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Research Article

Almost Automorphic Mild Solutions to Neutral Parabolic Nonautonomous Evolution Equations with Nondense Domain

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Combining the exponential dichotomy of evolution family, composition theorems for almost automorphic functions with Banach fixed point theorem, we establish new existence and uniqueness theorems for almost automorphic mild solutions to neutral parabolic nonautonomous evolution equations with nondense domain. A unified framework is set up to investigate the existence and uniqueness of almost automorphic mild solutions to some classes of parabolic partial differential equations and neutral functional differential equations.

1. Introduction

In this paper, we are interested in the existence and uniqueness of almost automorphic mild solutions to the following neutral parabolic evolution equations in Banach space $\mathbb{X}$:

$$\frac{d}{dt}\left[u(t)+f(t,u(t))\right] = A(t)\left[u(t)+f(t,u(t))\right] + g(t,u(t)), \quad t\in\mathbb{R}, \tag{1}$$

$$\frac{d}{dt}\left[u(t)+f(t,Bu(t))\right] = A(t)\left[u(t)+f(t,Bu(t))\right] + g(t,Cu(t)), \quad t\in\mathbb{R}, \tag{2}$$

where sectorial operators $A(t) : D(A(t)) \subset \mathbb{X} \to \mathbb{X}$ have a domain $D(A(t))$ not necessarily dense in $\mathbb{X}$ and satisfy "Acquistapace-Terreni" conditions, $f, g : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ are almost automorphic in the first argument and Lipschitz in the second argument, and $B, C : \mathbb{X} \mapsto \mathbb{X}$ are bounded linear operators.

Bochner has shown in the seminal work [1] that in certain situations it is possible to establish the almost periodicity of an object by first establishing its almost automorphy and then invoking auxiliary conditions which, when coupled with almost automorphy, give almost periodicity. From then on, automorphy has been widely investigated. Fundamental properties of almost automorphic functions on groups and abstract almost automorphic minimal flows were studied by Veech [2, 3] and others. Afterwards, Zaki [4] extended the notion of scalar-valued almost automorphy to the one of vector-valued almost automorphic functions, paving the road to many applications to differential equations and dynamical systems. Among other things, Shen and Yi [5] showed that almost automorphy is essential and fundamental in the qualitative study of almost periodic differential equations in the sense that almost automorphic solutions are the right class for almost periodic systems. We refer the readers to the monographs [6, 7] by N'Guérékata for more information on this topic.

In the autonomous case, namely $A(t) = A$, the existence and uniqueness of almost automorphic mild solutions to evolution equation (1) with $f = 0$ have been successfully investigated in [6–13] in the framework of semigroups of bounded linear operators. In [13], N'Guérékata studied the existence and uniqueness of almost automorphic solutions for semilinear evolution equation

$$\frac{d}{dt}u(t) = Au(t) + g(t,u(t)), \quad t\in\mathbb{R}, \tag{3}$$

where A generates an exponentially stable semigroup on Banach space $\mathbb{X}$ and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is almost automorphic. The author proved that the unique bounded mild solution $u : \mathbb{R} \to \mathbb{X}$ of (3) is almost automorphic. In [8], Boulite et al. studied the existence and uniqueness of almost automorphic solutions for evolution equation (3), assuming that A generates a hyperbolic semigroup on Banach space $\mathbb{X}$ and $g : \mathbb{R} \times \mathbb{X}_\alpha \to \mathbb{X}$ is almost automorphic, where $\mathbb{X}_\alpha$ is an intermediate space between $D(A)$ and $\mathbb{X}$. The authors proved that the unique bounded mild solution $u : \mathbb{R} \to \mathbb{X}_\alpha$ of (3) is almost automorphic. Cieutat and Ezzinbi [9] studied the existence of bounded and compact almost automorphic solutions for semilinear evolution equation (3). The main methods are through the minimizing of some subvariant functionals. They gave sufficient conditions ensuring the existence of an almost automorphic mild solution when there is at least one bounded mild solution on $\mathbb{R}^+$.

In the nonautonomous case, almost automorphic mild solutions to evolution equation (1) with $f = 0$ have been successfully investigated in [14–16] in the framework of evolution family. Among others, Baroun et al. [14] generalized the main results of [8] to the nonautonomous case. The authors proved that the unique bounded mild solution $u : \mathbb{R} \to \mathbb{X}_\alpha$ of the semilinear evolution equation

$$\frac{d}{dt} u(t) = A(t) u(t) + g(t, u(t)), \quad t \in \mathbb{R}, \tag{4}$$

is almost automorphic, assuming that the evolution family $\{U(t,s)\}_{t \geq s}$ generated by $A(t)$ has an exponential dichotomy and $g : \mathbb{R} \times \mathbb{X}_\alpha \to \mathbb{X}$ is almost automorphic. Ding et al. [15] established the existence and uniqueness theorem of almost automorphic mild solutions to the evolution equation (4), where the evolution family $\{U(t,s)\}_{t \geq s}$ generated by $A(t)$ has an exponential dichotomy and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is almost automorphic. Liu and Song [16] proved the existence and uniqueness of an almost automorphic or a weighted pseudo almost automorphic mild solution to (4), assuming that the evolution family $\{U(t,s)\}_{t \geq s}$ generated by $A(t)$ is exponentially stable and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is almost automorphic or weighted pseudo almost automorphic.

A rich source of the literature exists on almost automorphic mild solutions to linear and semilinear evolution equations. However, to the best of our knowledge, there are few results available on the existence and uniqueness of almost automorphic mild solutions to neutral parabolic nonautonomous evolution equations (1) and (2), especially in the case of not necessarily dense domain and bounded perturbations. Nondensity occurs in many situations, from restrictions made on the space where the equation is considered or from boundary conditions. For example, the space $C^2[0,\pi]$ of twice continuously differential functions with null value on the boundary is nondense in $C[0,\pi]$, the space of continuous functions. One can refer for this to [17–19] or Section 5 for more details. We further remark that our first main result (Theorem 18) recovers partly Theorem 2.2 in [15] and Theorem 3.2 in [16] in the parabolic case. Moreover, a unified framework is set up in the second main result (Theorem 21) to study the existence and uniqueness of almost automorphic mild solutions to some classes of parabolic partial differential equations and neutral functional differential equations. As one will see, the additional neutral term $f(t, u(t))$ greatly widens the applications of the main result since (2) is general enough to incorporate some classes of parabolic partial differential equations and neutral functional differential equations as special cases.

As a preparation, in Section 2 we fix our notation and collect some basic facts on evolution family and almost automorphy. Section 3 deals with the proof of the existence and uniqueness theorem of almost automorphic mild solutions to evolution equation (1). In Section 4, we study the existence and uniqueness of almost automorphic mild solutions to evolution equation (2) with bounded perturbations. Finally, the abstract results are applied to some classes of parabolic partial differential equations and neutral functional differential equations.

2. Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the sets of positive integer, integer, real, and complex numbers, and $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space. If $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ is another Banach space, the space $B(\mathbb{X}, \mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with the uniform operator topology. The resolvent operator $R(\lambda, A)$ is defined by $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$, the resolvent set of a linear operator A.

2.1. Evolution Family and Exponential Dichotomy

Definition 1 (see [20, 21]). A family of bounded linear operators $\{U(t,s)\}_{t \geq s}$ on a Banach space $\mathbb{X}$ is called an evolution family if

(1) $U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$;

(2) the map $(t,s) \mapsto U(t,s)x$ is continuous for all $x \in \mathbb{X}$, $t > s$, and $t, s \in \mathbb{R}$.

Definition 2 (see [20, 21]). An evolution family $\{U(t,s)\}_{t \geq s}$ on a Banach space $\mathbb{X}$ has an exponential dichotomy (or is called hyperbolic) if there exist projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t and constants $M > 0$, $\delta > 0$ such that

(1) $U(t,s)P(s) = P(t)U(t,s)$ for $t \geq s$ and $t, s \in \mathbb{R}$;

(2) the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible for $t \geq s$ (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$);

(3) $\|U(t,s)P(s)\|_{B(\mathbb{X})} \leq Me^{-\delta(t-s)}$, $\|U_Q(s,t)Q(t)\|_{B(\mathbb{X})} \leq Me^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

Here and below we set $Q := I - P$.

Remark 3. Exponential dichotomy is a classical concept in the study of the long-term behavior of evolution equations, combining forward exponential stability on some subspaces with backward exponential stability on their complements. Its importance relies in particular on the robustness; that is, exponential dichotomy persists under small linear or nonlinear perturbations (see, e.g., [20–24]).

Definition 4 (see [20, 21]). Given an evolution family $\{U(t,s)\}_{t\geq s}$ with an exponential dichotomy, one defines its Green's function by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t\geq s,\ t,s\in\mathbb{R}, \\ -U_Q(t,s)Q(s), & t<s,\ t,s\in\mathbb{R}. \end{cases} \tag{5}$$

2.2. Almost Automorphy and Bi-Almost Automorphy. Let $C(\mathbb{R},\mathbb{X})$ denote the collection of continuous functions from $\mathbb{R}$ into $\mathbb{X}$. Let $BC(\mathbb{R},\mathbb{X})$ denote the Banach space of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ equipped with the sup norm $\|u\|_\infty := \sup_{t\in\mathbb{R}}\|u(t)\|$. Similarly, $C(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ denotes the collection of all jointly continuous functions from $\mathbb{R}\times\mathbb{X}$ into $\mathbb{Y}$, and $BC(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ denotes the collection of all bounded and jointly continuous functions $f:\mathbb{R}\times\mathbb{X}\to\mathbb{Y}$.

Definition 5 (Bochner). A function $f\in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for any sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+s_n-s_m) = f(t) \tag{6}$$

pointwise for each $t\in\mathbb{R}$. This limit means that

$$g(t) = \lim_{n\to\infty} f(t+s_n) \tag{7}$$

is well defined for each $t\in\mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t-s_n) \tag{8}$$

for each $t\in\mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{X})$.

Example 6 (Levitan). The function $f(t)=\sin(1/(2+\cos t+\cos\pi t))$, $t\in\mathbb{R}$, is almost automorphic but not almost periodic.

Remark 7. An almost automorphic function may not be uniformly continuous, while an almost periodic function must be uniformly continuous.

Lemma 8 (see [6, 7]). *Assume that $f,g:\mathbb{R}\to\mathbb{X}$ are almost automorphic and λ is any scalar. Then the following holds true:*

(1) *$f+g$, λf are almost automorphic;*

(2) *the range R_f of f is precompact, so f is bounded;*

(3) *f_τ defined by $f_\tau(t)=f(t+\tau)$, $\tau\in\mathbb{R}$, is almost automorphic.*

Lemma 9 (see [6, 7]). *If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n\to f$ $(n\to\infty)$ uniformly on $\mathbb{R}$, then f is almost automorphic.*

Lemma 10 (see [6]). *The space $AA(\mathbb{X})$ equipped with sup norm $\|u\|_\infty=\sup_{t\in\mathbb{R}}\|u(t)\|$ is a Banach space.*

Definition 11 (see [25]). A function $f\in C(\mathbb{R}\times\mathbb{X},\mathbb{X})$ is said to be almost automorphic if f is almost automorphic in $t\in\mathbb{R}$ for each $u\in\mathbb{X}$. That is to say, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+s_n-s_m,u) = f(t,u) \tag{9}$$

pointwise on $\mathbb{R}$ for each $u\in\mathbb{X}$. Denote by $AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ the collection of all such functions.

Lemma 12 (see [6, Theorem 2.2.6]). *Assume that $f\in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$ and there exists a constant $L_f>0$ such that for all $t\in\mathbb{R}$ and $u,v\in\mathbb{X}$,*

$$\|f(t,u)-f(t,v)\| \leq L_f\|u-v\|. \tag{10}$$

If $\phi(\cdot)\in AA(\mathbb{X})$, then $f(\cdot,\phi(\cdot))\in AA(\mathbb{X})$.

Corollary 13 (see [6, Corollary 2.1.6]). *Assume that $u\in AA(\mathbb{X})$ and $B\in B(\mathbb{X})$. If for each $t\in\mathbb{R}$, $v(t)=Bu(t)$, then $v\in AA(\mathbb{X})$.*

Definition 14 (see [26]). A function $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{X})$ is called bi-almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, one can extract a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t,s) = \lim_{n\to\infty} f(t+s_n, s+s_n) \tag{11}$$

is well defined for each $t,s\in\mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n, s-s_n) = f(t,s) \tag{12}$$

for each $t,s\in\mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

In other words, a function $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{X})$ is said to be bi-almost automorphic if for any sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} f(t+s_n-s_m, s+s_n-s_m) = f(t,s) \tag{13}$$

pointwise for each $t,s\in\mathbb{R}$.

3. Neutral Parabolic Nonautonomous Evolution Equation

In this section, we will establish the existence and uniqueness theorem of almost automorphic mild solutions to neutral parabolic nonautonomous evolution equation (1) under assumptions (H1)–(H5) listed below:

(H1) there exist constants $\lambda_0\geq 0$, $\theta\in(\pi/2,\pi)$, L_0, $K_0\geq 0$, and α, $\beta\in(0,1]$ with $\alpha+\beta>1$ such that

$$\begin{gathered} \Sigma_\theta\cup\{0\}\subset\rho(A(t)-\lambda_0), \\ \|R(\lambda,A(t)-\lambda_0)\|_{B(\mathbb{X})} \leq \frac{K_0}{1+|\lambda|}, \\ \|(A(t)-\lambda_0)R(\lambda,A(t)-\lambda_0) \\ \times[R(\lambda_0,A(t))-R(\lambda_0,A(s))]\|_{B(\mathbb{X})} \\ \leq L_0|t-s|^\alpha|\lambda|^{-\beta} \end{gathered} \tag{14}$$

for $t,s\in\mathbb{R}$, $\lambda\in\Sigma_\theta:=\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg\lambda|\leq\theta\}$, (H2) the evolution family $\{U(t,s)\}_{t\geq s}$ generated by $A(t)$ has an exponential dichotomy with dichotomy constants $M > 0$, $\delta > 0$, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function $\Gamma(t,s)$,

(H3) $\Gamma(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$ for each $x \in \mathbb{X}$,

(H4) $f \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and there exists a constant $L_f > 0$ such that for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$,

$$\left\|f(t,u) - f(t,v)\right\| \leq L_f \|u - v\|, \tag{15}$$

(H5) $g \in AA(\mathbb{R}\times\mathbb{X},\mathbb{X})$, and there exists a constant $L_g > 0$ such that for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$,

$$\left\|g(t,u) - g(t,v)\right\| \leq L_g \|u - v\|. \tag{16}$$

Remark 15. Assumption (H1) is usually called "Acquistapace-Terreni" conditions, which was first introduced in [27] for $\lambda_0 = 0$. If (H1) holds, then there exists a unique evolution family $\{U(t,s)\}_{t\geq s}$ on $\mathbb{X}$ such that $(t,s) \mapsto U(t,s) \in B(\mathbb{X})$ is strongly continuous for $t > s$, $U(\cdot,s) \in C^1((s,\infty),B(\mathbb{X}))$, $\partial_t U(t,s) = A(t)U(t,s)$ for $t > s$. These assertions are established in Theorem 2.3 of [28]. See also [27, 29, 30].

Definition 16. A mild solution to (1) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying integral equation

$$\begin{aligned} u(t) = &-f(t,u(t)) + U(t,s)\left[u(s) + f(s,u(s))\right] \\ &+ \int_s^t U(t,\sigma)\, g(\sigma,u(\sigma))\, d\sigma, \end{aligned} \tag{17}$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Lemma 17. *Assume that (H1)–(H3) and (H5) hold. Define nonlinear operator Λ on $AA(\mathbb{X})$ by*

$$\begin{aligned} (\Lambda u)(t) = &\int_{-\infty}^t U(t,s)P(s)\,g(s,u(s))\,ds \\ &- \int_t^{+\infty} U_Q(t,s)Q(s)\,g(s,u(s))\,ds, \quad t \in \mathbb{R}. \end{aligned} \tag{18}$$

Then Λ maps $AA(\mathbb{X})$ into itself.

Proof. Combining the ideas from Theorem 4.28 in [22], the technique of exponential dichotomy, composition theorem of almost automorphic functions, and the Lebesgue dominated convergence theorem, we strive for a more self-contained proof. Let $u \in AA(\mathbb{X})$. Then it follows from Lemma 12 [6, Theorem 2.2.6] that $h := g(\cdot,u(\cdot)) \in AA(\mathbb{X})$, in view of (H5). Hence, $(\Lambda u)(t)$ can be rewritten as

$$\begin{aligned} (\Lambda u)(t) = &\int_{-\infty}^t U(t,s)P(s)h(s)\,ds \\ &- \int_t^{+\infty} U_Q(t,s)Q(s)h(s)\,ds, \quad t \in \mathbb{R}. \end{aligned} \tag{19}$$

From triangle inequality and exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, it follows that

$$\begin{aligned} \|(\Lambda u)(t)\| &\leq M\int_{-\infty}^t e^{-\delta(t-s)}\|h(s)\|\,ds \\ &\quad + M\int_t^{+\infty} e^{-\delta(s-t)}\|h(s)\|\,ds \\ &\leq M\|h\|_\infty \int_{-\infty}^t e^{-\delta(t-s)} ds \\ &\quad + M\|h\|_\infty \int_t^{+\infty} e^{-\delta(s-t)} ds \\ &\leq \frac{2M}{\delta}\|h\|_\infty. \end{aligned} \tag{20}$$

Hence, Λ is well defined for each $t \in \mathbb{R}$. To show $\Lambda u \in C(\mathbb{R},\mathbb{X})$, we will verify that

$$\lim_{\Delta t\to 0}\|(\Lambda u)(t+\Delta t) - (\Lambda u)(t)\| = 0, \quad \text{for each } t \in \mathbb{R}. \tag{21}$$

By $h \in C(\mathbb{R},\mathbb{X})$ and the strong continuity of $\Gamma(t,s)$, we have for each $t \in \mathbb{R}$, $x \in \mathbb{X}$, $\sigma > 0$,

$$\begin{aligned} &\lim_{\Delta t\to 0}\|h(t+\Delta t) - h(t)\| = 0, \\ &\lim_{\Delta t\to 0}\|U(t+\Delta t, t+\Delta t-\sigma)P(t+\Delta t-\sigma)x \\ &\qquad - U(t,t-\sigma)P(t-\sigma)x\| = 0, \\ &\lim_{\Delta t\to 0}\left\|U_Q(t+\Delta t, t+\Delta t+\sigma)Q(t+\Delta t+\sigma)x \right. \\ &\qquad \left. - U_Q(t,t+\sigma)Q(t+\sigma)x\right\| = 0. \end{aligned} \tag{22}$$

Transforming (19) into another form, we have, for $\sigma > 0$,

$$\begin{aligned} (\Lambda u)(t) = &\int_0^{+\infty} U(t,t-\sigma)P(t-\sigma)h(t-\sigma)\,d\sigma \\ &- \int_0^{+\infty} U_Q(t,t+\sigma)Q(t+\sigma)h(t+\sigma)\,d\sigma, \\ &t \in \mathbb{R}. \end{aligned} \tag{23}$$

In view of (23), triangle inequality and exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, we have

$$\begin{aligned} &\|(\Lambda u)(t+\Delta t) - (\Lambda u)(t)\| \\ &\leq \left\|\int_0^{+\infty} U(t+\Delta t, t+\Delta t-\sigma)P(t+\Delta t-\sigma) \right. \\ &\qquad \times h(t+\Delta t-\sigma)\,d\sigma \\ &\qquad \left. - \int_0^{+\infty} U(t,t-\sigma)P(t-\sigma)h(t-\sigma)\,d\sigma\right\| \end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^{+\infty} U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma) \times h(t+\Delta t+\sigma)\, d\sigma - \int_0^{+\infty} U_Q(t, t+\sigma) Q(t+\sigma) h(t+\sigma)\, d\sigma\right\| \\
&\leq \left\|\int_0^{+\infty} U(t+\Delta t, t+\Delta t-\sigma) P(t+\Delta t-\sigma) \times [h(t+\Delta t-\sigma)-h(t-\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U(t+\Delta t, t+\Delta t-\sigma) P(t+\Delta t-\sigma) - U(t, t-\sigma) P(t-\sigma)]\, h(t-\sigma)\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma) \times [h(t+\Delta t+\sigma)-h(t+\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma) - U_Q(t, t+\sigma) Q(t+\sigma)]\, h(t+\sigma)\, d\sigma\right\| \\
&\leq M \int_0^{+\infty} e^{-\delta\sigma} \|h(t+\Delta t-\sigma)-h(t-\sigma)\|\, d\sigma \\
&+\int_0^{+\infty} \|U(t+\Delta t, t+\Delta t-\sigma) P(t+\Delta t-\sigma) h(t-\sigma) - U(t, t-\sigma) P(t-\sigma) h(t-\sigma)\|\, d\sigma \\
&+M \int_0^{+\infty} e^{-\delta\sigma} \|h(t+\Delta t+\sigma)-h(t+\sigma)\|\, d\sigma \\
&+\int_0^{+\infty} \|U_Q(t+\Delta t, t+\Delta t+\sigma) Q(t+\Delta t+\sigma) h(t+\sigma) - U_Q(t, t+\sigma) Q(t+\sigma) h(t+\sigma)\|\, d\sigma.
\end{aligned} \tag{24}
$$

Thus, in the Lebesgue dominated convergence theorem, (22) leads to (21) and therefore to $\Lambda u \in C(\mathbb{R}, \mathbb{X})$.

To show $\Lambda u \in AA(\mathbb{X})$, let us take a sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ and show that there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$
\lim_{m\to\infty}\lim_{n\to\infty} \|(\Lambda u)(t+s_n-s_m) - (\Lambda u)(t)\| = 0, \quad \text{for each } t \in \mathbb{R}. \tag{25}
$$

By $h \in AA(\mathbb{X})$ and (H3), there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that for each $t \in \mathbb{R}$, $x \in \mathbb{X}$, $\sigma > 0$,

$$
\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty} \|h(t+s_n-s_m) - h(t)\| = 0, \\
&\lim_{m\to\infty}\lim_{n\to\infty} \|U(t+s_n-s_m, t+s_n-s_m-\sigma) \times P(t+s_n-s_m-\sigma)\, x - U(t, t-\sigma) P(t-\sigma)\, x\| = 0, \\
&\lim_{m\to\infty}\lim_{n\to\infty} \|U_Q(t+s_n-s_m, t+s_n-s_m+\sigma) \times Q(t+s_n-s_m+\sigma)\, x - U_Q(t, t+\sigma) P(t+\sigma)\, x\| = 0.
\end{aligned} \tag{26}
$$

Again, in view of (23), triangle inequality and exponential dichotomy of $\{U(t,s)\}_{t\geq s}$, we obtain

$$
\begin{aligned}
&\|(\Lambda u)(t+s_n-s_m) - (\Lambda u)(t)\| \\
&\leq \left\|\int_0^{+\infty} U(t+s_n-s_m, t+s_n-s_m-\sigma) \times P(t+s_n-s_m-\sigma) h(t+s_n-s_m-\sigma)\, d\sigma - \int_0^{+\infty} U(t, t-\sigma) P(t-\sigma) h(t-\sigma)\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} U_Q(t+s_n-s_m, t+s_n-s_m+\sigma) \times Q(t+s_n-s_m+\sigma) h(t+s_n-s_m+\sigma)\, d\sigma - \int_0^{+\infty} U_Q(t, t+\sigma) Q(t+\sigma) h(t+\sigma)\, d\sigma\right\| \\
&\leq \left\|\int_0^{+\infty} U(t+s_n-s_m, t+s_n-s_m-\sigma) \times P(t+s_n-s_m-\sigma) \cdot [h(t+s_n-s_m-\sigma) - h(t-\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U(t+s_n-s_m, t+s_n-s_m-\sigma) \times P(t+s_n-s_m-\sigma) - U(t, t-\sigma) P(t-\sigma)]\, h(t-\sigma)\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} U_Q(t+s_n-s_m, t+s_n-s_m+\sigma) \times Q(t+s_n-s_m+\sigma) \cdot [h(t+s_n-s_m+\sigma) - h(t+\sigma)]\, d\sigma\right\| \\
&+\left\|\int_0^{+\infty} [U_Q(t+s_n-s_m, t+s_n-s_m+\sigma) \times Q(t+s_n-s_m+\sigma) - U_Q(t, t+\sigma) Q(t+\sigma)]\, h(t+\sigma)\, d\sigma\right\| \\
&\leq M \int_0^{+\infty} e^{-\delta\sigma} \|h(t+s_n-s_m-\sigma) - h(t-\sigma)\|\, d\sigma \\
&+\int_0^{+\infty} \|U(t+s_n-s_m, t+s_n-s_m-\sigma) \times P(t+s_n-s_m-\sigma) h(t-\sigma) - U(t, t-\sigma) P(t-\sigma) h(t-\sigma)\|\, d\sigma
\end{aligned}
$$

$$+ M \int_0^{+\infty} e^{-\delta\sigma} \left\| h\left(t + s_n - s_m + \sigma\right) - h\left(t + \sigma\right) \right\| d\sigma$$
$$+ \int_0^{+\infty} \left\| U_Q\left(t + s_n - s_m, t + s_n - s_m + \sigma\right) \times Q\left(t + s_n - s_m + \sigma\right) h\left(t + \sigma\right) - U_Q\left(t, t + \sigma\right) Q\left(t + \sigma\right) h\left(t + \sigma\right) \right\| d\sigma. \quad (27)$$

Thus, in the Lebesgue dominated convergence theorem, (26) leads to (25) and therefore to $\Lambda u \in AA(\mathbb{X})$. Here we used the translation invariance of almost automorphic functions, which is collected in Lemma 8(3). The proof is complete. □

Now we are in a position to state and prove the first main result of this paper.

Theorem 18. *Suppose that (H1)–(H5) hold. If* $\Theta = L_f + (2ML_g/\delta) < 1$*, then there exists a unique mild solution* $u \in AA(\mathbb{X})$ *to* (1) *such that*

$$u(t) = -f(t, u(t)) + \int_{-\infty}^{t} U(t, \sigma) P(\sigma) g(\sigma, u(\sigma)) d\sigma - \int_t^{+\infty} U_Q(t, \sigma) Q(\sigma) g(\sigma, u(\sigma)) d\sigma. \quad (28)$$

Proof. Firstly, define nonlinear operator Γ on $AA(\mathbb{X})$ by

$$(\Gamma u)(t) = -f(t, u(t)) + \int_{-\infty}^{t} U(t, \sigma) P(\sigma) g(\sigma, u(\sigma)) d\sigma - \int_t^{+\infty} U_Q(t, \sigma) Q(\sigma) g(\sigma, u(\sigma)) d\sigma. \quad (29)$$

Let $u \in AA(\mathbb{X})$, then it follows from Lemma 12 [6, Theorem 2.2.6] that $f(\cdot, u(\cdot)) \in AA(\mathbb{X})$, in view of (H4). Together with Lemma 17, we deduce that the operator Γ is well defined and maps $AA(\mathbb{X})$ into itself.

Secondly, we will prove that Γ is a strict contraction on $AA(\mathbb{X})$. Let $v, w \in AA(\mathbb{X})$. By (H2), (H4), and (H5), we have

$$\begin{aligned}
&\|(\Gamma v)(t) - (\Gamma w)(t)\| \\
&\le L_f \|v(t) - w(t)\| + M \int_{-\infty}^{t} e^{-\delta(t-s)} \|g(s, v(s)) - g(s, w(s))\| ds + M \int_t^{+\infty} e^{-\delta(s-t)} \|g(s, v(s)) - g(s, w(s))\| ds \\
&\le L_f \|v - w\|_\infty + ML_g \int_{-\infty}^{t} e^{-\delta(t-s)} \|v(s) - w(s)\| ds + ML_g \int_t^{+\infty} e^{-\delta(s-t)} \|v(s) - w(s)\| ds \\
&\le \left(L_f + \frac{2ML_g}{\delta}\right) \|v - w\|_\infty.
\end{aligned} \quad (30)$$

Hence,

$$\|\Gamma v - \Gamma w\|_\infty \le \left(L_f + \frac{2ML_g}{\delta}\right) \|v - w\|_\infty. \quad (31)$$

If $\Theta = L_f + (2ML_g/\delta) < 1$, then the operator Γ becomes a strict contraction on $AA(\mathbb{X})$. Since the space $AA(\mathbb{X})$ equipped with sup norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$ is a Banach space by Lemma 10, an application of Banach fixed point theorem shows that there exists a unique $u \in AA(\mathbb{X})$ such that (28) holds.

Finally, to prove that u satisfies (17) for all $t \ge s$, all $s \in \mathbb{R}$. For this, we let

$$u(s) = -f(s, u(s)) + \int_{-\infty}^{s} U(s, \sigma) P(\sigma) g(\sigma, u(\sigma)) d\sigma - \int_s^{+\infty} U_Q(s, \sigma) Q(\sigma) g(\sigma, u(\sigma)) d\sigma. \quad (32)$$

Multiplying both sides of (32) by $U(t, s)$ for all $t \ge s$, we have

$$\begin{aligned}
U(t, s) u(s) &= -U(t, s) f(s, u(s)) + \int_{-\infty}^{s} U(t, \sigma) P(\sigma) g(\sigma, u(\sigma)) d\sigma - \int_s^{+\infty} U_Q(t, \sigma) Q(\sigma) g(\sigma, u(\sigma)) d\sigma \\
&= -U(t, s) f(s, u(s)) + \int_{-\infty}^{t} U(t, \sigma) P(\sigma) g(\sigma, u(\sigma)) d\sigma - \int_s^{t} U(t, \sigma) P(\sigma) g(\sigma, u(\sigma)) d\sigma \\
&\quad - \int_t^{+\infty} U_Q(t, \sigma) Q(\sigma) g(\sigma, u(\sigma)) d\sigma - \int_s^{t} U_Q(t, \sigma) Q(\sigma) g(\sigma, u(\sigma)) d\sigma \\
&= -U(t, s) f(s, u(s)) + u(t) + f(t, u(t)) - \int_s^{t} U(t, \sigma) g(\sigma, u(\sigma)) d\sigma.
\end{aligned} \quad (33)$$

Hence, $u \in AA(\mathbb{X})$ is a unique mild solution to (1). The proof is complete. □

4. Bounded Perturbations

In this section, we consider neutral parabolic nonautonomous evolution equation (2). For this, we need assumptions (H1)–(H5) listed in the previous section and the following assumption:

(H6) $B, C \in B(\mathbb{X})$ with $\max\{\|B\|_{B(\mathbb{X})}, \|C\|_{B(\mathbb{X})}\} = K$.

Definition 19. A mild solution to (2) is a continuous function $u : \mathbb{R} \to \mathbb{X}$ satisfying integral equation

$$u(t) = -f(t, Bu(t)) + U(t,s)[u(s) + f(s, Bu(s))] + \int_s^t U(t,\sigma) g(\sigma, Cu(\sigma)) d\sigma \quad (34)$$

for all $t \geq s$ and all $s \in \mathbb{R}$.

Lemma 20. *Let assumptions (H1)–(H3), (H5), and (H6) hold. Define nonlinear operator Λ_1 on $AA(\mathbb{X})$ by*

$$(\Lambda_1 u)(t) = \int_{-\infty}^t U(t,s) P(s) g(s, Cu(s)) ds - \int_t^{+\infty} U_Q(t,s) Q(s) g(s, Cu(s)) ds, \quad t \in \mathbb{R}. \quad (35)$$

Then Λ_1 maps $AA(\mathbb{X})$ into itself.

Proof. Let $u(\cdot) \in AA(\mathbb{X})$. By (H6) and Corollary 13, we obtain $Cu(\cdot) \in AA(\mathbb{X})$. Then it follows from Lemma 12 [6, Theorem 2.2.6] that $h_1 := g(\cdot, Cu(\cdot)) \in AA(\mathbb{X})$, in view of (H5). The left is almost same as the proof of Lemma 17, remembering to replace Λ, h by Λ_1, and h_1, respectively. This ends the proof. □

Now we are in a position to state and prove the second main result of this paper.

Theorem 21. *Suppose that (H1)–(H6) hold. If $\Theta_1 = K(L_f + (2ML_g/\delta)) < 1$, then there exists a unique mild solution $u \in AA(\mathbb{X})$ to (2) such that*

$$u(t) = -f(t, Bu(t)) + \int_{-\infty}^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma - \int_t^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma. \quad (36)$$

Proof. Firstly, define nonlinear operator Γ_1 on $AA(\mathbb{X})$ by

$$(\Gamma_1 u)(t) = -f(t, Bu(t)) + \int_{-\infty}^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma - \int_t^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma. \quad (37)$$

Let $u \in AA(\mathbb{X})$. By (H6) and Corollary 13, we obtain $Bu \in AA(\mathbb{X})$. Hence, it follows from Lemma 12 [6, Theorem 2.2.6] that $f(\cdot, Bu(\cdot)) \in AA(\mathbb{X})$, in view of (H4). Together with Lemma 20, we deduce that the operator Γ_1 is well defined and maps $AA(\mathbb{X})$ into itself.

Secondly, we will prove that Γ_1 is a strict contraction on $AA(\mathbb{X})$ and apply Banach fixed point theorem. Let $v, w \in AA(\mathbb{X})$. Then it follows from (H2), (H4), (H5), and (H6) that

$$\begin{aligned}
&\|(\Gamma_1 v)(t) - (\Gamma_1 w)(t)\| \\
&\leq KL_f \|v(t) - w(t)\| \\
&\quad + M \int_{-\infty}^t e^{-\delta(t-s)} \|g(s, Cv(s)) - g(s, Cw(s))\| ds \\
&\quad + M \int_t^{+\infty} e^{-\delta(s-t)} \|g(s, Cv(s)) - g(s, Cw(s))\| ds \\
&\leq KL_f \|v - w\|_\infty + MKL_g \int_{-\infty}^t e^{-\delta(t-s)} \|v(s) - w(s)\| ds \\
&\quad + MKL_g \int_t^{+\infty} e^{-\delta(s-t)} \|v(s) - w(s)\| ds \\
&\leq K \left(L_f + \frac{2ML_g}{\delta} \right) \|v - w\|_\infty.
\end{aligned} \quad (38)$$

Hence,

$$\|\Gamma_1 v - \Gamma_1 w\|_\infty \leq K \left(L_f + \frac{2ML_g}{\delta} \right) \|v - w\|_\infty. \quad (39)$$

If $\Theta_1 = K(L_f + (2ML_g/\delta)) < 1$, then the operator Γ_1 becomes a strict contraction on $AA(\mathbb{X})$. Since the space $AA(\mathbb{X})$ equipped with sup norm $\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|$ is a Banach space by Lemma 10, an application of Banach fixed point theorem shows that there exists a unique $u \in AA(\mathbb{X})$ such that (36) holds.

Finally, to prove that u satisfies (34) for all $t \geq s$, all $s \in \mathbb{R}$. For this, we let

$$u(s) = -f(s, Bu(s)) + \int_{-\infty}^s U(s,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma - \int_s^{+\infty} U_Q(s,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma. \quad (40)$$

Multiplying both sides of (40) by $U(t,s)$ for all $t \geq s$, then

$$\begin{aligned}
U(t,s) u(s) &= -U(t,s) f(s, Bu(s)) \\
&\quad + \int_{-\infty}^s U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma \\
&\quad - \int_s^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma)) d\sigma \\
&= -U(t,s) f(s, Bu(s)) \\
&\quad + \int_{-\infty}^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma \\
&\quad - \int_s^t U(t,\sigma) P(\sigma) g(\sigma, Cu(\sigma)) d\sigma
\end{aligned}$$

$$\begin{aligned}
&- \int_t^{+\infty} U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma))\, d\sigma \\
&- \int_s^t U_Q(t,\sigma) Q(\sigma) g(\sigma, Cu(\sigma))\, d\sigma \\
&= -U(t,s) f(s, Bu(s)) + u(t) + f(t, Bu(t)) \\
&- \int_s^t U(t,\sigma) g(\sigma, Cu(\sigma))\, d\sigma.
\end{aligned} \tag{41}$$

Hence, $u \in AA(\mathbb{X})$ is a unique mild solution to (2). The proof is complete. □

5. Applications to Parabolic Partial Differential Equations and Neutral Functional Differential Equations

In this section, two examples are given to illustrate the effectiveness and flexibility of Theorem 21. By a mild solution to a partial or neutral functional differential equation, we mean a mild solution to the corresponding evolution equation.

Example 22. Consider the following parabolic partial differential equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,x) + f(t, q(x) u(t,x))\right] \\
&\quad = \frac{\partial^2}{\partial x^2}\left[u(t,x) + f(t, q(x) u(t,x))\right] \\
&\qquad + (-3 + \sin t + \sin \pi t)\left[u(t,x) + f(t, q(x) u(t,x))\right] \\
&\qquad + g(t, q(x) u(t,x)), \quad t \in \mathbb{R},\ x \in [0,\pi], \\
&\left[u(t,x) + f(t, q(x) u(t,x))\right]\big|_{x=0} \\
&\quad = \left[u(t,x) + f(t, q(x) u(t,x))\right]\big|_{x=\pi} = 0, \quad t \in \mathbb{R},
\end{aligned} \tag{42}$$

where q is continuous on $[0,\pi]$.

Let $\mathbb{X} := C[0,\pi]$ denote the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ equipped with the sup norm and define the operator A by

$$\begin{aligned}
&D(A) := \left\{\varphi \in C^2[0,\pi] : \varphi(0) = \varphi(\pi) = 0\right\}, \\
&A\varphi := \varphi'', \ \varphi \in D(A).
\end{aligned} \tag{43}$$

It is known (see [18, Example 14.4]) that A is sectorial, $D(A)$ is not dense in $C[0,\pi]$, and A is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ not strongly continuous at 0. Since the spectrum of A consists of the sequence of eigenvalues $\lambda_n = -n^2$, $n \in \mathbb{N}$, it can be easily checked that $\|T(t)\| \leq e^{-t}$ for $t \geq 0$, remembering that the spectral bound $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ of A coincides with its growth bound

$$\omega_A = \inf\left\{\gamma \in \mathbb{R} : \exists M > 0 \text{ s.t. } \|T(t)\| \leq M e^{\gamma t}, t \geq 0\right\}. \tag{44}$$

Define a family of linear operators $A(t)$, $t \in \mathbb{R}$ by

$$\begin{aligned}
&D(A(t)) = D(A(0)) = D(A), \\
&A(t)\varphi = (A - 3 + \sin t + \sin \pi t)\varphi, \quad \forall \varphi \in D(A).
\end{aligned} \tag{45}$$

In the case that $A(t) : D(A(t)) \subset \mathbb{X} \to \mathbb{X}$ have a constant domain $D(A(t)) \equiv D(A(0))$ and $\lambda_0 = 0$, it is known that [31, 32] assumption (H1) can be replaced by the following assumption (ST).

(ST) There exist constants $\theta \in (\pi/2, \pi)$, $L_0, K_0 \geq 0$, and $\alpha \in (0,1]$ such that

$$\begin{aligned}
&\Sigma_\theta \cup \{0\} \subset \rho(A(t)), \qquad \|R(\lambda, A(t))\|_{B(\mathbb{X})} \leq \frac{K_0}{1+|\lambda|}, \\
&\left\|(A(t) - A(s)) A(r)^{-1}\right\|_{B(\mathbb{X})} \leq L_0 |t-s|^\alpha
\end{aligned} \tag{46}$$

for $t, s, r \in \mathbb{R}, \lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \theta\}$.

Now, it is not hard to verify that $A(t)$ satisfy (H1). By Theorem 2.3 of [28], $A(t)$ generate an evolution family $\{U(t,s)\}_{t\geq s}$ that is strongly continuous for $t > s$. Furthermore,

$$U(t,s)\varphi = T(t-s) e^{\int_s^t (-3+\sin\tau + \sin \pi\tau) d\tau} \varphi. \tag{47}$$

Hence,

$$\|U(t,s)\| \leq e^{-2(t-s)} \quad \text{for } t \geq s, \tag{48}$$

and (H2) is satisfied with $M = 1, \delta = 2, P(s) = I$.

As for (H3), it is obvious that $U(t,s)\varphi \in C(\mathbb{R}\times\mathbb{R}, C[0,\pi])$ for each $\varphi \in C[0,\pi]$.

To show that $U(t,s)\varphi \in bAA(\mathbb{R}\times\mathbb{R}, C[0,\pi])$ for each $\varphi \in C[0,\pi]$, let us take a sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ and show that there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} U(t + s_n - s_m, s + s_n - s_m)\varphi = U(t,s)\varphi \tag{49}$$

pointwise for each $t, s \in \mathbb{R}$.

By $-3 + \sin\tau + \sin\pi\tau \in AA(\mathbb{R})$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that pointwise for each $\tau \in \mathbb{R}$,

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\left(-3 + \sin(\tau + s_n - s_m) + \sin\pi(\tau + s_n - s_m)\right) \\
&\quad = -3 + \sin\tau + \sin\pi\tau.
\end{aligned} \tag{50}$$

In view of (47) and (50), an application of the Lebesgue dominated convergence theorem shows that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty} U(t + s_n - s_m, s + s_n - s_m)\varphi \\
&\quad = \lim_{m\to\infty}\lim_{n\to\infty} T(t-s) e^{\int_{s+s_n-s_m}^{t+s_n-s_m}(-3+\sin\tau+\sin\pi\tau)d\tau}\varphi \\
&\quad = T(t-s) \\
&\qquad \times e^{\lim_{m\to\infty}\lim_{n\to\infty}\int_s^t(-3+\sin(\tau+s_n-s_m)+\sin\pi(\tau+s_n-s_m))d\tau}\varphi \\
&\quad = T(t-s) \\
&\qquad \times e^{\int_s^t \lim_{m\to\infty}\lim_{n\to\infty}(-3+\sin(\tau+s_n-s_m)+\sin\pi(\tau+s_n-s_m))d\tau}\varphi \\
&\quad = T(t-s) e^{\int_s^t(-3+\sin\tau+\sin\pi\tau)d\tau}\varphi = U(t,s)\varphi
\end{aligned} \tag{51}$$

pointwise for each $t, s \in \mathbb{R}$. Hence, (H3) is satisfied.

Define the operators $B,\ C$ by

$$D(B) = D(C) = C[0,\pi], \qquad B\varphi = C\varphi = q(\xi)\varphi, \quad \xi \in [0,\pi],\ \varphi \in C[0,\pi], \tag{52}$$

then $\|B\|_{B(C[0,\pi])} = \|C\|_{B(C[0,\pi])} = \|q\|_{\infty} := \max_{\xi\in[0,\pi]}\{q(\xi)\}$.

In view of the above, (42) can be transformed into the abstract form (2), and assumptions (H1)–(H3) and (H6) are satisfied.

We add the following assumptions:

(H4a) $f : \mathbb{R} \times C[0,\pi] \to C[0,\pi], (t,u) \mapsto f(t,u)$ is almost automorphic, and there exists a constant $L_f > 0$ such that for all $t \in \mathbb{R}$, $u(t,\cdot), v(t,\cdot) \in C[0,\pi]$,

$$\|f(t,u(t,\cdot)) - f(t,v(t,\cdot))\|_{C[0,\pi]} \le L_f \|u(t,\cdot) - v(t,\cdot)\|_{C[0,\pi]}, \tag{53}$$

(H5a) $g : \mathbb{R} \times C[0,\pi] \to C[0,\pi], (t,u) \mapsto g(t,u)$ is almost automorphic, and there exists a constant $L_g > 0$ such that for all $t \in \mathbb{R}$, $u(t,\cdot), v(t,\cdot) \in C[0,\pi]$,

$$\|g(t,u(t,\cdot)) - g(t,v(t,\cdot))\|_{C[0,\pi]} \le L_g \|u(t,\cdot) - v(t,\cdot)\|_{C[0,\pi]}. \tag{54}$$

Now, the following proposition is an immediate consequence of Theorem 21.

Proposition 23. *Under assumptions (H4a) and (H5a), parabolic partial differential equation* (42) *admits a unique almost automorphic mild solution if*

$$\|q\|_{\infty}\left(L_f + L_g\right) < 1. \tag{55}$$

Furthermore, if one takes

$$\begin{aligned} f(t,u) &= u \sin \frac{1}{-3 + \sin t + \sin \pi t}, \\ &\qquad t \in \mathbb{R},\ u \in C[0,\pi], \\ g(t,u) &= \frac{1}{8} u \left(1 + \sin \frac{1}{-3 + \sin t + \sin \pi t}\right), \\ &\qquad t \in \mathbb{R},\ u \in C[0,\pi]. \end{aligned} \tag{56}$$

A simple computation shows that $f, g \in AA(\mathbb{R} \times C[0,\pi], C[0,\pi])$, (H4a), and (H5a) are satisfied with $L_f = 1$, $L_g = 1/4$. By Proposition 23, (42) *admits a unique almost automorphic mild solution whenever $\|q\|_{\infty} < 4/5$.*

Example 24. Consider neutral functional differential equation

$$\begin{aligned} &\frac{\partial}{\partial t}\left[u(t,x) + f(t,u(t-\tau,x))\right] \\ &\quad = \frac{\partial^2}{\partial x^2}\left[u(t,x) + f(t,u(t-\tau,x))\right] \\ &\qquad + (-3 + \sin t + \sin \pi t)\left[u(t,x) + f(t,u(t-\tau,x))\right] \\ &\qquad + g(t,u(t-\tau,x)), \quad t \in \mathbb{R},\ x \in [0,\pi], \\ &\left[u(t,x) + f(t,u(t-\tau,x))\right]\big|_{x=0} \\ &\quad = \left[u(t,x) + f(t,u(t-\tau,x))\right]\big|_{x=\pi} = 0, \quad t \in \mathbb{R}, \end{aligned} \tag{57}$$

where $\tau \in \mathbb{R}$ is a fixed constant.

Take $\mathbb{X}$, A, $A(t)$, $t \in \mathbb{R}$, (H4a), and (H5a) as in Example 22. Define the operators $B,\ C$ by

$$D(B) = D(C) = C[0,\pi], \qquad B\varphi(\cdot) = C\varphi(\cdot) = \varphi(\cdot - \tau), \quad \varphi(\cdot) \in C[0,\pi], \tag{58}$$

then $\|B\|_{B(C[0,\pi])} = \|C\|_{B(C[0,\pi])} = 1$.

Now, (57) can be transformed into the abstract form (2) and assumptions (H1)–(H3) and (H6) are satisfied. Hence, Theorem 21 leads also to the following proposition.

Proposition 25. *Under assumptions (H4a) and (H5a), neutral functional differential equation* (57) *admits a unique almost automorphic mild solution if*

$$L_f + L_g < 1. \tag{59}$$

Furthermore, if one takes

$$\begin{aligned} f(t,u) &= \frac{1}{2} u \sin \frac{1}{-3 + \sin t + \sin \pi t}, \\ &\qquad t \in \mathbb{R},\ u \in C[0,\pi], \\ g(t,u) &= \frac{1}{8} u \left(1 + \sin \frac{1}{-3 + \sin t + \sin \pi t}\right), \\ &\qquad t \in \mathbb{R},\ u \in C[0,\pi]. \end{aligned} \tag{60}$$

A simple computation shows that $f, g \in AA(\mathbb{R} \times C[0,\pi], C[0,\pi])$, (H4a), and (H5a) are satisfied with $L_f = 1/2$, $L_g = 1/4$. By Proposition 25, (57) *admits a unique almost automorphic mild solution.*

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