in TN_\theta$ and $U \in TN_\perp$, we have

$$\left(\overline{\nabla}_X F\right)U = \nabla_X^\perp FU - (X\ln f)FU. \tag{4.14}$$

Taking the inner product FW_1 for any $W_1 \in TN_\perp$ in (4.14) and using (2.1) and (2.2), we derive

$$g\left(\left(\overline{\nabla}_X F\right)U, FW_1\right) = g\left(\nabla_X^\perp FU, FW_1\right) - (X\ln f)g(U, W_1). \tag{4.15}$$

By virtue of (4.13), the above equation yields

$$g\left(\left(\overline{\nabla}_X F\right)U, FW_1\right) = 0, \quad \text{for any } X \in TN_\theta,\ U, W_1 \in TN_\perp. \tag{4.16}$$

Similarly, if any $W_2 \in TN_\theta$, then from (2.13), we obtain

$$g\left(\left(\overline{\nabla}_X F\right)U, \phi W_2\right) = g\left(\nabla_X^\perp FU, \phi W_2\right) - g(F\nabla_X U, \phi W_2). \tag{4.17}$$

Since the product of tangential component with normal is zero and N_θ is a proper slant submanifold, we may conclude from (4.17) that

$$g\left(\left(\overline{\nabla}_X F\right)U, \phi W_2\right) = 0 \quad \text{for any } X, W_2 \in TN_\theta,\ U \in TN_\perp. \tag{4.18}$$

From (4.16) and (4.18), it follows that $(\overline{\nabla}_X F)(U) \in \mu$ and hence the proof is complete. □

5. Examples on LP-Sasakian Manifolds

Example 5.1. We consider a 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by

$$E_1 = e^z \frac{\partial}{\partial x}, \qquad E_2 = e^{z-ax}\frac{\partial}{\partial y}, \qquad E_3 = -\frac{\partial}{\partial z}, \tag{5.1}$$

where a is a nonzero constant such that $a \neq 1$. Let g be the Lorentzian metric defined by $g(E_1, E_3) = g(E_2, E_3 = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in TM$. Let θ be the (1,1) tensor field defined by $\eta E_1 = -E_1$, $\phi E_2 = -E_2$, and $\phi E_3 = 0$. Then, using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$, and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in TM$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, we have

$$[E_1, E_2] = -ae^z E_2, \qquad [E_1, E_3] = -E_1, \qquad [E_2, E_3] = -E_2. \tag{5.2}$$

Using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\begin{gathered} \nabla_{E_1} E_1 = -E_3, \qquad \nabla_{E_1} E_2 = 0, \qquad \nabla_{E_1} E_3 = -E_1, \\ \nabla_{E_2} E_1 = ae^z E_2, \qquad \nabla_{E_2} E_2 = -ae^z E_1 - E_3, \qquad \nabla_{E_2} E_3 = -E_2, \\ \nabla_{E_3} E_1 = 0, \qquad \nabla_{E_3} E_2 = 0, \qquad \nabla_{E_3} E_3 = 0. \end{gathered} \tag{5.3}$$

From the above computations, it can be easily seen that for $E_3 = \xi$, (ϕ, ξ, η, g) is an LP-Sasakian structure on M. Consequently, $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold.

Example 5.2 (see [16]). Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Define

$$\begin{gathered} \eta = ds - y\,dx - t\,dz, \qquad \xi = \frac{\partial}{\partial s}, \\ g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2, \\ \phi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \qquad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}, \\ \phi\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \qquad \phi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \qquad \phi\left(\frac{\partial}{\partial s}\right) = 0, \end{gathered} \tag{5.4}$$

the structure (ϕ, η, ξ, g) becomes an LP-Sasakian structure in $\mathbb{R}^5$.

Example 5.3. Consider a 4-dimensional submanifold N of $\mathbb{R}^7$ with the cordinate system $(x_1, x_2, \ldots, x_6, t)$ and the structure is defined as

$$\begin{gathered} \phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad (i = 1, 2, 3), \\ \phi\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_j}, \quad (j = 4, 5, 6), \\ \eta = dt, \qquad \xi = -\frac{\partial}{\partial t}, \qquad \phi\left(\frac{\partial}{\partial t}\right) = 0, \\ g = dx_i^2 + dx_j^2 + \eta \otimes \eta. \end{gathered} \tag{5.5}$$

Hence, the structure (ϕ, ξ, η, g) is an LP-contact structure on $\mathbb{R}^7$. Now, for any $\alpha \in (0, \pi/2)$ and nonzero u and v, we define the submanifold N as follows:

$$\omega(u, v, \alpha, t) = 2(u, v, u\cos\alpha, -v\sin\alpha, u\sin\alpha, v\cos\alpha, t). \tag{5.6}$$

Then, the tangent space TN is spanned by the vectors:

$$\begin{aligned}
e_1 &= \frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_5}, \\
e_2 &= \frac{\partial}{\partial x_2} - \sin\alpha\frac{\partial}{\partial x_4} + \cos\alpha\frac{\partial}{\partial x_6}, \\
e_3 &= -u\sin\alpha\frac{\partial}{\partial x_3} - v\cos\alpha\frac{\partial}{\partial x_4} + u\cos\alpha\frac{\partial}{\partial x_5} - v\sin\alpha\frac{\partial}{\partial x_6}, \\
e_4 &= -\frac{\partial}{\partial t}.
\end{aligned} \tag{5.7}$$

Then the distributions $D_\theta = \operatorname{span}\{e_1, e_2, e_4\}$ is a slant distribution tangent to $\xi = e_4$ and $D^\perp = \operatorname{span}\{e_3\}$ is an anti-invariant distribution, respectively. Let us denote by N_θ and $N_\perp$ their integral submanifolds, then the metric g on N is given by

$$g = 2\left(du^2 + dv^2\right) + \left(u^2 + v^2\right)d\alpha^2. \tag{5.8}$$

Hence, the submanifold $N = N_\theta \times_f N_\perp$ is a hemi-slant-warped product submanifold of $\mathbb{R}^7$ with the warping function $f = \sqrt{(u^2 + v^2)}$.

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