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Research Article

Ideal Theory in BCK/BCI-Algebras Based on Soft Sets and $\mathcal{N}$-Structures

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Based on soft sets and $\mathcal{N}$-structures, the notion of (closed) $\mathcal{N}$-ideal over a BCI-algebra is introduced, and related properties are investigated. Relations between $\mathcal{N}$-BCI-algebras and $\mathcal{N}$-ideals are established. Characterizations of a (closed) $\mathcal{N}$-ideal over a BCI-algebra are provided. Conditions for an $\mathcal{N}$-ideal to be an $\mathcal{N}$-BCI-algebra are considered.

1. Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [1]. Maji et al. [2] and Molodtsov [1] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [1] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [2] described the application of soft set theory to a decision-making problem. Maji et al. [3] also studied several operations on the theory of soft sets. Chen et al. [4] presented a new definition of soft set parametrization reduction and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [5]. Roy and Maji [6] presented some results on an application of fuzzy soft sets in decision making problem. Feng et al. [7] provided a framework to combine fuzzy sets, rough sets, and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets, and soft rough fuzzy sets. Feng et al. [8] gave deeper insights into decision making based on fuzzy soft sets. Feng et al. [9] initiated the notion of soft rough sets, which can be seen as a generalized rough set model based on soft sets. Aygünoğlu et al. [10] introduced the notion of fuzzy soft group and studied its properties. Ali et al. [11] discussed new operations in soft set theory. Jun [12] applied the notion of soft set to BCK/BCI-algebras, and Jun et al. [13] considered applications of soft set theory in the ideals of d-algebras. Han et al. [14] discussed the fuzzy set theory of fated filters in R_0-algebras based on fuzzy points.

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalizations of the crisp set have been conducted on the unit interval $[0,1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fits the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [15] introduced a new function, which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [16] considered closed ideals in BCH-algebras based on $\mathcal{N}$-structures. Jun et al. [17] introduced the notion of $\mathcal{N}$-soft sets which are a soft set based on $\mathcal{N}$-structures, and then they applied it to both a decision-making problem and a BCK/BCI-algebra.

In this paper, we introduce the notion of (closed) $\mathcal{N}$-ideal over a BCI-algebra based on soft sets and $\mathcal{N}$-structures and investigate related properties. We establish relations between $\mathcal{N}$-BCI-algebras and $\mathcal{N}$-ideals. We also provide characterizations of a (closed) $\mathcal{N}$-ideal over a BCI-algebra and consider conditions for an $\mathcal{N}$-ideal to be an $\mathcal{N}$-BCI-algebra.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2,0)$. An element $X \in K(\tau)$ is called a *BCI-algebra* if it satisfies the following axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,

(a2) $(x * (x * y)) * y = 0$,

(a3) $x * x = 0$,

(a4) $x * y = y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*.

We can define a partial ordering $\leq$ on X by

$$(\forall x, y \in X) \quad (x \leq y \Longleftrightarrow x * y = 0). \tag{2.1}$$

In a BCK/BCI-algebra X, the following hold:

(b1) $x * 0 = x$,

(b2) $(x * y) * z = (x * z) * y$,

(b3) $0 * (0 * (0 * x)) = 0 * x$,

(b4) $0 * (x * y) = (0 * x) * (0 * y)$

for all $x, y, z \in X$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

(c1) $0 \in A$,

(c2) $(\forall x, y \in X)$ $(x * y \in A,\ y \in A \Rightarrow x \in A)$.

We refer the reader to the books [18, 19] for further information regarding BCK/BCI-algebras.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X.

Definition 2.1 (see [15]). By a *subalgebra* of a BCK/BCI-algebra X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-subalgebra* of X) are means an $\mathcal{N}$-structure (X, f) in which f satisfies the following assertion:

$$(\forall x, y \in X) \quad \left(f(x * y) \leq \bigvee\{f(x), f(y)\}\right). \tag{2.2}$$

Definition 2.2 (see [15]). By an *ideal* of a BCK/BCI-algebra X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-ideal* of X) are means an $\mathcal{N}$-structure (X, f) in which f satisfies the following assertion:

$$(\forall x, y \in X) \quad \left(f(0) \leq f(x) \leq \bigvee\{f(x * y), f(y)\}\right). \tag{2.3}$$

Definition 2.3 (see [15]). Let X be a BCI-algebra. An $\mathcal{N}$-ideal (X, f) is said to be *closed* if it is also an $\mathcal{N}$-subalgebra of X.

3. $\mathcal{N}$-Soft BCK/BCI-Algebras and $\mathcal{N}$-Soft Ideals

In what follows let E denote a set of attributes unless otherwise specified. We will use the terminology "soft machine" which means that it produces a BCK/BCI-algebra.

Definition 3.1 (see [17]). Let X be an initial universe set and E a set of attributes. By an *$\mathcal{N}$-soft set* over X we mean a pair (f, A) where $A \subset E$ and f is a mapping from A to $\mathcal{F}(X, [-1, 0])$; that is, for each $a \in A$, $f(a) := f_a$ is an $\mathcal{N}$-function on X.

Denote by $\mathcal{N}(X, E)$ the collection of all $\mathcal{N}$-soft sets over X with attributes from E, and we call it an $\mathcal{N}$-soft class.

Definition 3.2 (see [17]). Let (f, A) and (g, B) be $\mathcal{N}$-soft sets in $\mathcal{N}(X, E)$. Then (f, A) is called an *$\mathcal{N}$-soft subset* of (g, B), denoted by $(f, A)\widetilde{\subseteq}(g, B)$, if it satisfies

(i) $A \subseteq B$,

(ii) $(\forall e \in A)$ $(f_e \subseteq g_e,$ i.e., $f_e(x) \leq g_e(x)$ $\forall x \in X)$.

Definition 3.3 (see [17]). Let (f, A) be an $\mathcal{N}$-soft set over a BCK/BCI-algebra X where A is a subset of E. If there exists an attribute $u \in A$ for which the $\mathcal{N}$-structure (X, f_u) is an $\mathcal{N}$-subalgebra of X, then we say that (f, A) is an *$\mathcal{N}$-soft BCK/BCI-algebra* related to the attribute u (briefly, *N_u-soft BCK/BCI-algebra*). If (f, A) is an $\mathcal{N}_u$-soft BCK/BCI-algebra for all $u \in A$, we say that (f, A) is an *$\mathcal{N}$-soft BCK/BCI-algebra*.

Definition 3.4. Let (f, A) be an $\mathcal{N}$-soft set over a BCK/BCI-algebra X where A is a subset of E. If there exists an attribute $u \in A$ for which the $\mathcal{N}$-structure (X, f_u) is an $\mathcal{N}$-ideal of X, then we say that (f, A) is an *$\mathcal{N}$-soft ideal* of X related to the attribute u (briefly, *$\mathcal{N}_u$-soft ideal*). If (f, A) is an $\mathcal{N}_u$-soft ideal of X for all $u \in A$, we say that (f, A) is an *$\mathcal{N}$-soft ideal* of X.

Example 3.5. Let $U := \{\text{apple}, \text{banana}, \text{carrot}, \text{peach}, \text{radish}\}$ be a universe, and consider a soft machine \$ which produces the following products:

$$
\begin{gathered}
\text{apple } \$\ x = \text{apple} \quad \forall x \in U, \\
\text{banana } \$\ y = \begin{cases} \text{apple} & \text{if } y \in \{\text{banana}, \text{peaqch}, \text{radish}\}, \\ \text{banana} & \text{if } y \in \{\text{apple}, \text{carrot}\}, \end{cases} \\
\text{carrot } \$\ z = \begin{cases} \text{carrot} & \text{if } z \in \{\text{apple}, \text{banana}, \text{radish}\}, \\ \text{apple} & \text{if } z \in \{\text{carrot}, \text{peach}\}, \end{cases} \\
\text{peach } \$\ u = \begin{cases} \text{peach} & \text{if } u = \text{apple}, \\ \text{banana} & \text{if } u = \text{carrot}, \\ \text{apple} & \text{if } u = \text{peach}, \\ \text{carrot} & \text{if } u \in \{\text{banana}, \text{radish}\}, \end{cases} \\
\text{radish } \$\ v = \begin{cases} \text{apple} & \text{if } v = \text{radish}, \\ \text{radish} & \text{if } v \in \{\text{apple}, \text{carrot}\}, \\ \text{banana} & \text{if } v \in \{\text{banana}, \text{peach}\}. \end{cases}
\end{gathered}
\tag{3.1}
$$

Then U is a BCK-algebra under the soft machine \$. Consider a set of attributes

$$
A := \{\text{cat}, \text{cow}, \text{dog}, \text{duck}, \text{horse}, \text{pig}\}, \tag{3.2}
$$

Table 1: Tabular representation of (f, A).

(f, A)	Apple	Banana	Carrot	Peach	Radish
Cat	−0.5	−0.3	−0.3	−0.3	−0.3
Cow	−0.6	−0.4	−0.6	−0.4	−0.4
Dog	−0.7	−0.7	−0.2	−0.2	−0.7
Duck	−0.3	−0.1	−0.3	−0.1	−0.1
Horse	−0.4	−0.4	−0.9	−0.6	−0.3
Pig	−0.5	−0.4	−0.1	−0.1	−0.4

and let (f, A) be an $\mathcal{N}$-soft set over the BCK-algebra U with the tabular representation which is given by Table 1. Then f_{cat}, f_{cow}, f_{dog}, f_{duck}, and f_{fig} are $\mathcal{N}$-soft ideals over U. But f_{hourse} is not an $\mathcal{N}$-soft ideal over U since

$$\begin{aligned} f_{\text{hourse}}(\text{radish}) &= -0.3 > -0.4 \\ &= \bigvee\{f_{\text{hourse}}(\text{radish } \$ \text{ peach}), f_{\text{hourse}}(\text{peach})\}. \end{aligned} \tag{3.3}$$

In general, we know that horses like carrots best of all. In the above example, we know that (f, A) is not an $\mathcal{N}$-soft ideal over U based on attribute “horse.” This means that if a horse like a carrot better than the others, then (f, A) cannot be an $\mathcal{N}$-soft ideal over U.

Obviously, every $\mathcal{N}$-soft ideal is an $\mathcal{N}$-soft BCK-algebra in a BCK-algebra, but the converse is not true as seen in the following example.

Example 3.6. Let

$$U := \{\text{apple}, \text{banana}, \text{carrot}, \text{peach}, \text{radish}\} \tag{3.4}$$

be a universe, and consider a soft machine $\Diamond$ which produces the following products:

$$\begin{gathered} \text{apple} \Diamond x = \begin{cases} \text{apple} & \text{if } x \in \{\text{carrot}, \text{peach}, \text{radish}\}, \\ \text{carrot} & \text{if } x \in \{\text{apple}, \text{banana}\}, \end{cases} \\ \text{banana} \Diamond y = \begin{cases} \text{banana} & \text{if } y \in \{\text{apple}, \text{carrot}, \text{peach}, \text{radish}\}, \\ \text{carrot} & \text{if } x = \text{banana}, \end{cases} \\ \text{carrot} \Diamond z = \text{carrot } \forall z \in U, \\ \text{peach} \Diamond u = \begin{cases} \text{peach} & \text{if } u \in \{\text{carrot}, \text{radish}\}, \\ \text{carrot} & \text{if } u \in \{\text{apple}, \text{peach}, \text{banana}\}, \end{cases} \\ \text{radish} \Diamond v = \begin{cases} \text{radish} & \text{if } v \in \{\text{carrot}, \text{peach}\}, \\ \text{carrot} & \text{if } v \in \{\text{apple}, \text{banana}, \text{radish}\}. \end{cases} \end{gathered} \tag{3.5}$$

Then U is a BCK-algebra under the soft machine $\Diamond$. Consider a set of parameters

$$B := \{\text{duck}, \text{horse}, \text{pig}\}. \tag{3.6}$$

Table 2: Tabular representation of (g, B).

(g, B)	Apple	Banana	Carrot	Peach	Radish
Duck	−0.2	−0.2	−0.6	−0.4	−0.2
Horse	−0.4	−0.1	−0.5	−0.1	−0.1
Pig	−0.3	−0.3	−0.7	−0.6	−0.5

Let (g, B) be an $\mathcal{N}$-soft set over the BCK-algebra U with the tabular representation which is given by Table 2. It is easy to verify that (g, B) is an $\mathcal{N}$-soft BCK-algebra. But it is not an $\mathcal{N}$-soft ideal over U because (g, B) is not an $\mathcal{N}_{\text{horse}}$-soft ideal since

$$g_{\text{horse}}(\text{peach}) = -0.1 > -0.4 = \bigvee\{g_{\text{horse}}(\text{peach} \diamondsuit \text{apple}), g_{\text{horse}}(\text{apple})\}. \tag{3.7}$$

We discuss characterizations of an $\mathcal{N}$-soft ideal.

Theorem 3.7. *For a BCK/BCI-algebra X, let (f, A) be an $\mathcal{N}$-soft set in (X, E) such that $f_u(0) \le f_u(x)$ for all $x \in X$ and $u \in A$. Then (f, A) is an $\mathcal{N}$-soft ideal of X if and only if the following assertion is valid:*

$$(\forall x, y, z \in X) \quad (\forall u \in A) \quad \left(x * y \le z \Longrightarrow f_u(x) \le \bigvee\{f_u(y), f_u(z)\}\right). \tag{3.8}$$

Proof. Suppose that (f, A) is an $\mathcal{N}$-soft ideal of X. Let $u \in A$ and $x, y, z \in X$ be such that $x * y \le z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} f_u(x * y) &\le \bigvee\{f_u((x * y) * z), f_u(z)\} \\ &= \bigvee\{f_u(0), f_u(z)\} = f_u(z). \end{aligned} \tag{3.9}$$

It follows that $f_u(x) \le \bigvee\{f_u(x * y), f_u(y)\} \le \bigvee\{f_u(z), f_u(y)\}$.

Conversely, assume that (3.8) holds. Note that $x * (x * y) \le y$ for all $x, y \in X$. It follows from (3.8) that

$$f_u(x) \le \bigvee\{f_u(x * y), f_u(y)\} \tag{3.10}$$

for all $x, y \in X$ and $u \in A$. Hence (f, A) is an $\mathcal{N}_u$-soft ideal of X for all $u \in A$, and so (f, A) is an $\mathcal{N}$-soft ideal of X. □

Lemma 3.8 (see [17]). *Every $\mathcal{N}$-soft BCK/BCI-algebra (f, A) over a BCK/BCI-algebra X satisfies the following inequality:*

$$(\forall x \in X) \quad (\forall u \in A) \quad (f_u(0) \le f_u(x)). \tag{3.11}$$

Theorem 3.9. *Let (f, A) be an $\mathcal{N}$-BCK/BCI-algebra of a BCK/BCI-algebra X. Then (f, A) is an $\mathcal{N}$-soft ideal of X if and only if it satisfies* (3.8).

Proof. Necessity is by Theorem 3.7. Conversely, assume that assertion (3.8) is valid. Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows that $f_u(x) \leq \bigvee\{f_u(x * y), f_u(y)\}$ for all $x, y \in X$ and $u \in A$. Combining this and Lemma 3.8, we know that (f, A) is an $\mathcal{N}$-soft ideal of X. □

Proposition 3.10. *Every $\mathcal{N}$-soft ideal (f, A) of a BCI-algebra X satisfies the following inequality:*

$$(\forall x \in X) \quad (\forall u \in A) \quad \left(f_u(0 * (0 * x)) \leq f_u(x)\right). \tag{3.12}$$

Proof. Let (f, A) be an $\mathcal{N}$-soft ideal of a BCI-algebra X. Then

$$\begin{aligned} f_u(0 * (0 * x)) &\leq \bigvee\{f_u((0 * (0 * x)) * x), f_u(x)\} \\ &= \bigvee\{f_u(0), f_u(x)\} = f_u(x) \end{aligned} \tag{3.13}$$

for all $x \in X$ and $u \in A$. □

The following example shows that there exists an attribute $u \in A$ such that an $\mathcal{N}_u$-soft ideal of a BCI-algebra may not be an $\mathcal{N}_u$-soft BCI-algebra.

Example 3.11. Let $U = \mathbb{Q} - \{0\}$ be a universe where $\mathbb{Q}$ is the set of all rational numbers. Let $\bigstar$ be a soft machine which is established by

$$(\forall x, y \in U) \quad \left(x \bigstar y = \frac{x}{y}\right). \tag{3.14}$$

Then U is a BCI-algebra under the soft machine $\bigstar$. Let (f, A) be an $\mathcal{N}$-soft set in $\mathcal{N}(U, E)$ which is defined by

$$(\forall x \in U) \quad (\forall u \in A) \quad \left(f_u(x) = \begin{cases} -0.6 & \text{if } x \in \mathbb{Z} - \{0\} \\ -0.2 & \text{otherwise} \end{cases}\right). \tag{3.15}$$

Then (f, A) is an $\mathcal{N}$-soft ideal over U, but it is not an $\mathcal{N}$-soft BCI-algebra over U since

$$f_u(3 \bigstar 2) = -0.2 > -0.6 = \bigvee\{f_u(3), f_u(2)\}. \tag{3.16}$$

Definition 3.12. Let (f, A) be an $\mathcal{N}$-soft set over a BCI-algebra X, where A is a subset of E. If there exists an attribute $u \in A$ for which the $\mathcal{N}$-structure (X, f_u) is a closed $\mathcal{N}$-ideal of X, then we say that (f, A) is a *closed $\mathcal{N}$-soft ideal* over X related to the attribute u (briefly, *closed $\mathcal{N}_u$-soft ideal*). If (f, A) is a closed $\mathcal{N}_u$-soft ideal over X for all $u \in A$, we say that (f, A) is a *closed $\mathcal{N}$-soft ideal* over X.

Table 3: Tabular representation of (f, A).

(f, A)	White	Blackish	Reddish	Green	Yellow
Beautiful	−0.9	−0.6	−0.4	−0.3	−0.3
Fine	−0.6	−0.4	−0.6	−0.5	−0.4
Moderate	−0.7	−0.3	−0.5	−0.4	−0.3

Example 3.13. Suppose there are five colors in the universe U, that is,

$$U := \{\text{white}, \text{blackish}, \text{reddish}, \text{green}, \text{yellow}\}. \tag{3.17}$$

Let $\uplus$ be a soft machine to mix two colors according to order in such a way that we have the following results:

$$\begin{gathered}
x \uplus \text{white} = x \quad \forall x \in U, \\
x \uplus \text{blackish} = \begin{cases} \text{white} & \text{if } x \in \{\text{white}, \text{blackish}\}, \\ \text{green} & \text{if } x \in \{\text{green}, \text{yellow}\}, \\ x & \text{if } x = \text{reddish}, \end{cases} \\
x \uplus \text{reddish} = \begin{cases} \text{white} & \text{if } x \in \{\text{reddish}\}, \\ x & \text{if } x \in \{\text{white}, \text{blackish}, \text{green}, \text{yellow}\}, \end{cases} \\
x \uplus \text{green} = \begin{cases} \text{green} & \text{if } x \in \{\text{white}, \text{reddish}\}, \\ \text{yellow} & \text{if } x = \text{blackish}, \\ \text{white} & \text{if } x = \text{green}, \\ \text{blackish} & \text{if } x = \text{yellow}, \end{cases} \\
x \uplus \text{yellow} = \begin{cases} \text{green} & \text{if } x \in \{\text{white}, \text{blackish}, \text{reddish}\}, \\ \text{white} & \text{if } x \in \{\text{green}, \text{yellow}\}. \end{cases}
\end{gathered} \tag{3.18}$$

Then U is a BCI-algebra under the soft machine $\uplus$. Consider a set of attributes

$$A := \{\text{beautiful}, \text{fine}, \text{moderate}\}, \tag{3.19}$$

and let (f, A) be an $\mathcal{N}$-soft set over U with the tabular representation which is given by Table 3. Then (f, A) is a closed $\mathcal{N}$-soft ideal over U.

Theorem 3.14. *A closed $\mathcal{N}$-soft ideal of a BCI-algebra X related to an attribute is an $\mathcal{N}$-soft BCI-algebra over X related to the same attribute.*

Proof. Let (f, A) be a closed $\mathcal{N}$-soft ideal over a BCI-algebra X related to an attribute $u \in A$. Then $f_u(0 * x) \le f_u(x)$ for all $x \in X$. It follows that

$$\begin{aligned} f_u(x * y) &\le \bigvee\{f_u((x * y) * x), f_u(x)\} = \bigvee\{f_u((x * x) * y), f_u(x)\} \\ &= \bigvee\{f_u(0 * y), f_u(x)\} \le \bigvee\{f_u(y), f_u(x)\} \end{aligned} \tag{3.20}$$

for all $x, y \in X$. Hence (f, A) is an $\mathcal{N}_u$-soft *BCI*-algebra over X. □

We provide a characterization of a closed $\mathcal{N}$-soft ideal.

Theorem 3.15. *For an $\mathcal{N}$-soft ideal (f, A) over a BCI-algebra X, the following are equivalent:*

(1) *(f, A) is closed,*

(2) *(f, A) is an $\mathcal{N}$-soft BCI-algebra over X.*

Proof. (1) ⇒ (2). It follows from Theorem 3.14.

(2) ⇒ (1). Suppose that (f, A) is an $\mathcal{N}$-soft BCI-algebra over X. Then $f_u(0) \le f_u(x)$ for all $x \in X$ and $u \in A$, and so

$$f_u(0 * x) \le \bigvee\{f_u(0), f_u(x)\} \le \bigvee\{f_u(x), f_u(x)\} = f_u(x). \tag{3.21}$$

Thus (f, A) is a closed $\mathcal{N}_u$-soft ideal over X for all $u \in A$, and therefore (f, A) is a closed $\mathcal{N}$-soft ideal over X. □

We consider a condition for an $\mathcal{N}$-soft ideal to be closed.

Theorem 3.16. *Given an attribute $u \in A$, if an $\mathcal{N}_u$-soft ideal (f, A) over a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X) \quad \left(f_u(0 * x) \le f_u(x)\right), \tag{3.22}$$

then (f, A) is a closed $\mathcal{N}_u$-soft ideal over X.

Proof. For all $x, y \in X$, we have $(x * y) * x = 0 * y$. Hence

$$\begin{aligned} f_u(x * y) &\le \bigvee\{f_u((x * y) * x), f_u(x)\} \\ &= \bigvee\{f_u(0 * y), f_u(x)\} \\ &\le \bigvee\{f_u(x), f_u(y)\}. \end{aligned} \tag{3.23}$$

Thus (f, A) is an $\mathcal{N}_u$-soft BCI-algebra over X. It follows from Theorem 3.15 that (f, A) is a closed $\mathcal{N}_u$-soft ideal over X. □

Corollary 3.17. *If an $\mathcal{N}$-soft ideal (f, A) over a BCI-algebra X satisfies the following assertion:*

$$(\forall x \in X) \quad (\forall u \in A) \quad \left(f_u(0 * x) \leq f_u(x)\right), \tag{3.24}$$

then (f, A) is a closed $\mathcal{N}$-soft ideal over X.

Definition 3.18 (see [17]). Let (f, A) and (g, B) be two $\mathcal{N}$-soft sets in (X, E). The *union* of (f, A) and (g, B) is defined to be the $\mathcal{N}$-soft set (h, C) in (X, E) satisfying the following conditions:

(i) $C = A \cup B$,

(ii) for all $x \in C$,

$$h_x = \begin{cases} f_x & \text{if } x \in A \setminus B, \\ g_x & \text{if } x \in B \setminus A, \\ f_x \cup g_x & \text{if } x \in A \cap B. \end{cases} \tag{3.25}$$

In this case, we write $(f, A) \tilde{\cup} (g, B) = (h, C)$.

Lemma 3.19 (see [15]). *If (X, f) and (X, g) are $\mathcal{N}$-ideals of a BCK/BCI -algebra X, then the union $(X, f \cup g)$ of (X, f) and (X, g) is an $\mathcal{N}$-ideal of X.*

Theorem 3.20. *If (f, A) and (g, B) are $\mathcal{N}$-soft ideals over a BCK/BCI-algebra X, then the union of (f, A) and (g, B) is an $\mathcal{N}$-soft ideal over X.*

Proof. Let $(f, A) \tilde{\cup} (g, B) = (h, C)$ be the union of (f, A) and (g, B). Then $C = A \cup B$. For any $x \in C$, if $x \in A \setminus B$ (resp., $x \in B \setminus A$), then $(X, h_x) = (X, f_x)$ (resp., $(X, h_x) = (X, g_x)$) is an $\mathcal{N}$-ideal of X. If $A \cap B \neq \emptyset$, then $(X, h_x) = (X, f_x \cup g_x)$ is an $\mathcal{N}$-ideal of X for all $x \in A \cap B$ by Lemma 3.19. Therefore (h, C) is an $\mathcal{N}$-soft ideal over a BCK/BCI-algebra X. □

Definition 3.21 (see [17]). Let (f, A) and (g, B) be two $\mathcal{N}$-soft sets in (X, E). The *intersection* of (f, A) and (g, B) is the $\mathcal{N}$-soft set (h, C) in (X, E), where $C = A \cup B$ and for every $x \in C$,

$$h_x = \begin{cases} f_x & \text{if } x \in A \setminus B, \\ g_x & \text{if } x \in B \setminus A, \\ f_x \cap g_x & \text{if } x \in A \cap B. \end{cases} \tag{3.26}$$

In this case, we write $(f, A) \tilde{\cap} (g, B) = (h, C)$.

Theorem 3.22. *Let (f, A) and (g, B) be $\mathcal{N}$-soft ideals over a BCK/BCI-algebra X. If A and B are disjoint, then the intersection of (f, A) and (g, B) is an $\mathcal{N}$-soft ideal over X.*

Proof. Let $(f, A) \tilde{\cap} (g, B) = (h, C)$ be the intersection of (f, A) and (g, B). Then $C = A \cup B$. Since $A \cap B = \emptyset$, if $x \in C$, then either $x \in A \setminus B$ or $x \in B \setminus A$. If $x \in A \setminus B$, then $(X, h_x) = (X, f_x)$ is an $\mathcal{N}$-ideal of X. If $x \in B \setminus A$, then $(X, h_x) = (X, g_x)$ is an $\mathcal{N}$-ideal of X. Hence (h, C) is an $\mathcal{N}$-soft ideal over a BCK/BCI-algebra X. □

The following example shows that Theorem 3.22 is not valid if A and B are not disjoint.

Example 3.23. Let U be an initial universe set that consists of "white," "blackish," "reddish," "green" and "yellow." Consider a soft machine "¥" which produces the following products:

$$
\begin{aligned}
\text{white} \ ¥ \ x &= \begin{cases} \text{white} & \text{if } x \in \{\text{white}, \text{blackish}\}, \\ x & \text{if } x \in \{\text{reddish}, \text{green}, \text{yellow}\}, \end{cases} \\
\text{blackish} \ ¥ \ y &= \begin{cases} \text{blackish} & \text{if } y = \text{white}, \\ \text{white} & \text{if } y = \text{blackish}, \\ y & \text{if } y \in \{\text{reddish}, \text{green}, \text{yellow}\}, \end{cases} \\
\text{reddish} \ ¥ \ z &= \begin{cases} \text{white} & \text{if } z = \text{reddish}, \\ \text{reddish} & \text{if } z \in \{\text{white}, \text{blackish}\}, \\ \text{yellow} & \text{if } z = \text{green}, \\ \text{green} & \text{if } z = \text{yellow}, \end{cases} \\
\text{green} \ ¥ \ u &= \begin{cases} \text{white} & \text{if } u = \text{green}, \\ \text{green} & \text{if } u \in \{\text{white}, \text{blackish}\}, \\ \text{yellow} & \text{if } u = \text{reddish}, \\ \text{reddish} & \text{if } u = \text{yellow}, \end{cases} \\
\text{yellow} \ ¥ \ v &= \begin{cases} \text{white} & \text{if } v = \text{yellow}, \\ \text{reddish} & \text{if } v = \text{green}, \\ \text{green} & \text{if } v = \text{reddish}, \\ \text{yellow} & \text{if } v \in \{\text{white}, \text{blackish}\}. \end{cases}
\end{aligned}
\tag{3.27}
$$

Then U is a BCI-algebra under the soft machine ¥. Consider sets of attributes:

$$
\begin{aligned}
A &:= \{\text{beautiful}, \text{fine}, \text{smart}\}, \\
B &:= \{\text{smart}, \text{chaste}\}.
\end{aligned}
\tag{3.28}
$$

Let (f, A) and (g, B) be $\mathcal{N}$-soft sets over U with the tabular representations which are given by Tables 4 and 5, respectively. Then (f, A) and (g, B) are $\mathcal{N}$-soft ideals over U. But we have

$$
\begin{aligned}
\left(f_{\text{smart}} \cap g_{\text{smart}}\right)(\text{reddish}) &= -0.5 > -0.6 \\
&= \bigvee\left\{\left(\mathrm{f}_{\text{smart}} \cap g_{\text{smart}}\right)(\text{reddish} \ ¥ \ \text{green}), \left(f_{\text{smart}} \cap g_{\text{smart}}\right)(\text{green})\right\},
\end{aligned}
\tag{3.29}
$$

and so $(f, A) \,\tilde{\cap}\, (g, B)$ is not an $\mathcal{N}$-soft ideal over U.

4. Conclusions

We have introduced the notion of (closed) $\mathcal{N}$-ideal over a *BCI*-algebra based on soft sets and $\mathcal{N}$-structures. We have investigated several properties and established relations between $\mathcal{N}$-*BCI*-algebras and $\mathcal{N}$-ideals. We have provided characterizations of a (closed) $\mathcal{N}$-ideal over

Table 4: Tabular representation of (f, A).

(f, A)	White	Blackish	Reddish	Green	Yellow
Beautiful	−0.8	−0.6	−0.5	−0.4	−0.4
Fine	−0.7	−0.6	−0.3	−0.5	−0.3
Smart	−0.9	−0.8	−0.2	−0.2	−0.6

Table 5: Tabular representation of (g, B).

(g, B)	White	Blackish	Reddish	Green	Yellow
Smart	−0.8	−0.7	−0.5	−0.6	−0.5
Chaste	−0.6	−0.5	−0.1	−0.1	−0.3

a *BCI*-algebra and considered conditions for an $\mathcal{N}$-ideal to be an $\mathcal{N}$-BCI-algebra. Based on these results, we will apply the $\mathcal{N}$-structure to the other type of ideals/filters in BCK/BCI-algebras, MV-algebras, MTL-algebras, R_0-algebras BL-algebras, and so forth, in the future study.

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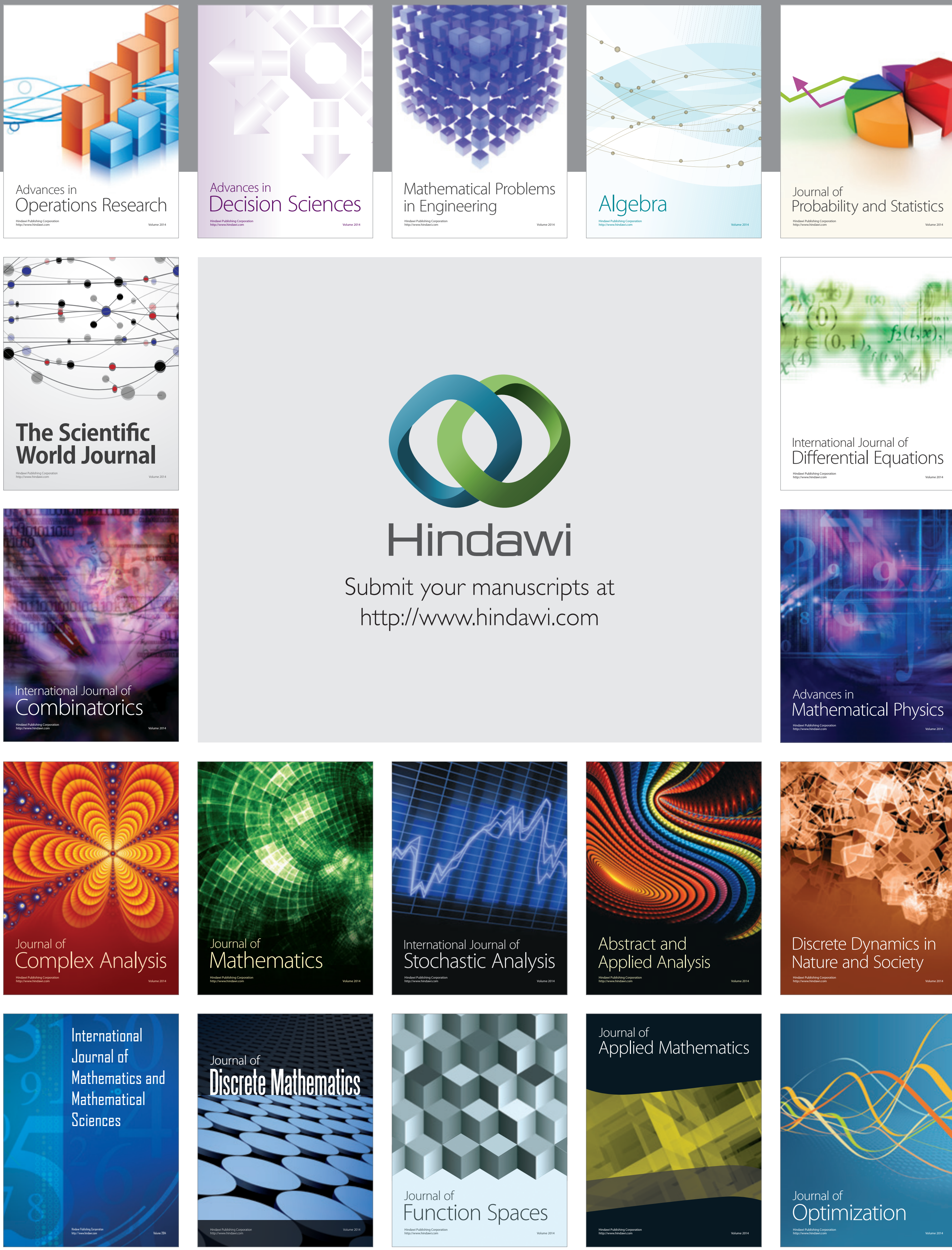
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