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Research Article

The Solutions of Mixed Monotone Fredholm-Type Integral Equations in Banach Spaces

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By introducing new definitions of ϕ convex and $-\varphi$ concave quasioperator and v_0 quasilower and u_0 quasiupper, by means of the monotone iterative techniques without any compactness conditions, we obtain the iterative unique solution of nonlinear mixed monotone Fredholm-type integral equations in Banach spaces. Our results are even new to ϕ convex and $-\varphi$ concave quasi operator, and then we apply these results to the two-point boundary value problem of second-order nonlinear ordinary differential equations in the ordered Banach spaces.

1. Introduction

In this paper, we will consider the following nonlinear Fredholm integral equation:

$$u(t) = \int_I H(t, s, u(s))\, ds, \quad t \in I, \tag{1}$$

where $I = [a, b]$ and $H \in C[I \times I \times E, E]$, E is a real Banach space with the norm $\|\cdot\|$, and there exists a function $G \in C[I \times I \times E \times E, E]$ such that for any $(t, s, x) \in I \times I \times E$

$$H(t, s, x) = G(t, s, x, x). \tag{2}$$

Guo and Lakshmikantham [1] introduced the definition of mixed monotone operator and coupled fixed point; there are many good results (see [1–13]). In the special case where $H(t, s, x)$ is nondecreasing in x for fixed $t, s \in I$, Guo [2] established an existence theorem of the maximal and minimal solutions for (1) in the ordered Banach spaces by means of monotone iterative techniques. Recently, Jingxian and Lishan [3] and Lishan [4] obtained iterative sequences that converge uniformly to solutions and coupled minimal and maximal quasisolutions of the nonlinear Fredholm integral equations in ordered Banach spaces by using the Möuch fixed point theorem and establishing new comparison results. But these all required the compactness conditions and the monotone conditions in the above papers, and furthermore they did not obtain the unique solutions. In addition, extensive studies have also been carried out to study the global or iterative solutions of initial value problems [8–13].

In this paper, by introducing new definitions of ϕ convex and $-\varphi$ concave quasioperator and v_0 quasilower and u_0 quasiupper, by means of the monotone iterative techniques without any compactness conditions which are of the essence in [2–4, 7, 8, 14], we obtain the iterative unique solution of nonlinear mixed monotone Fredholm-type integral equations in Banach spaces and then apply these results to the two-point boundary value problem of second-order nonlinear ordinary differential equations.

2. Preliminaries and Definitions

Let P be a cone in E, that is, a closed convex subset such that $\lambda P \subset P$ for any $\lambda \geq 0$ and $P \cap \{-P\} = \{\theta\}$. By means of P, a partial order $\leq$ is defined as $x \leq y$ if and only if $y - x \in P$. A cone P is said to be normal if there exists a constant $N > 0$ such that $x, y \in E$, $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$, where θ denotes the zero element of E (see [2, 14]), and we call the smallest number N the normal constant of P and denote N_P. The cone P is normal if and only if every ordered interval $[x, y] = \{z \in E : x \leq z \leq y\}$ is bounded.

Let $P_I = \{u \in C[I, E] : u(t) \geq \theta \text{ for all } t \in I\}$, where $C[I, E]$ denotes the Banach space of all the continuous mapping $u : I \longrightarrow E$ with the norm $\| u\|_C = \max_{t\in I}|u(t)|$. It is clear that P_I is a cone of space $C[I,E]$, and so it defines a partial ordering in $C[I,E]$. Obviously, the normality of P implies the normality of P_I and the normal constants of P_I, and P are the same.

Let $u_0, v_0 \in C[I,E]$. Then, u_0, v_0 are said to be coupled lower and upper quasi-solutions of (1) if

$$u_0(t) \le \int_I G(t,s,u_0(s),v_0(s))\,ds, \quad t \in I,$$
$$v_0(t) \ge \int_I G(t,s,v_0(s),u_0(s))\,ds, \quad t \in I. \quad (3)$$

If the equality in (3) holds, then u_0, v_0 are said to be coupled quasi-solutions of (1).

We will always assume in this paper that P is a normal cone of E. For any $u_0, v_0 \in C[I,E]$ such that $v_0 \le w_0$, we define the ordered interval $D = [u_0, v_0] = \{u \in C[I,E] : u_0 \le u \le v_0\}$.

Next, we will give the new definition of ϕ convex and $-\varphi$ concave quasi operator and v_0 quasi-lower and u_0 quasi-upper.

Definition 1. Suppose that, $G \in C[I \times I \times E \times E, E]$. Then G is called ϕ convex and $-\varphi$ concave quasi operator, if there exist functions

$$\phi : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty),$$
$$\varphi : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty), \quad (4)$$

such that

(1) $G(t,s,\alpha u,\beta v) \ge \phi(\alpha,\beta)G(t,s,u,v)$, $\alpha < \beta$, $\alpha, \beta \in (0,\infty)$, for all $u, v \in E$,

(2) $G(t,s,\alpha u,\beta v) \le \varphi(\alpha,\beta)G(t,s,u,v)$, $\alpha \ge \beta$, $\alpha, \beta \in (0,\infty)$, for all $u, v \in E$.

Definition 2. Suppose that $G \in C[I \times I \times E \times E, E]$, $u_0 \in P$. Then, G is called u_0 quasi-upper, if for any $u, v \in E$, $u, v < u_0$ such that $\int_I G(t,s,u,v)ds < u_0$.

Definition 3. Suppose that $G \in C[I \times I \times E \times E, E]$, $v_0 \in E$. Then, G is called v_0 quasi-lower, if for any $u, v \in E$, $u, v > v_0$ such that $\int_I G(t,s,u,v)ds > v_0$.

Let us list the following assumption for convenience.

(H_1) G is uniformly continuous on $I \times I \times E \times E$, and G is ϕ convex and $-\varphi$ concave quasi operator.

(H_2) $G(t,s,x,y)$ is nondecreasing in $x \in E$ for fixed $(t,s,y) \in I \times I \times E$. $G(t,s,x,y)$ is nonincreasing in $y \in E$ for fixed $(t,s,x) \in I \times I \times E$.

(H_3) $\phi(\alpha,\beta)$, $\varphi(\alpha,\beta)$ are all increasing in α, decreasing in β, and $\phi(\alpha_0,\beta_0) \ge \alpha_0$, $\varphi(\beta_0,\alpha_0) \le \beta_0$ and for $\alpha, \beta \in [\alpha_0, \beta_0]$, $\alpha < \beta$,

$$\varphi(\beta,\alpha) - \phi(\alpha,\beta) \le l(\beta - \alpha), \quad 0 < l < 1. \quad (5)$$

3. The Main Result

The main results of this paper are the following three theorems.

Theorem 4. *Let P be a normal cone of E, let $u_0, v_0 \in P_I$ be coupled lower and upper quasi-solutions of* (1). *Assume that conditions (H_1), (H_2), and (H_3) hold and*

(H_4) *There exists $w_0 \in P_I$ such that $u_0 \le w_0 \le v_0$, and for $\alpha_0, \beta_0 \in (0,\infty)$ of (H_3) such that $u_0 \ge \alpha_0 w_0$, $\beta_0 w_0 \ge v_0$.*

Then, (1) *has a unique solution $x^*(t) \in D = [u_0, v_0]$, and for any initial $x_0, y_0 \in [u_0, v_0]$, one has*

$$x_n(t) \longrightarrow x^*(t), \quad y_n(t) \longrightarrow x^*(t),$$
$$\text{uniformly on } t \in I \quad \text{as } n \longrightarrow \infty, \quad (6)$$

where $\{x_n(t)\}$, $\{y_n(t)\}$ are defined as

$$x_n(t) = \int_I G(t,s,x_{n-1}(s),y_{n-1}(s))\,ds,$$
$$y_n(t) = \int_I G(t,s,y_{n-1}(s),x_{n-1}(s))\,ds, \quad t \in I. \quad (7)$$

Proof. We first define the operator $A : [u_0, v_0] \times [u_0, v_0] \longrightarrow C[I,E]$ by the formula

$$A(u,v) = \int_I G(t,s,u(s),v(s))\,ds. \quad (8)$$

It follows from the assumption (H_2) that A is a mixed monotone operator, that is, $A(u,v)$ is nondecreasing in $u \in [u_0, v_0]$ and nonincreasing in $v \in [u_0, v_0]$, and $u_0 \le A(u_0, v_0)$, $A(v_0, u_0) \le v_0$.

By (7), we have $x_n(t) = A(x_{n-1}(t), y_{n-1}(t))$, $y_n(t) = A(y_{n-1}(t), x_{n-1}(t))$ and set $w_n(t) = A(w_{n-1}(t), w_{n-1}(t))$ for initial w_0 in (H_4), and we also define that

$$u_n(t) = A(u_{n-1}(t), v_{n-1}(t)),$$
$$v_n(t) = A(v_{n-1}(t), u_{n-1}(t)). \quad (9)$$

Since A is a mixed monotone operator, it is easy to see that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0,$$
$$u_n \le w_n \le v_n. \quad (10)$$

Obviously, by induction, it is easy to see that

$$u_n \ge \alpha_n w_n, \quad v_n \le \beta_n w_n, \quad n = 0, 1, \ldots, \quad (11)$$

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \cdots \le 1 \le \cdots \le \beta_n \le \cdots \le \beta_1 \le \beta_0, \quad (12)$$

where $\alpha_n = \phi(\alpha_{n-1}, \beta_{n-1})$, $\beta_n = \varphi(\beta_{n-1}, \alpha_{n-1})$, $n = 1, 2, \ldots$.

In fact, by the assumption (H_4), we have that inequality (11) holds as $n = 0$. Suppose that inequality (11) holds as $n = k$, that is, $u_k \geq \alpha_k w_k, v_k \leq \beta_k w_k$. Then, as $n = k+1$, by the assumption (H_3), we have

$$\begin{aligned} u_{k+1} &= A(u_k, v_k) = \int_I G(t, s, u_k(s), v_k(s))\, ds \\ &\geq \int_I G(t, s, \alpha_k w_k, \beta_k w_k)\, ds \\ &\geq \phi(\alpha_k, \beta_k) \int_I G(t, s, w_k(s), w_k(s))\, ds = \alpha_{k+1} w_{k+1}, \\ v_{k+1} &= A(v_k, u_k) = \int_I G(t, s, v_k(s), u_k(s))\, ds \\ &\leq \int_I G(t, s, \beta_k w_k, \alpha_k w_k)\, ds \\ &\leq \varphi(\beta_k, \alpha_k) \int_I G(t, s, w_k(s), w_k(s))\, ds = \beta_{k+1} w_{k+1}. \end{aligned} \tag{13}$$

Then, it is easy to show by induction that inequality (11) holds.

For inequality (12), by $u_{k+1} \leq w_{k+1} \leq v_{k+1}$ and the above discussion, we have $0 < \alpha_{k+1} \leq 1 \leq \beta_{k+1}$. Obviously, it follows from the assumption (H_4) that $\alpha_0 \leq \alpha_1, \beta_1 \leq \beta_0$. Suppose that $\alpha_{k-1} \leq \alpha_k, \beta_k \leq \beta_{k-1}$, so it is easy to show by (H_3) that

$$\begin{aligned} \phi(\alpha_{k-1}, \beta_{k-1}) &\leq \phi(\alpha_k, \beta_k), \\ \varphi(\beta_k, \alpha_k) &\leq \varphi(\beta_{k-1}, \alpha_{k-1}), \end{aligned} \tag{14}$$

that is, $\alpha_k \leq \alpha_{k+1}, \beta_{k+1} \leq \beta_k$. Then, it is easy to show by induction that inequality (12) holds.

Then, it follows from the inequality (12) that there exist limits of the sequences $\{\alpha_n\}, \{\beta_n\}$. Suppose that there exist α, β such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, and $n \to \infty$, and by (H_3), we also have

$$\begin{aligned} 0 \leq \beta_n - \alpha_n &= \varphi(\beta_{n-1}, \alpha_{n-1}) - \phi(\alpha_{n-1}, \beta_{n-1}) \\ &\leq l(\beta_{n-1} - \alpha_{n-1}) \leq \cdots \leq l^n(\beta_0 - \alpha_0), \end{aligned} \tag{15}$$

they $0 < l < 1$, and taking limits in the above inequality as $n \to \infty$, we have $\alpha = \beta$.

Next, we will show that the sequences $\{u_n\}, \{v_n\}$ are all Cauchy sequences on D.

In fact, by (10) and (11), for any natural number p, we know that

$$\begin{aligned} \theta \leq u_{n+p} - u_n \leq v_n - u_n \leq (\beta_n - \alpha_n) u_0, \\ \theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (\beta_n - \alpha_n) u_0. \end{aligned} \tag{16}$$

By the normality of P_I and (15), we have

$$\begin{aligned} \|u_{n+p} - u_n\|_C \leq N_P l^n (\beta_0 - \alpha_0) \|u_0\|_C, \\ \|v_n - v_{n+p}\|_C \leq N_P l^n (\beta_0 - \alpha_0) \|u_0\|_C, \end{aligned} \tag{17}$$

where N_P is a normal constant. So $\{u_n\}$, $\{v_n\}$ are all Cauchy sequences on D, and then there exists $u^*, v^* \in [u_0, v_0]$ such that $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$.

It is easy to know by (10) and (11) that

$$\theta \leq v_n - u_n \leq \beta_n w_n - \alpha_n w_n \leq (\beta_n - \alpha_n) u_0 \leq l^n (\beta_0 - \alpha_0) u_0, \tag{18}$$

so by the normality of P_I, we have

$$\|v_n - u_n\|_C \leq N_P l^n (\beta_0 - \alpha_0) \|u_0\|_C, \tag{19}$$

and taking limits in the above inequality as $n \to \infty$, we have $x^* = u^* = v^* \in [u_0, v_0]$, and for any natural number n, we also have $u_n \leq x^* \leq v_n$, $t \in I$.

Then, by the mixed monotone quality of A we have

$$u_{n+1} = A(u_n, v_n) \leq A(x^*, x^*) \leq A(v_n, u_n) = v_{n+1}, \tag{20}$$

and taking limits in above inequality as $n \to \infty$, we know that

$$x^* = A(x^*, x^*), \tag{21}$$

that is, $x^* \in [u_0, v_0]$ is the fixed point of A; thus, x^* is the solution of (1) on $D = [u_0, v_0]$.

Furthermore, we will show that the solution is unique. Suppose that $y^* \in [u_0, v_0]$ satisfy $y^* = A(y^*, y^*)$. Then, by the mixed monotone quality of A and induction, for any natural number n, it is easy to have that $u_n \leq y^* \leq v_n$. Then, by the normality of P_I and taking limits in the above inequality as $n \to \infty$ and the above discussion, we have $y^* = x^*$.

For any initial $x_0, y_0 \in [u_0, v_0]$, by (7) and (8), the mixed monotone quality of A and induction, for any natural number n, we have $u_n(t) \leq x_n(t) \leq v_n(t), u_n(t) \leq y_n(t) \leq v_n(t)$, $t \in I$. Then, the normality of P_I and (19) imply that

$$\begin{aligned} \|x_n - u_n\|_C \leq N_P l^n (\beta_0 - \alpha_0) \|u_0\|_C, \\ \|y_n - u_n\|_C \leq N_P N_P l^n (\beta_0 - \alpha_0) \|u_0\|_C. \end{aligned} \tag{22}$$

Thus, the sequence $\{x_n(t)\}$, $\{y_n(t)\}$ all converges uniformly to $x^*(t)$ on $t \in I$. This completes the proof of Theorem 4. □

Theorem 5. *Let P be a normal cone of E, let $u_0, v_0 \in P_I$ be coupled lower and upper quasi-solutions of* (1). *Assume that conditions $(H_1), (H_2)$, and (H_3) hold.*

- (H_4') *G is u_0 quasi-upper, and there exists $w_0 \in P_I$ such that $w_0 < u_0 < v_0$, and there exist $\alpha_0 = \sup\{\alpha > 0 : u_0 \geq \alpha w_0\}$, $\beta_0 = \inf\{\beta > 0 : v_0 \leq \beta w_0\}$.*

Then, (1) *has a unique solution $x^*(t) \in D = [u_0, v_0]$, and for any initial $x_0, y_0 \in [u_0, v_0]$, one has*

$$\begin{aligned} x_n(t) \longrightarrow x^*(t), \quad y_n(t) \longrightarrow x^*(t), \\ \text{uniformly on } t \in I \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{23}$$

where $\{x_n(t)\}$, $\{y_n(t)\}$ are defined as

$$\begin{aligned} x_n(t) &= \int_I G(t, s, x_{n-1}(s), y_{n-1}(s))\, ds, \\ y_n(t) &= \int_I G(t, s, y_{n-1}(s), x_{n-1}(s))\, ds, \quad t \in I. \end{aligned} \tag{24}$$

Proof. We first define the operator $A : [u_0, v_0] \times [u_0, v_0] \to C[I, E]$ by the formula

$$A(u, v) = \int_I G(t, s, u(s), v(s))\, ds. \qquad (8')$$

It follows from the assumption (H_2) that A is a mixed monotone operator, that is, $A(u, v)$ is nondecreasing in $u \in [u_0, v_0]$ and nonincreasing in $v \in [u_0, v_0]$ and $u_0 \leq A(u_0, v_0)$, $A(v_0, u_0) \leq v_0$. By (7), we have $x_n(t) = A(x_{n-1}(t), y_{n-1}(t))$, $y_n(t) = A(y_{n-1}(t), x_{n-1}(t))$ and set $w_n(t) = A(w_{n-1}(t), w_{n-1}(t))$, and we also define

$$\begin{aligned} u_n(t) &= A(u_{n-1}(t), v_{n-1}(t)), \\ v_n(t) &= A(v_{n-1}(t), u_{n-1}(t)). \end{aligned} \qquad (25)$$

Since A is a mixed monotone operator, it is easy to see that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \qquad (10')$$

Because G is u_0 quasi-upper and $w_0 < u_0$, we have

$$\begin{aligned} w_1(t) &= A(w_0(t), w_0(t)) \\ &= \int_I G(t, s, w_0(s), w_0(s))\, ds < u_0. \end{aligned} \qquad (26)$$

So for any natural number n, by induction, we know that $w_n(t) = A(w_{n-1}(t), w_{n-1}(t)) < u_0$.

It is easy to see by induction that

$$u_k \geq \alpha_k w_k, \qquad v_k \leq \beta_k w_k, \qquad (11')$$

$$a_0 \leq a_1 \leq \cdots \leq a_k \leq \cdots \leq b_k \leq \cdots \leq b_1 \leq b_0, \qquad (12')$$

where $\alpha_k = \phi(\alpha_{k-1}, \beta_{k-1})$, $\beta_k = \varphi(\beta_{k-1}, \alpha_{k-1})$, $k = 1, 2, \ldots$.

In fact, by the assumptions (H_1) and (H_3) and the above discussion, as $n = 0$, we have

$$\begin{aligned} u_1 &= A(u_0, v_0) = \int_I G(t, s, u_0(s), v_0(s))\, ds \\ &\geq \int_I G(t, s, \alpha_0 w_0, \beta_0 w_0)\, ds \\ &\geq \phi(\alpha_0, \beta_0) \int_I G(t, s, w_0(s), w_0(s))\, ds = \alpha_1 w_1, \end{aligned} \qquad (27)$$

$$\begin{aligned} v_1 &= A(v_0, u_0) = \int_I G(t, s, v_0(s), u_0(s))\, ds \\ &\leq \int_I G(t, s, \beta_0 w_0, \alpha_0 w_0)\, ds \\ &\leq \varphi(\beta_0, \alpha_0) \int_I G(t, s, w_0(s), w_0(s))\, ds = \beta_1 w_1. \end{aligned} \qquad (28)$$

By the above two inequalities and assumption (H_3), we have

$$a_0 \leq \alpha_1 = \phi(\alpha_0, \beta_0) \leq \varphi(\beta_0, \alpha_0) = \beta_1 \leq b_0. \qquad (29)$$

Suppose that for $k - 1$ we have $u_{k-1} \geq \alpha_{k-1} w_{k-1}$, $v_{k-1} \leq \beta_{k-1} w_{k-1}$, and $\alpha_{k-2} \leq \alpha_{k-1} \leq \beta_{k-1} \leq \beta_{k-2}$. Then, for $k + 1$, by the assumption (H_3), we have

$$\begin{aligned} u_k &= A(u_{k-1}, v_{k-1}) = \int_I G(t, s, u_{k-1}(s), v_{k-1}(s))\, ds \\ &\geq \int_I G(t, s, \alpha_{k-1} w_{k-1}, \beta_{k-1} w_{k-1})\, ds \\ &\geq \phi(\alpha_{k-1}, \beta_{k-1}) \int_I G(t, s, w_{k-1}(s), w_{k-1}(s))\, ds = \alpha_k w_k, \\ v_k &= A(v_{k-1}, u_{k-1}) = \int_I G(t, s, v_{k-1}(s), u_{k-1}(s))\, ds \\ &\leq \int_I G(t, s, \beta_{k-1} w_{k-1}, \alpha_{k-1} w_{k-1})\, ds \\ &\leq \varphi(\beta_{k-1}, \alpha_{k-1}) \int_I G(t, s, w_{k-1}(s), w_{k-1}(s))\, ds = \beta_k w_k. \end{aligned} \qquad (30)$$

By the above two inequalities and assumption (H_3), we have

$$\begin{aligned} \alpha_{k-1} &= \phi(\alpha_{k-2}, \beta_{k-2}) \leq \phi(\alpha_{k-1}, \beta_{k-1}) = \alpha_k \leq \beta_k \\ &= \varphi(\beta_{k-1}, \alpha_{k-1}) \leq \varphi(\beta_{k-2}, \alpha_{k-2}) = \beta_{k-1}. \end{aligned} \qquad (31)$$

Then, it is easy to show by induction that inequalities $(11')$ and $(12')$ hold.

The following proof is similar to that of Theorem 4. This completes the proof of Theorem 5. □

By a similar argument to that of Theorem 5, we obtain the following results.

Theorem 6. *Let P be a normal cone of E, and let $u_0, v_0 \in P_I$ be coupled lower and upper quasi-solutions of* (1). *Assume that condition* (H_1), (H_2), *and* (H_3) *hold.*

(H_4'') *G is v_0 quasi-lower, and there exists $w_0 \in P_I$ such that $u_0 < v_0 < w_0$, and there exist $\alpha_0 = \sup\{\alpha > 0 : u_0 \geq \alpha w_0\}$, $\beta_0 = \inf\{\beta > 0 : v_0 \leq \beta w_0\}$.*

Then, (1) *has a unique solution $x^*(t) \in D = [u_0, v_0]$, and for any initial $x_0, y_0 \in [u_0, v_0]$, one has*

$$\begin{gathered} x_n(t) \longrightarrow x^*(t), \quad y_n(t) \longrightarrow x^*(t), \\ \text{uniformly on } t \in I \quad \text{as } n \longrightarrow \infty, \end{gathered} \qquad (32)$$

where $\{x_n(t)\}$, $\{y_n(t)\}$ are defined as

$$\begin{aligned} x_n(t) &= \int_I G(t, s, x_{n-1}(s), y_{n-1}(s))\, ds, \\ y_n(t) &= \int_I G(t, s, y_{n-1}(s), x_{n-1}(s))\, ds, \quad t \in I. \end{aligned} \qquad (33)$$

4. Applications

Consider the following two-point BVP in the Banach space:

$$-u'' = f(t,u), \quad t \in J = [0,1], \qquad u(0) = u(1) = 0, \tag{34}$$

where $f \in C[J \times P, P]$, P is a cone in a real Banach space E. Suppose that there exists a mapping $g \in C[J \times P \times P, P]$ such that $f(t,x) = g(t,x,x)$, and that g satisfies the following conditions:

(C_1) g is uniformly continuous on $J \times P \times P$, and G is ϕ convex and $-\varphi$ concave quasi operator,

(C_2) $g(t,x,y)$ is nondecreasing in $x \in P$ for fixed $(t,y) \in J \times P$, and $g(t,x,y)$ is nonincreasing in $y \in P$, for fixed $(t,x) \in J \times P$,

(C_3) there exist the bounded nonnegative Lebesgue integrable functions $a(t)$, $b(t)$, $c(t)$, and $d(t)$ satisfying $\int_J a(s)ds < 8$, $\int_J c(s)ds < 8$ such that

$$a(t)x + b(t) \le g(t,x,y) \le c(t)x + d(t), \quad t \in J, \ x, y \in P. \tag{35}$$

It is well known that $u \in C^2[J,P]$ is a solution of BVP(34) in $C^2[J,P]$ if and only if $u \in C[J,P]$ is a solution of the following integral equation:

$$u(t) = \int_J h(t,s)\, g(s, u(s), u(s))\, ds, \quad t \in J, \tag{36}$$

where

$$h(t,s) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & t > s. \end{cases} \tag{37}$$

Lemma 7. *If assumption (C_3) holds, then there exists $u_0, v_0 \in C[J,P]$ such that*

$$\begin{aligned} u_0(t) &\le \int_J h(t,s)\, g(s, u_0(s), v_0(s))\, ds, \quad t \in J, \\ v_0(t) &\ge \int_J h(t,s)\, g(s, v_0(s), u_0(s))\, ds, \quad t \in J. \end{aligned} \tag{38}$$

Proof. In fact, let

$$\begin{aligned} L_1 u(t) &= \int_J h(t,s)\, a(s)\, u(s)\, ds, \\ x_0(t) &= \int_J h(t,s)\, b(s)\, ds, \quad t \in J, \\ L_2 v(t) &= \int_J h(t,s)\, c(s)\, v(s)\, ds, \\ y_0(t) &= \int_J h(t,s)\, d(s)\, ds, \quad t \in J. \end{aligned} \tag{39}$$

Obviously, by assumption (C_3), we can get that $\| L_1 \| \le \max_{t\in I}(t(1-t)/2) \int_J a(s)ds = (1/8)\int_J a(s)ds < 1$, then the equation $(I - L_1)u = x_0$ has a unique solution

$$u_0(t) = (I - L_1)^{-1} x_0 = \sum_{n=0}^{\infty} L_1^n x_0 \in P_I. \tag{40}$$

Similarly, the equation $(I - L_2)v = y_0$ has a unique solution

$$v_0(t) = (I - L_2)^{-1} y_0 = \sum_{n=0}^{\infty} L_2^n y_0 \in P_I. \tag{41}$$

Thus, by assumption (C_3), for any $t \in J$, we have

$$\begin{aligned} &\int_J h(t,s)\, g(s, u_0(s), v_0(s))\, ds \\ &\quad \ge \int_J h(t,s)\,(a(s)x + b(s))\, ds \\ &\quad = L_1 u_0(t) + x_0(t) = u_0(t), \\ &\int_J h(t,s)\, g(s, v_0(s), u_0(s))\, ds \\ &\quad \le \int_J h(t,s)\,(c(s) v_0(s) + b(s))\, ds \\ &\quad = L_2 v_0(t) + y_0(t) = v_0(t), \end{aligned} \tag{42}$$

that is, (38) holds. □

Theorem 8. *Let P be a normal cone of E. Assume that (C_1) and (C_3) hold,*

(C_4) *there exists $w_0 \in P_I$ and u_0, v_0 in (38) of Lemma 7 such that $u_0 < w_0 < v_0$, and also there exists $\alpha_0, \beta_0 \in (0, \infty)$ such that $u_0 \ge \alpha_0 w_0$, $\beta_0 w_0 \ge v_0$,*

(C_5) *$\phi(\alpha,\beta)$, $\varphi(\alpha,\beta)$ are all increasing in α, decreasing in β and $\phi(\alpha_0,\beta_0) \ge \alpha_0$, $\varphi(\beta_0,\alpha_0) \le \beta_0$, for $\alpha, \beta \in [\alpha_0, \beta_0]$, $\alpha < \beta$,*

$$\varphi(\beta,\alpha) - \phi(\alpha,\beta) \le l(\beta - \alpha), \quad 0 < l < 1. \tag{43}$$

Then, (34) has a unique solution $x^*(t) \in D = [u_0, v_0]$, and for any initial $x_0, y_0 \in [u_0, v_0]$, one has

$$x_n(t) \longrightarrow x^*(t), \quad y_n(t) \longrightarrow x^*(t), \quad \text{uniformly on } t \in I \quad \text{as } n \longrightarrow \infty, \tag{44}$$

where $\{x_n(t)\}$, $\{y_n(t)\}$ are defined as

$$\begin{aligned} x_n(t) &= \int_J h(t,s)\, g(s, x_{n-1}(s), y_{n-1}(s))\, ds, \\ y_n(t) &= \int_J h(t,s)\, g(s, y_{n-1}(s), x_{n-1}(s))\, ds, \quad t \in J. \end{aligned} \tag{45}$$

Proof. It is easy to see by conditions (C_1) and (C_2) that $G(t,s,x,y) = h(t,s)g(s,x,y)$ satisfy the conditions (H_1) and (H_2) of Theorem 4. By (C_3) and (38), we have $u_0, v_0 \in C[I,P]$ as coupled lower and upper quasi-solutions of (34).

Thus, the assumption (H_1)–(H_4) of Theorem 4 is satisfied from the assumption (C_1)–(C_5) of Theorem 8. The conclusion of Theorem 8 follows from Theorem 4. □

Example 9. In fact, we can construct the function $f(t,x)$ in Theorem 8.

Let

$$f(t,x) = g(t,x,y) = x + \frac{1}{y}, \qquad t \in [0,1],$$
$$\phi(\alpha,\beta) = \sin\alpha + \frac{1}{2\beta}, \qquad \alpha \in \left[0, \frac{\pi}{2}\right], \qquad \varphi(\alpha,\beta) = 3\alpha - 5\beta, \tag{46}$$

then

$$\begin{aligned} G(t,s,\alpha x,\beta y) &= h(t,s)\, g(s,\alpha x,\beta y) \\ &= h(t,s)\left(\alpha x + \frac{1}{\beta y}\right) \\ &\geq h(t,s)\left(\sin\alpha + \frac{1}{2\beta}\right)\left(x + \frac{1}{y}\right) \\ &= \phi(\alpha,\beta)\, G(t,s,x,y), \end{aligned} \tag{47}$$

$$\begin{aligned} G(t,s,\alpha x,\beta y) &= h(t,s)\, g(s,\alpha x,\beta y) \\ &= h(t,s)\left(\alpha x + \frac{1}{\beta y}\right) \\ &\leq h(t,s)(3\alpha - 5\beta)\left(x + \frac{1}{y}\right) \\ &= \varphi(\alpha,\beta)\, G(t,s,x,y). \end{aligned} \tag{48}$$

Thus, G is ϕ convex and $-\varphi$ concave quasi operator and thus satisfies (C_1).

It is easy to check that $g(t,x,y)$ is nondecreasing in x for fixed (t,y) and is nonincreasing in y for fixed (t,x) and thus satisfies (C_2).

There exist $a(t) = t/2$, $b(t) = t/100$, $c(t) = 2t$, and $d(t) = 1000t$ satisfying

$$\int_0^1 a(s)\, ds = \frac{1}{2}\int_0^1 t\, dt = \frac{1}{4} < 8, \qquad \int_0^1 c(s)\, ds = 2\int_0^1 s\, ds = 1 < 8, \tag{49}$$

such that

$$a(t)x + b(t) \leq g(t,x,y) \leq c(t)x + d(t). \tag{50}$$

Thus, (C_3) holds.

There exist

$$u_0 = \int_0^1 h(t,s)\left[u_0(s) + \frac{1}{v_0(s)}\right] ds, \qquad v_0 = 2u_0 = \int_0^1 h(t,s)\left[v_0(s) + \frac{1}{u_0(s)}\right] ds. \tag{51}$$

Choose $w_0 = (3/2)u_0$ such that $u_0 < w_0 < v_0$, and also there exist $\alpha_0 = 2/3$, $\beta_0 = 4/3$ such that

$$u_0 = \alpha_0 \frac{3}{2} u_0 = \alpha_0 w_0, \qquad \beta_0 \frac{3}{2} u_0 = 2u_0 = v_0. \tag{52}$$

Thus, (C_4) is satisfied.

$\phi(\alpha,\beta)$, $\varphi(\alpha,\beta)$ are all increasing in α and nondecreasing in β,

$$\phi\left(\frac{2}{3},\frac{4}{3}\right) = \sin\frac{2}{3} + \frac{1}{2\times(4/3)} \geq \frac{2}{3} = \alpha_0, \qquad \varphi\left(\frac{4}{3},\frac{2}{3}\right) = 3\times\frac{4}{3} - 5\times\frac{2}{3} = \frac{2}{3} \leq \frac{4}{3} = \beta_0, \tag{53}$$

and for $\alpha, \beta \in [2/3, 4/3]$, $\alpha < \beta$, we have

$$\varphi(\beta,\alpha) - \phi(\alpha,\beta) = 3\beta - 5\alpha - \sin\alpha - \frac{1}{2\beta} \leq \frac{99}{100}(\beta - \alpha). \tag{54}$$

Thus, (C_5) also holds.

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