QUASICONTRACTION NONSELF-MAPPINGS ON CONVEX METRIC SPACES AND COMMON FIXED POINT THEOREMS

LJILJANA GAJIĆ AND VLADIMIR RAKOČEVIĆ

Received 29 September 2004 and in revised form 24 January 2005

We consider quasicontraction nonself-mappings on Takahashi convex metric spaces and common fixed point theorems for a pair of maps. Results generalizing and unifying fixed point theorems of Ivanov, Jungck, Das and Naik, and Ćirić are established.

1. Introduction and preliminaries

Let X be a complete metric space. A map $T : X \mapsto X$ such that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$

$$d(Tx,Ty) \le \lambda \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1.1}$$

is called *quasicontraction*. Let us remark that Ćirić [1] introduced and studied quasicontraction as one of the most general contractive type map. The well known Ćirić's result (see, e.g., [1, 6, 11]) is that quasicontraction T possesses a unique fixed point.

For the convenience of the reader we recall the following recent Ćirić's result.

Theorem 1.1 [2, Theorem 2.1]. *Let X be a Banach space, C a nonempty closed subset of X, and ∂C the boundary of C. Let $T : C \mapsto X$ be a nonself mapping such that for some constant $\lambda \in (0,1)$ and for every $x, y \in C$*

$$d(Tx,Ty) \le \lambda \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{1.2}$$

Suppose that

$$T(\partial C) \subset C. \tag{1.3}$$

Then T has a unique fixed point in C.

Following Ćirić [3], let us remark that *problem to extend the known fixed point theorem for self mappings $T : C \mapsto C$, defined by (1.1), to corresponding nonself mappings $T : C \mapsto X$, $C \ne X$, was open more than* 20 *years*.

In 1970, Takahashi [15] introduced the definition of convexity in metric space and generalized same important fixed point theorems previously proved for Banach spaces. In

Fixed Point Theory and Applications 2005:3 (2005) 365–375
DOI: 10.1155/FPTA.2005.365

this paper we consider quasicontraction nonself-mappings on Takahashi convex metric spaces and common fixed point theorems for a pair of maps. Results generalizing and unifying fixed point theorems of Ivanov [7], Jungck [8], Das and Naik [3], Ćirić [2], Gajić [5] and Rakočević [12] are established.

Let us recall that (see Jungck [9]) the self maps f and g on a metric space (X,d) are said to be a *compatible pair* if

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0 \tag{1.4}$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = x \tag{1.5}$$

for some x in X.

Following Sessa [14] we will say that $f,g : X \mapsto X$ are *weakly commuting* if

$$d(fgx,gfx) \le d(fx,gx) \quad \text{for every } x \in X. \tag{1.6}$$

Clearly weak commutativity of f and g is a generalization of the conventional commutativity of f and g, and the concept of compatibility of two mappings includes weakly commuting mappings as a proper subclass.

We recall the following definition of a convex metric space (see [15]).

Definition 1.2. Let X be a metric space and $I = [0,1]$ the closed unit interval. A Takahashi convex structure on X is a function $W : X \times X \times I \mapsto X$ which has the property that for every $x, y \in X$ and $\lambda \in I$

$$d(z, W(x, y,\lambda)) \le \lambda d(z,x) + (1-\lambda)d(z, y) \tag{1.7}$$

for every $z \in X$. If (X,d) is equipped with a Takahashi convex structure, then X is called a Takahashi convex metric space.

If (X,d) is a Takahashi convex metric space, then for $x, y \in X$ we set

$$\text{seg}[x, y] = \{W(x, y,\lambda) : \lambda \in [0,1]\}. \tag{1.8}$$

Let us remark that any convex subset of normed space is a convex metric space with $W(x, y,\lambda) = \lambda x + (1-\lambda)y$.

2. Main results

The next theorem is our main result.

THEOREM 2.1. *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable, C a nonempty closed subset of X and ∂C the boundary of C. Let $g : C \mapsto X$, $f : X \mapsto X$ and $f : C \mapsto C$. Suppose that $\partial C \neq \emptyset$, f is continuous, and let us assume that f and g satisfy the following conditions.*

(i) *For every $x, y \in C$*

$$d(gx,gy) \leq M_\omega(x,y), \tag{2.1}$$

where

$$\begin{aligned} M_\omega(x,y) = \max\{&\omega[d(fx,fy)],\omega[d(fx,gx)],\omega[d(fy,gy)],\\ &\omega[d(fx,gy)],\omega[d(fy,gx)]\}, \end{aligned} \tag{2.2}$$

$\omega : [0,+\infty) \mapsto [0,+\infty)$ *is a nondecreasing semicontinuous function from the right, such that $\omega(r) < r$, for $r > 0$, and $\lim_{r\to\infty}[r - \omega(r)] = +\infty$.*

(ii) *f and g are a compatible pair on C, that is,*

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0 \tag{2.3}$$

whenever $\{x_n\}$ is a sequence in C such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = x \tag{2.4}$$

for some x in X.

(iii)

$$g(C)\bigcap C \subset f(C). \tag{2.5}$$

(iv)

$$g(\partial C) \subset C. \tag{2.6}$$

(v)

$$f(\partial C) \supset \partial C. \tag{2.7}$$

Then f and g have a unique common fixed point z in C.

Proof. Starting with an arbitrary $x_0 \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows. By (2.6) $g(x_0) \in C$. Hence, (2.5) implies that there is $x_1 \in C$ such that $f(x_1) = g(x_0)$. Let us consider $g(x_1)$. If $g(x_1) \in C$, again by (2.5) there is $x_2 \in C$ such that $f(x_2) = g(x_1)$. Suppose that $g(x_1) \notin C$. Now, because W is continuous in the third variable, there exists $\lambda_{11} \in [0,1]$ such that

$$W(f(x_1),g(x_1),\lambda_{11}) \in \partial C \bigcap \operatorname{seg}[f(x_1),g(x_1)]. \tag{2.8}$$

By (2.7) there is $x_2 \in \partial C$ such that $f(x_2) = W(f(x_1),g(x_1),\lambda_{11})$.

Hence, by induction we construct a sequence $\{x_n\}$ of points in C as follows. If $g(x_n) \in C$, than by (2.5) $f(x_{n+1}) = g(x_n)$ for some $x_{n+1} \in C$; if $g(x_n) \notin C$, then there exists $\lambda_{nn} \in [0,1]$ such that

$$W(f(x_n),g(x_n),\lambda_{nn}) \in \partial C \bigcap \operatorname{seg}[f(x_n),g(x_n)]. \tag{2.9}$$

Now, by (2.7) pick $x_{n+1} \in \partial C$ such that

$$f(x_{n+1}) = W(f(x_n),g(x_n),\lambda_{nn}). \tag{2.10}$$

Let us remark (see [6]) that for every $x,y \in X$ and every $\lambda \in [0,1]$

$$d(x,y) = d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y). \tag{2.11}$$

Furthermore, if $u \in X$ and $z = W(x,y,\lambda) \in \operatorname{seg}[x,y]$ then

$$d(u,z) = d(u,W(x,y,\lambda)) \le \max\{d(u,x),d(u,y)\}. \tag{2.12}$$

First let us prove that

$$f(x_{n+1}) \ne g(x_n) \Longrightarrow f(x_n) = g(x_{n-1}). \tag{2.13}$$

Suppose the contrary that $f(x_n) \ne g(x_{n-1})$. Then $x_n \in \partial C$. Now, by (2.5) $g(x_n) \in C$, hence $f(x_{n+1}) = g(x_n)$, a contradiction. Thus we prove (2.13).

We will prove that $g(x_n)$ and $f(x_n)$ are Cauchy sequences. First we will prove that these sequences are bounded, that is that the set

$$A = \left(\bigcup_{i=0}^{\infty}\{f(x_i)\}\right)\bigcup\left(\bigcup_{i=0}^{\infty}\{g(x_i)\}\right) \tag{2.14}$$

is bounded.

For each $n \ge 1$ set

$$\begin{gathered} A_n = \left(\bigcup_{i=0}^{n-1}\{f(x_i)\}\right)\bigcup\left(\bigcup_{i=0}^{n-1}\{g(x_i)\}\right), \\ a_n = \operatorname{diam}(A_n). \end{gathered} \tag{2.15}$$

We will prove that

$$a_n = \max\{d(f(x_0),g(x_i)) : 0 \le i \le n-1\}. \tag{2.16}$$

If $a_n = 0$, then $f(x_0) = g(x_0)$. We will prove that $g(x_0)$ is a common fixed point for f and g. By (2.3) it follows that

$$fg(x_0) = gf(x_0) = gg(x_0). \tag{2.17}$$

Now we obtain

$$d(gg(x_0),g(x_0)) \leq M_\omega(gx_0,x_0) = \omega(d(gg(x_0),g(x_0))), \tag{2.18}$$

and hence $gg(x_0) = g(x_0)$. From (2.17), we conclude that $g(x_0) = z$ is also a fixed point of f. To prove the uniqueness of the common fixed point, let us suppose that $fu = gu = u$ for some $u \in C$. Now, by (2.1) we have

$$d(z,u) = d(gz,gu) \leq M_\omega(z,u) = \omega(d(z,u)), \tag{2.19}$$

and so, $z = u$.

Suppose that $a_n > 0$. To prove (2.16) we have to consider three cases.

Case 1. Suppose that $a_n = d(fx_i,gx_j)$ for some $0 \leq i, j \leq n-1$.

(1i) Now, if $i \geq 1$ and $fx_i = gx_{i-1}$, we have

$$a_n = d(fx_i,gx_j) = d(gx_{i-1},gx_j) \leq M_\omega(x_{i-1},x_j) \leq \omega(a_n) < a_n. \tag{2.20}$$

and we get a contradiction. Hence $i = 0$.

(1ii) If $i \geq 1$ and $fx_i \neq gx_{i-1}$, we have $i \geq 2$, and $fx_{i-1} = gx_{i-2}$. Hence

$$fx_i \in \operatorname{seg}[g(x_{i-2}),g(x_{i-1})], \tag{2.21}$$

we have

$$\begin{aligned} a_n = d(fx_i,gx_j) &\leq \max\{d(gx_{i-2},gx_j),d(gx_{i-1},gx_j)\} \\ &\leq \max\{M_\omega(x_{i-2},x_j),M_\omega(x_{i-1},x_j)\} \leq \omega(a_n) < a_n \end{aligned} \tag{2.22}$$

and we get a contradiction.

Case 2. Suppose that $a_n = d(fx_i, fx_j)$ for some $0 \leq i, j \leq n-1$.

(2i) If $fx_j = gx_{j-1}$, then Case (2i) reduces to Case (1i).

(2ii) If $fx_j \neq gx_{j-1}$, then as in the Case (1ii) we have $j \geq 2$, $fx_{j-1} = gx_{j-2}$, and

$$fx_j \in \partial C \bigcap \operatorname{seg}[gx_{j-2},gx_{j-1}]. \tag{2.23}$$

Hence

$$a_n = d(fx_i, fx_j) \leq \max\{d(fx_i,gx_{j-2}),d(fx_i,gx_{j-1})\} \tag{2.24}$$

and Case (2ii) reduces to Case (1i).

Case 3. The remaining case $a_n = d(gx_i, gx_j)$ for some $0 \le i, j \le n-1$, is not possible (see Case (1i)). Hence we proved (2.16).

Now

$$a_n = d(fx_0, gx_i) \le d(fx_0, gx_0) + d(gx_0, gx_i) \le d(fx_0, gx_0) + \omega(a_n), \tag{2.25}$$

$$a_n - \omega(a_n) \le d(fx_0, gx_0). \tag{2.26}$$

By (i) there is $r_0 \in [0, +\infty)$ such that

$$r - \omega(r) > d(fx_0, gy_0), \quad \text{for } r > r_0. \tag{2.27}$$

Thus, by (2.26)

$$a_n \le r_0, \quad n = 1, 2, \dots, \tag{2.28}$$

and clearly

$$a = \lim_{n\to\infty} a_n = \operatorname{diam}(A) \le r_0. \tag{2.29}$$

Hence we proved that gx_n and fx_n are bounded sequences.

To prove that gx_n and fx_n are Cauchy sequences, let us consider the set

$$B_n = \left(\bigcup_{i=n}^{\infty}\{fx_i\}\right) \bigcup \left(\bigcup_{i=n}^{\infty}\{gx_i\}\right), \quad n = 2, 3, \dots. \tag{2.30}$$

By (2.16) we have

$$b_n \equiv \operatorname{diam}(B_n) = \sup_{j \ge n} d(fx_n, gx_j), \quad n = 1, 2, \dots. \tag{2.31}$$

If $fx_n = gx_{n-1}$, then as in Case (1i) for each $j \ge n$

$$b_n = d(fx_n, gx_j) = d(gx_{n-1}, gx_j) \le \omega(b_{n-1}), \quad n = 1, 2, \dots. \tag{2.32}$$

If $fx_n \ne gx_{n-1}$, then as in Case (1ii) for each $n \ge 1$ and $j \ge n$

$$b_n = d(fx_n, gx_j) \le \max\{d(gx_{n-2}, gx_j), d(gx_{n-1}, gx_j)\} \le \omega(b_{n-2}). \tag{2.33}$$

By (2.32) and (2.33) we get

$$b_n \le \omega(b_{n-2}), \quad n = 2, 3, \dots. \tag{2.34}$$

Clearly, $b_n \ge b_{n+1}$ for each n, and set $\lim_n b_n = b$. We will prove that $b = 0$. If $b > 0$, then (2.34) and (i) imply $b \le \omega(b) < b$, and we get a contradiction. It follows that both fx_n and gx_n are Cauchy sequences. Since $fx_n \in C$ and C is a closed subset of a complete metric space X we conclude that $\lim_n fx_n = y \in C$. Furthermore,

$$d(f(x_n), g(x_n)) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{2.35}$$

implies $\lim g(x_n) = y$. Hence,

$$\lim g(x_n) = \lim f(x_n) = y \in C. \tag{2.36}$$

By continuity of f

$$\lim f(g(x_n)) = \lim f(f(x_n)) = f(y) \in C. \tag{2.37}$$

Now, by (2.3), we have

$$d(gf(x_n), f(y)) \le d(gf(x_n), fg(x_n)) + d(fg(x_n), f(y)) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{2.38}$$

that is

$$\lim(gf)(x_n) = f(y). \tag{2.39}$$

Now,

$$\begin{gathered} M_\omega(fx_n, y) \longrightarrow \omega(d(fy, gy)) \quad n \longrightarrow \infty, \\ d(gfx_n, gy) \le M_\omega(fx_n, y) \quad n \longrightarrow \infty, \end{gathered} \tag{2.40}$$

implies

$$d(fy, gy) \le \omega(d(fy, gy)). \tag{2.41}$$

Hence, $f(y) = g(y)$, and gy is a common fixed point of f and g (see (2.17)). □

In the special case, when $\omega(r) = \lambda \cdot r$ where $0 < \lambda < 1$, we obtain the following result.

Theorem 2.2. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable, C a nonempty closed subset of X and ∂C the boundary of C. Let $g : C \mapsto X$, $f : X \mapsto X$ and $f : C \mapsto C$. Suppose that $\partial C \neq \emptyset$, f is continuous, and let us assume that f and g satisfy the following conditions.*

(i) *There exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in C$*

$$d(gx, gy) \le \lambda \cdot M(x, y), \tag{2.42}$$

where

$$M(x, y) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \tag{2.43}$$

Suppose that the conditions (ii)–(v) in Theorem 2.1 are satisfied. Then f and g have a unique common fixed point z in C and g is continuous at z. Moreover, if $z_n \in C$, $n = 1, 2, \ldots$, then

$$\lim d(fz_n, gz_n) = 0 \quad \textit{iff} \quad \lim_n z_n = z. \tag{2.44}$$

Proof. By Theorem 2.1 we know that f and g have a unique common fixed point z in C. Now, we show that g is continuous at z. Let $\{y_n\}$ be a sequence in C such that $y_n \to z$.

Now we have

$$\begin{aligned} d(gy_n,gz) &\le \lambda\cdot M(y_n,z)\\ &= \lambda\cdot \max\{d(fy_n,fz),d(fy_n,gy_n),d(fz,gy_n)\}\\ &= \lambda\cdot \max\{d(fy_n,fz),d(fy_n,gy_n)\}\\ &\le \lambda\cdot (d(fy_n,fz)+d(fz,gy_n)), \end{aligned} \tag{2.45}$$

that is

$$d(gy_n,gz) \le (1-\lambda)^{-1}\lambda\cdot d(fy_n,fz). \tag{2.46}$$

Therefore, we have $gy_n \to gz$ and so g is continuous at z. To prove (2.44), let us suppose that $w\in C$. Now, since $fz=gz=z$, we have

$$\begin{aligned} d(fw,gw) &\le d(fw,fz)+d(gw,gz)\le d(fw,fz)+\lambda\cdot M(w,z)\\ &\le d(fw,fz)+\lambda\cdot \max\{d(fw,fz),d(fw,gw),d(fz,gw)\}\\ &\le d(fw,fz)+\lambda\cdot (d(fw,fz)+d(fw,gw)), \end{aligned} \tag{2.47}$$

that is

$$(1-\lambda)d(fw,gw) \le (1+\lambda)d(fw,fz). \tag{2.48}$$

Let us remark that

$$\begin{aligned} d(fw,fz) &\le d(fw,gw)+d(gw,gz)\le d(fw,gw)+\lambda\cdot M(w,z)\\ &\le d(fw,gw)+\lambda\cdot \max\{d(fw,fz),d(fw,gw),d(fz,gw)\}\\ &\le d(fw,gw)+\lambda\cdot (d(fw,fz)+d(fw,gw)), \end{aligned} \tag{2.49}$$

that is

$$(1-\lambda)d(fw,fz) \le (1+\lambda)d(fw,gw). \tag{2.50}$$

By (2.48) and (2.50) we obtain

$$\begin{aligned} (1-\lambda)d(fw,gw) &\le (1+\lambda)d(fw,fz)\\ &\le (1-\lambda)^{-1}(1+\lambda)^2 d(fw,gw). \end{aligned} \tag{2.51}$$

Clearly (2.51) implies (2.44). □

Remark 2.3. Let (K,ρ) be a bounded metric space. It is said that the fixed point problem for a mapping $A: K\mapsto K$ is *well posed* if there exists a unique $x_A\in K$ such that $Ax_A=x_A$ and the following property holds: If $\{x_n\}\subset K$ and $\rho(x_n,Ax_n)\to 0$ as $n\to\infty$, then $\rho(x_n,x_A)\to 0$ as $n\to\infty$. Let us remark that condition (2.44) is related to the notion of well posed fixed point problem, and the notion of well-posedness is of central importance in many areas of Mathematics and its applications ([4, 10, 13]).

Remark 2.4. If in Theorem 2.1 we let f be the identity map on X and $\omega(r) = \lambda \cdot r$ where $0 < \lambda < 1$, we get Ćirić's Theorem 1.1 (Gajić's theorem [5]) stated for a Banach (convex complete metric) space X.

Remark 2.5. If in Theorem 2.1 we let f be the identity map on X and $C = X$, we get Ivanov's result [6, 7] stated for a Banach space X.

Remark 2.6. Let us recall that the first part of Theorem 2.2, that is the existence of the unique common fixed point of f and g was proved by Rakočević [12].

By the proof of Theorem 2.1 we can recover some results of Das and Naik [3] and Jungck [8].

Corollary 2.7 [3, Theorem 2.1]. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy*

$$g(X) \subset f(X) \tag{2.52}$$

and there exists a constant $\lambda \in (0,1)$ such that for every $x, y \in X$

$$d(gx, gy) \le \lambda \cdot M(x,y), \tag{2.53}$$

where

$$M(x,y) = \max\{d(fx,fy), d(fx,gx), d(fy,gy), d(fx,gy), d(fy,gx)\}. \tag{2.54}$$

Then f and g have a unique fixed point.

Proof. We follow the proof of Theorem 2.1. Let us remark that the condition (2.52) implies that starting with an arbitrary $x_0 \in X$, we construct a sequence $\{x_n\}$ of points in X such that $f(x_{n+1}) = g(x_n)$, $n = 0, 1, 2, \ldots$. The rest of the proof follows by the proof of Theorem 2.1. □

Corollary 2.8 [3, Theorem 3.1]. *Let X be a complete metric space. Let f^2 be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy*

$$gf(X) \subset f^2(X) \tag{2.55}$$

and $f(g(x)) = g(f(x))$ whenever both sides are defined. Further, let there exist a constant $\lambda \in (0,1)$ such that for every $x, y \in f(X)$

$$d(gx, gy) \le \lambda \cdot M(x,y), \tag{2.56}$$

where

$$M(x,y) = \max\{d(fx,fy), d(fx,gx), d(fy,gy), d(fx,gy), d(fy,gx)\}. \tag{2.57}$$

Then f and g have a unique common fixed point.

Proof. Again, we follow the proof of Theorem 2.1. By (2.55) starting with an arbitrary $x_0 \in f(X)$, we construct a sequence $\{x_n\}$ of points in $f(X)$ such that $f(x_{n+1}) = g(x_n) = y_n$, $n = 0,1,2,\ldots$. Now $f(y_n) = f(g(x_n)) = g(f(x_n)) = g(y_{n-1}) = z_n$, $n = 1,2,\ldots$, and from the proof of Theorem 2.1 we conclude that $\{z_n\}$ is a Cauchy sequence in X and hence convergent to some $z \in X$. Now, for each $n \geq 1$

$$
\begin{aligned}
d(&f^2g(x_n),gf(z)) \\
&= d(gf^2(x_n),gf(z)) \leq \lambda \cdot M(f^2(x_n),f(z)) \\
&= \lambda \cdot \max\Big\{d(f^2f(x_n),f^2(z)),d(f^2f(x_n),f^2g(x_n)), \\
&\qquad d(f^2(z),gf(z)),d(f^2f(x_n),gf(z)),d(f^2(z),f^2g(x_n))\Big\}.
\end{aligned}
\tag{2.58}
$$

Now, by continuity of f^2

$$d(f^2(z),gf(z)) \leq \lambda \cdot d(f^2(z),gf(z)). \tag{2.59}$$

Whence, $f^2(z) = gf(z)$, and gfz is a unique common fixed of f and g. □

Let us remark that from Theorem 2.1 and the proof of Corollary 2.7, we get the following.

Corollary 2.9. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that weakly commutes with f. Further let f and g satisfy (2.52) and (2.53). Then f and g have a unique common fixed point.*

Now as a corollary we get the following result of Jungck [8].

Corollary 2.10. *Let X be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further let f and g satisfy (2.52) and there exists a constant $\lambda \in (0,1)$ such that for every $x, y \in X$*

$$d(gx,gy) \leq \lambda \cdot d(fx,fy). \tag{2.60}$$

Then f and g have a unique common fixed point.

Corollary 2.11. *Let X be a convex complete metric space, C a nonempty compact subset of X, and ∂C the boundary of C. Let $g : C \mapsto X$, $f : X \mapsto X$ and $f : C \mapsto C$. Suppose that g and f are continuous, f and g satisfy the conditions* (ii)–(v) *in Theorem 2.1, and for all $x, y \in C$, $x \neq y$*

$$d(gx,gy) < M(x,y), \tag{2.61}$$

where

$$M(x,y) = \max\{d(fx,fy),d(fx,gx),d(fy,gy),d(fx,gy),d(fy,gx)\}. \tag{2.62}$$

Then f and g have a unique common fixed point in C.

Proof. By Theorem 2.2 and the proof of [12, Theorem 4]. □

Acknowledgment

The authors are grateful to the referees for some helpful comments and suggestions.

References

[1] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.

[2] ———, *Quasi contraction non-self mappings on Banach spaces*, Bull. Cl. Sci. Math. Nat. Sci. Math. **23** (1998), 25–31.

[3] K. M. Das and K. V. Naik, *Common fixed-point theorems for commuting maps on a metric space*, Proc. Amer. Math. Soc. **77** (1979), no. 3, 369–373.

[4] F. S. De Blasi and J. Myjak, *Sur la porosité de l'ensemble des contractions sans point fixe* [*On the porosity of the set of contractions without fixed points*], C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), no. 2, 51–54 (French).

[5] Lj. Gajić, *Quasi-contractive nonself mappings on Takahashi convex metric spaces*, Novi Sad J. Math. **30** (2000), no. 3, 41–46.

[6] O. Hadžić, *Foundations of Fixed Point Theory*, Institut za Matematiku, Novi Sad, 1978.

[7] A. A. Ivanov, *Fixed points of mappings of metric spaces. Studies in topology, II*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **66** (1976), 5–102, 207 (Russian).

[8] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), no. 4, 261–263.

[9] ———, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), no. 4, 771–779.

[10] W. A. Kirk and L. M. Saliga, *Some results on existence and approximation in metric fixed point theory*, J. Comput. Appl. Math. **113** (2000), no. 1-2, 141–152.

[11] V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd, 1994.

[12] ———, *Quasi contraction nonself mappings on Banach spaces and common fixed point theorems*, Publ. Math. Debrecen **58** (2001), no. 3, 451–460.

[13] S. Reich and A. J. Zaslavski, *Well-posedness of fixed point problems*, Far East J. Math. Sci. (FJMS), (2001), Special Volume, Part III, 393–401.

[14] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.) **32(46)** (1982), 149–153.

[15] W. Takahashi, *A convexity in metric space and nonexpansive mappings. I*, Kōdai Math. Sem. Rep. **22** (1970), 142–149.

Ljiljana Gajić: Institute of Mathematics, Faculty of Science, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia and Montenegro

E-mail address: gajic@im.ns.ac.yu

Vladimir Rakočević: Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia and Montenegro

E-mail address: vrakoc@bankerinter.net