

CONVERGENCE THEOREMS FOR A COMMON FIXED POINT OF A FINITE FAMILY OF NONSELF NONEXPANSIVE MAPPINGS

C. E. CHIDUME, HABTU ZEGEYE, AND NASEER SHAHZAD

Received 10 September 2003 and in revised form 6 July 2004

Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \rightarrow E$, $i = 1, \dots, r$, be a family of nonexpansive mappings which are weakly inward. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. A strong convergence theorem is proved for a common fixed point of a family of nonexpansive mappings provided that T_i , $i = 1, 2, \dots, r$, satisfy some mild conditions.

1. Introduction

Let K be a nonempty closed convex subset of a real Banach space E . A mapping $T : K \rightarrow E$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Let $T : K \rightarrow K$ be a nonexpansive self-mapping. For a sequence $\{\alpha_n\}$ of real numbers in $(0, 1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})Tx_n, \quad n \geq 0. \quad (1.1)$$

Halpern [5] was the first to study the convergence of the algorithm (1.1) in the framework of Hilbert spaces. Lions [6] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (iii) $\lim_{n \rightarrow \infty} ((\alpha_n - \alpha_{n-1})/\alpha_n^2) = 0$.

It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [12] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (iii)* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Reich [9] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps. Moreover, the sequence $\{\alpha_n\}$ is required to satisfy conditions (i) and (ii) and to be decreasing (and hence also satisfying (iii)*). Subsequently, Shioji and Takahashi [10] extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex subset of K has the fixed point property for nonexpansive mappings and $\{\alpha_n\}$ satisfies conditions (i), (ii), and (iii)*.

Xu [13] showed that the results of Halpern holds in *uniformly smooth Banach spaces* if $\{\alpha_n\}$ satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (iii)** $\lim_{n \rightarrow \infty} ((\alpha_n - \alpha_{n-1})/\alpha_n) = 0$.

As has been remarked in [13], conditions (iii) and (iii)* are not comparable. Also conditions (iii)* and (iii)** are not comparable. However, condition (iii) does not permit the natural choice $\alpha_n := (n + 1)^{-1}$ for all integers $n \geq 0$. Hence, conditions (iii)* and (iii)** are preferred.

In [2], Chidume et al. extended the results of Xu to Banach spaces which are more general than uniformly smooth spaces.

Next consider r nonexpansive mappings T_1, T_2, \dots, T_r . For a sequence $\{\alpha_n\} \subseteq (0, 1)$ and an arbitrary $u_0 \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \tag{1.2}$$

where $T_n = T_{n(\text{mod } r)}$.

In 1996, Bauschke [1] defined and studied the iterative process (1.2) in Hilbert spaces with conditions in (i), (ii), and (iii)* on the parameter $\{\alpha_n\}$.

Recently, Takahashi et al. [11] extended Bauschke's result to uniformly convex Banach spaces. More precisely, they proved the following result.

THEOREM 1.1 [11]. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_i : K \rightarrow K, i = 1, \dots, r$, be a family of nonexpansive mappings with $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \tag{1.3}$$

where $T_n := T_{n(\text{mod } r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (iii)* $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_r\}$. Further, if $Px_0 = \lim_{n \rightarrow \infty} x_n$ for each $x_0 \in K$, then P is a sunny nonexpansive retraction of K onto F .

More recently, O'Hara et al. [8] proved the following complementary result to Bauschke's theorem [1] with condition (iii)* replaced with (iii)** $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n) / \alpha_{n+r}) = 0$ (or equivalently, $\lim_{n \rightarrow \infty} (\alpha_n / \alpha_{n+r}) = 1$).

THEOREM 1.2 [8]. *Let K be a nonempty closed convex subset of a Hilbert space H and let $T_i : K \rightarrow K, i = 1, \dots, r$, be a family of nonexpansive mappings with $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \tag{1.4}$$

where $T_n := T_{n(\text{mod } r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (iii)** $\lim_{n \rightarrow \infty} (\alpha_n / \alpha_{n+r}) = 1$. Then $\{x_n\}$ converges strongly to Pu , where P is the projection of K onto F .

In the above work, the mappings T_1, T_2, \dots, T_r remain self-mappings of a nonempty closed convex subset K either of a Hilbert space or a uniformly convex space. If, however, the domain of $T_1, T_2, \dots, T_r, D(T_i) = K, i = 1, 2, \dots, r$, is a proper subset of E and T_i maps K into E , then the iteration process (1.4) may fail to be well defined (see also (1.3)).

It is our purpose in this paper to define an algorithm for nonself-mappings and to obtain a strong convergence theorem to a fixed point of a family of nonself nonexpansive mappings in Banach spaces more general than the spaces considered by Takahashi et al. [11] with $\{\alpha_n\}$ satisfying conditions (i), (ii), and (iii)*. We also show that our result holds if $\{\alpha_n\}$ satisfies conditions (i), (ii), and (iii)**. Our results extend and improve the corresponding results of O'Hara et al. [8], Takahashi et al. [11], and hence Bauschke [1] to more general Banach spaces and to the class of *nonself*-maps.

2. Preliminaries

Let E be a real Banach space with dual E^* . We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued. In the sequel, we will denote the single-valued normalized duality map by j .

The norm is said to be *uniformly Gâteaux differentiable* if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$, $\lim_{t \rightarrow 0} ((\|x + ty\| - \|x\|)/t)$ exists uniformly for $x \in S_1(0)$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see, e.g., [4]). Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality map is norm-to- w^* uniformly continuous on bounded subsets of E .

A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space E , we have that if $\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$, for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Let K be a nonempty subset of a Banach space E . For $x \in K$, the *inward set* of $x, I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \rightarrow E$ is called *weakly inward* if $Tx \in \text{cl}[I_K(x)]$ for all $x \in K$, where $\text{cl}[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $K \subseteq E$ be closed convex and Q a mapping of E onto K . Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, range of Q . A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K . If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H .

In the sequel, we will make use of the following lemma.

LEMMA 2.1. Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the relation

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \quad n \geq 0, \tag{2.2}$$

where (i) $0 < \alpha_n < 1$; (ii) $\sum_{n=1}^\infty \alpha_n = \infty$. Suppose, either (a) $\sigma_n = o(\alpha_n)$, or (b) $\sum_{n=1}^\infty \sigma_n < \infty$, or (c) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$ (see, e.g., [13]).

We will also need the following results.

LEMMA 2.2 (see, e.g., [7]). Let E be a real Banach space. Then the following inequality holds. For each $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall j(x + y) \in J(x + y). \tag{2.3}$$

THEOREM 2.3 [7, Theorem 1, Proposition 2(v)]. Let K be a nonempty closed convex subset of a reflexive Banach space E which has a uniformly Gâteaux differentiable norm. Let $T : K \rightarrow E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \rightarrow z_t, 0 < t < 1$, satisfying $z_t = tu + (1 - t)Tz_t$, for arbitrary but fixed $u \in K$, which converges strongly to a fixed point of T . Further, if $Pu = \lim_{t \rightarrow 0} z_t$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto $F(T)$.

3. Main results

We now prove the following theorem.

THEOREM 3.1. Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \rightarrow E, i = 1, \dots, r$, be a family of nonexpansive mappings which are weakly inward with $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^r F(QT_i) = F(QT_r QT_{r-1} \cdots QT_1) = F(QT_1 QT_r \cdots QT_2) = \cdots = F(QT_{r-1} QT_{r-2} \cdots QT_1 QT_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})QT_{n+1}x_n, \quad n \geq 0, \tag{3.1}$$

where $T_n := T_{n(\text{mod } r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^\infty \alpha_n = \infty$; and either (iii)* $\sum_{n=1}^\infty |\alpha_{n+r} - \alpha_n| < \infty$, or (iii)** $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point

of $\{T_1, T_2, \dots, T_r\}$. Further, if $Pu = \lim_{n \rightarrow \infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F .

Proof. For $x^* \in F$, one easily shows by induction that $\|x_n - x^*\| \leq \max\{\|x_0 - x^*\|, \|u - x^*\|\}$, for all integers $n \geq 0$, and hence $\{x_n\}$ and $\{QT_{n+1}x_n\}$ are bounded. But this implies that $\|x_{n+1} - QT_{n+1}x_n\| = \alpha_{n+1}\|u - QT_{n+1}x_n\| \rightarrow 0$ as $n \rightarrow \infty$. Now we show that

$$\|x_{n+r} - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.2}$$

From (3.1), we get that

$$\begin{aligned} \|x_{n+r} - x_n\| &= \|(\alpha_{n+r} - \alpha_n)(u - QT_n x_{n-1}) + (1 - \alpha_{n+r})(QT_{n+r} x_{n+r-1} - QT_n x_{n-1})\| \\ &= \|(\alpha_{n+r} - \alpha_n)(u - QT_n x_{n-1}) + (1 - \alpha_{n+r})(QT_n x_{n+r-1} - QT_n x_{n-1})\| \\ &\leq (1 - \alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + |\alpha_{n+r} - \alpha_n| M, \end{aligned} \tag{3.3}$$

for some $M > 0$. We consider two cases.

Case 1. Condition (iii)* is satisfied. Then,

$$\|x_{n+r} - x_n\| \leq (1 - \alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + \sigma_n, \tag{3.4}$$

where $\sigma_n := M|\alpha_{n+r} - \alpha_n|$ so that $\sum_{n=1}^\infty \sigma_n < \infty$.

Case 2. Condition (iii)** is satisfied. Then,

$$\|x_{n+r} - x_n\| \leq (1 - \alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + \sigma_n, \tag{3.5}$$

where $\sigma_n := \alpha_{n+r}\beta_n$ and $\beta_n := (|\alpha_{n+r} - \alpha_n|M/\alpha_{n+r})$ so that $\sigma_n = o(\alpha_{n+r})$.

In either case, by Lemma 2.1, we conclude that $\lim_{n \rightarrow \infty} \|x_{n+r} - x_n\| = 0$. Next we prove that

$$\lim_{n \rightarrow \infty} \|x_n - QT_{n+r} \cdots QT_{n+1} x_n\| = 0. \tag{3.6}$$

In view of (3.2), it suffices to show that $\lim_{n \rightarrow \infty} \|x_{n+r} - QT_{n+r} \cdots QT_{n+1} x_n\| = 0$. Since $\|x_{n+r} - QT_{n+r} x_{n+r-1}\| = \alpha_{n+r}\|u - QT_{n+r} x_{n+r-1}\|$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have that $x_{n+r} - QT_{n+r} x_{n+r-1} \rightarrow 0$. From

$$\begin{aligned} \|x_{n+r} - QT_{n+r} QT_{n+r-1} x_{n+r-2}\| &\leq \|x_{n+r} - QT_{n+r} x_{n+r-1}\| \\ &\quad + \|QT_{n+r} x_{n+r-1} - QT_{n+r} QT_{n+r-1} x_{n+r-2}\| \\ &\leq \|x_{n+r} - QT_{n+r} x_{n+r-1}\| + \|x_{n+r-1} - QT_{n+r-1} x_{n+r-2}\| \\ &= \|x_{n+r} - QT_{n+r} x_{n+r-1}\| + \alpha_{n+r-1} \|u - QT_{n+r-1} x_{n+r-2}\|, \end{aligned} \tag{3.7}$$

we also have $x_{n+r} - QT_{n+r} QT_{n+r-1} x_{n+r-2} \rightarrow 0$. Similarly, we obtain the conclusion. Let $z_t^n \in K$ be a continuous path satisfying

$$z_t^n = tu + (1 - t)QT_{n+r} QT_{n+r-1} \cdots QT_{n+1} z_t^n \tag{3.8}$$

guaranteed by Theorem 2.3. Also by Theorem 2.3, $z_t^n \rightarrow Pu$ as $t \rightarrow 0^+$, where P is the sunny nonexpansive retraction of K onto $\bigcap_{i=1}^r F(QT_i)$ (notice $\bigcap_{i=1}^r F(QT_i) = F(QT_{n+r}QT_{n+r-1}\dots QT_{n+1})$) and hence as $T_i, i = 1, \dots, r$, is weakly inward by [2, Remark 2.1], $Pu \in F = \bigcap_{i=1}^r F(T_i)$. Let $a = \limsup_{n \rightarrow \infty} \langle u - Pu, j(x_n - Pu) \rangle$. Now we show that $a \leq 0$. We can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $a = \lim_{i \rightarrow \infty} \langle u - Pu, j(x_{n_i} - Pu) \rangle$. We assume that $n_i \equiv k \pmod r$ for some $k \in \{1, 2, \dots, r\}$. Using Lemma 2.2, we have that

$$\begin{aligned} \|z_t^k - x_{n_i}\|^2 &= \|t(u - x_{n_i}) + (1 - t)(QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}z_t^k - x_{n_i})\|^2 \\ &\leq (1 - t)^2 \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}z_t^k - x_{n_i}\|^2 + 2t \langle u - x_{n_i}, j(z_t^k - x_{n_i}) \rangle \\ &\leq (1 - t)^2 (\|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}z_t^k - QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i}\| \\ &\quad + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|)^2 \\ &\quad + 2t (\|z_t^k - x_{n_i}\|^2 + \langle u - z_t^k, j(z_t^k - x_{n_i}) \rangle) \\ &\leq (1 + t^2) \|z_t - x_{n_i}\|^2 + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\| \\ &\quad \times (2\|z_t^k - x_{n_i}\| + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|) \\ &\quad + 2t \langle u - z_t^k, j(z_t^k - x_{n_i}) \rangle, \end{aligned} \tag{3.9}$$

and hence,

$$\begin{aligned} \langle u - z_t^k, j(x_{n_i} - z_t^k) \rangle &\leq \frac{t}{2} \|z_t^k - x_{n_i}\|^2 + \frac{\|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|}{2t} \\ &\quad \times (2\|z_t^k - x_{n_i}\| + \|QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i} - x_{n_i}\|). \end{aligned} \tag{3.10}$$

Since $\{x_{n_i}\}$ is bounded, we have that $\{QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i}\}$ is bounded and by (3.6), $\|x_{n_i} - QT_{n_i+r}QT_{n_i+r-1}\dots QT_{n_i+1}x_{n_i}\| \rightarrow 0$ as $i \rightarrow \infty$, then it follows from the last inequality that

$$\limsup_{t \rightarrow 0^+} \limsup_{i \rightarrow \infty} \langle u - z_t^k, j(x_{n_i} - z_t^k) \rangle \leq 0. \tag{3.11}$$

Moreover, j is norm-to- w^* uniformly continuous on bounded subsets of E . Thus, we obtain from (3.11) that

$$\limsup_{i \rightarrow \infty} \langle u - Pu, j(x_{n_i} - Pu) \rangle \leq 0, \tag{3.12}$$

and hence $\limsup_{n \rightarrow \infty} \langle u - Pu, j(x_n - Pu) \rangle \leq 0$. Furthermore, from (3.1), we have $x_{n+1} - Pu = \alpha_{n+1}(u - Pu) + (1 - \alpha_{n+1})(QT_{n+1}x_n - Pu)$. Thus using Lemma 2.2, we obtain that

$$\begin{aligned} \|x_{n+1} - Pu\|^2 &\leq (1 - \alpha_{n+1})^2 \|QT_{n+1}x_n - Pu\|^2 + 2\alpha_{n+1} \langle u - Pu, j(x_{n+1} - Pu) \rangle \\ &\leq (1 - \alpha_{n+1}) \|x_n - Pu\|^2 + \sigma_{n+1}, \end{aligned} \tag{3.13}$$

where $\sigma_{n+1} := \alpha_{n+1}\beta_{n+1}$ and $\limsup_{n \rightarrow \infty} \sigma_{n+1} \leq 0$, for $\beta_{n+1} := \langle u - Pu, j(x_{n+1} - Pu) \rangle$. Thus, by Lemma 2.1, $\{x_n\}$ converges strongly to a common fixed point Pu of $\{T_1, T_2, \dots, T_r\}$. The proof is complete. \square

If in Theorem 3.1, $T_i, i = 1, \dots, r$, are self-mappings then the projection operator Q is replaced with I , the identity map on E . Moreover, each T_i for $i \in \{1, 2, \dots, r\}$ is weakly inward. Thus, we have the following corollary.

COROLLARY 3.2. *Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \rightarrow K, i = 1, \dots, r$, be a family of nonexpansive mappings with $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^r F(T_i) = F(T_r T_{r-1} \dots T_1) = F(T_1 T_r \dots T_2) = \dots = F(T_{r-1} T_{r-2} \dots T_1 T_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T_{n+1}x_n, \quad n \geq 0, \tag{3.14}$$

where $T_n := T_{n(\text{mod } r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^\infty \alpha_n = \infty$; and either (iii)* $\sum_{n=1}^\infty |\alpha_{n+r} - \alpha_n| < \infty$, or (iii)** $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_r\}$. Further, if $Pu = \lim_{n \rightarrow \infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F .

In the sequel, we will use the following lemma.

LEMMA 3.3. *Let K be a nonempty closed convex subset of a strictly convex real Banach space E . Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \rightarrow E, i = 1, \dots, r$, be a family of nonexpansive mappings which are weakly inward with $\bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let $S_i : K \rightarrow E, i = 1, \dots, r$, be a family of mappings defined by $S_i := (1 - \lambda_i)I + \lambda_i T_i, 0 < \lambda_i < 1$ for each $i = 1, 2, \dots, r$. Then $\bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r F(S_i) = \bigcap_{i=1}^r F(QS_i)$ and $\bigcap_{i=1}^r F(S_i) = F(QS_r QS_{r-1} \dots QS_1) = F(QS_1 QS_r \dots QS_2) = \dots = F(QS_{r-1} QS_{r-2} \dots QS_1 QS_r)$.*

Proof. We note that, since T_i for each $i \in \{1, 2, \dots, r\}$ is weakly inward, then by [3, Remark 3.3], S_i is weakly inward. Moreover, by [2, Remark 2.1], $F(QS_i) = F(S_i)$. Furthermore, one easily shows that $F(S_i) = F(T_i)$ for each $i = 1, 2, \dots, r$. Now we show that $\bigcap_{i=1}^r F(S_i) = F(QS_r QS_{r-1} \dots QS_1) = F(QS_1 QS_r \dots QS_2) = \dots = F(QS_{r-1} QS_{r-2} \dots QS_1 QS_r)$. For simplicity, we prove for $r = 2$. It is clear that $F(S_1) \cap F(S_2) \subseteq F(QS_2 QS_1)$. Now, we show that $F(QS_2 QS_1) \subseteq F(S_1) \cap F(S_2)$. Let $z \in F(QS_2 QS_1)$ and $w \in F(S_1) \cap F(S_2) = F(T_1) \cap F(T_2)$. Then,

$$\begin{aligned} \|z - w\| &= \|QS_2 QS_1 z - w\| \\ &\leq \|(1 - \lambda_2)Q[(1 - \lambda_1)z + \lambda_1 T_1 z] + \lambda_2 T_2(Q[(1 - \lambda_1)z + \lambda_1 T_1 z]) - w\| \\ &\leq (1 - \lambda_2)\|(1 - \lambda_1)z + \lambda_1 T_1 z - w\| + \lambda_2\|(1 - \lambda_1)z + \lambda_1 T_1 z - w\| \\ &= \|(1 - \lambda_1)(z - w) + \lambda_1(T_1 z - w)\| \\ &\leq (1 - \lambda_1)\|z - w\| + \lambda_1\|T_1 z - w\| \leq \|z - w\|. \end{aligned} \tag{3.15}$$

Thus from the preceding inequalities and strict convexity of E , we obtain that $z - w = T_1 z - w$ and $T_2(Q[(1 - \lambda_1)z + \lambda_1 T_1 z]) - w = z - w$. Therefore, we obtain that $z = T_1 z = T_2 z$. This completes the proof. \square

THEOREM 3.4. *Let K be a nonempty closed convex subset of a strictly convex reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \rightarrow E, i = 1, \dots, r$, be a family of nonexpansive mappings which are weakly inward with $\bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let $S_i : K \rightarrow E, i = 1, \dots, r$, be a family of mappings defined by $S_i := (1 - \lambda_i)I + \lambda_i T_i, 0 < \lambda_i < 1$ for each $i = 1, 2, \dots, r$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})QS_{n+1}x_n, \quad n \geq 0, \tag{3.16}$$

where $S_n := S_{n(\text{mod } r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; and either (iii)* $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$, or (iii)** $\lim_{n \rightarrow \infty} ((\alpha_{n+r} - \alpha_n)/\alpha_{n+1}) = 0$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_r\}$. Further, if $Pu = \lim_{n \rightarrow \infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F .

Proof. By Lemma 3.3, $\bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r F(S_i) = \bigcap_{i=1}^r F(QS_i)$ and $\bigcap_{i=1}^r F(QS_i) = F(QS_r QS_{r-1} \cdots QS_1) = F(QS_1 QS_r \cdots QS_2) = \cdots = F(QS_{r-1} QS_{r-2} \cdots QS_1 QS_r)$. Thus, as in the proof of Theorem 3.1, $x_n \rightarrow x^* \in \bigcap_{i=1}^r F(T_i)$. The proof is complete. \square

If in Theorem 3.4, $T_i, i = 1, \dots, r$, are self-mappings, the following corollary follows.

COROLLARY 3.5. *Let K be a nonempty closed convex subset of a strictly convex reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \rightarrow K, i = 1, \dots, r$, be a family of nonexpansive mappings with $\bigcap_{i=1}^r F(T_i) \neq \emptyset$. Let $S_i : K \rightarrow K, i = 1, \dots, r$, be a family of mappings defined by $S_i := (1 - \lambda_i)I + \lambda_i T_i, 0 < \lambda_i < 1$ for each $i = 1, 2, \dots, r$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})S_{n+1}x_n, \quad n \geq 0, \tag{3.17}$$

where $S_n := S_{n(\text{mod } r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; and either (iii)* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, or (iii)** $\lim_{n \rightarrow \infty} ((\alpha_{n+1} - \alpha_n)/\alpha_{n+r}) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_r\}$. Further, if $Pu = \lim_{n \rightarrow \infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F .

Remark 3.6. Corollaries 3.2 and 3.5 are improvements of Theorems 1.1 and 1.2 to more general Banach spaces (having a uniformly Gâteaux differentiable norm) than uniformly convex spaces. Moreover, If E is a Hilbert space, Corollary 3.2 reduces to the result of Bauschke [1].

Acknowledgments.

This work was done while the authors Habtu Zegeye and Naseer Shahzad were visiting the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, the first as

a Postdoctoral Fellow and the second as a Junior Associate. They would like to thank the Centre for hospitality and financial support. The authors also thank the referee for valuable remarks.

References

- [1] H. H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl. **202** (1996), no. 1, 150–159.
- [2] C. E. Chidume, N. Shahzad, and H. Zegeye, *Strong convergence theorem for nonexpansive mappings in arbitrary Banach spaces*, submitted to Nonlinear Anal.
- [3] C. E. Chidume and H. Zegeye, *Strong convergence theorems for a finite family of nonexpansive mappings in Banach spaces*, Comm. Appl. Nonlinear Anal., **11** (2004), no. 2, 25–32.
- [4] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Mathematics and Its Applications, vol. 62, Kluwer Academic, Dordrecht, 1990.
- [5] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
- [6] P.-L. Lions, *Approximation de points fixes de contractions*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 21, A1357–A1359 (French).
- [7] C. H. Morales and J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3411–3419.
- [8] J. G. O'Hara, P. Pillay, and H.-K. Xu, *Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces*, Nonlinear Anal. **54** (2003), no. 8, 1417–1426.
- [9] S. Reich, *Approximating fixed points of nonexpansive mappings*, Panamer. Math. J. **4** (1994), no. 2, 23–28.
- [10] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 12, 3641–3645.
- [11] W. Takahashi, T. Tamura, and M. Toyoda, *Approximation of common fixed points of a family of finite nonexpansive mappings in Banach spaces*, Sci. Math. Jpn. **56** (2002), no. 3, 475–480.
- [12] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. (Basel) **58** (1992), no. 5, 486–491.
- [13] H.-K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. (2) **66** (2002), no. 1, 240–256.

C. E. Chidume: Mathematics Section, The Abdus Salam International Centre for Theoretical Physics, 34014 Trieste, Italy

E-mail address: chidume@ictp.trieste.it

Habtu Zegeye: Mathematics Section, The Abdus Salam International Centre for Theoretical Physics, 34014 Trieste, Italy

E-mail address: habtuzh@yahoo.com

Naseer Shahzad: Department of Mathematics, Faculty of Sciences, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

E-mail address: nshahzad@kau.edu.sa