STRONG CONVERGENCE TO COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS WITHOUT COMMUTATIVITY ASSUMPTION

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We introduce an iteration scheme for nonexpansive mappings in a Hilbert space and prove that the iteration converges strongly to common fixed points of the mappings without commutativity assumption.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. A mapping T of C into itself is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.1}$$

for each $x, y \in C$. For a mapping T of C into itself, we denote by $F(T)$ the set of fixed points of T. We also denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of positive integers and nonnegative real numbers, respectively.

Baillon [1] proved the first nonlinear ergodic theorem. Let C be a nonempty bounded convex closed subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for an arbitrary $x \in C$, $\{(1/(n+1))\sum_{i=0}^{n} T^i x\}_{n=0}^{\infty}$ converges weakly to a fixed point of T. Wittmann [9] studied the following iteration scheme, which has first been considered by Halpern [3]:

$$\begin{gathered} x_0 = x \in C, \\ x_{n+1} = \alpha_{n+1} x + (1 - \alpha_{n+1}) T x_n, \quad n \geq 0, \end{gathered} \tag{1.2}$$

where a sequence $\{\alpha_n\}$ in [0,1] is chosen so that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; see also Reich [7]. Wittmann proved that for any $x \in C$, the sequence

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$\{x_n\}$ defined by (1.2) converges strongly to the unique element $Px \in F(T)$, where P is the metric projection of H onto $F(T)$.

Recall that two mappings S and T of H into itself are called commutative if

$$ST = TS, \tag{1.3}$$

for all $x, y \in H$.

Recently, Shimizu and Takahashi [8] have first considered an iteration scheme for two commutative nonexpansive mappings S and T and proved that the iterations converge strongly to a common fixed point of S and T. They obtained the following result.

Theorem 1.1 (see [8]). *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let S and T be nonexpansive mappings of C into itself such that $ST = TS$ and $F(S)\bigcap F(T)$ is nonempty. Suppose that $\{\alpha_n\}_{n=0}^{\infty} \subseteq [0,1]$ satisfies*

(i) $\lim_{n\to\infty} \alpha_n = 0$, *and*

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, for an arbitrary $x \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by $x_0 = x$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S^i T^j x_n, \quad n \geq 0, \tag{1.4}$$

converges strongly to a common fixed point Px of S and T, where P is the metric projection of H onto $F(S)\bigcap F(T)$.

Remark 1.2. At this point, we note that the authors have imposed the commutativity on the mappings S and T. But there are many mappings, that do not satisfy $ST = TS$. For example, if $X = [-1/2, 1/2]$, and S and T of X into itself are defined by

$$S = x^2, \qquad T = \sin x, \tag{1.5}$$

then $ST = \sin^2 x$, whereas $TS = \sin x^2$.

In this paper, we deal with the strong convergence to common fixed points of two nonexpansive mappings in a Hilbert space. We consider an iteration scheme for nonexpansive mappings without commutativity assumption and prove that the iterations converge strongly to a common fixed point of the mappings T_i, $i = 1, 2$.

2. Preliminaries

Let C be a closed convex subset of a Hilbert space H and let S and T be nonexpansive mappings of C into itself. Then we consider the iteration scheme

$$\begin{gathered} x_0 = x \in C, \\ x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S^i T^j y_n, \\ y_n = \beta_n x_n + (1-\beta_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} T^i S^j x_n, \quad n \geq 0, \end{gathered} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1]. We know that a Hilbert space H satisfies Opial's condition [6], that is, if a sequence $\{x_n\}$ in H converges weakly to an element y of H and $y \neq z$, then

$$\liminf_{n\to\infty} ||x_n - y|| < \liminf_{n\to\infty} ||x_n - z||. \tag{2.2}$$

In what follows, we will use P_C to denote the metric projection from H onto C; that is, for each $x \in H$, P_C is the only point in C with the property

$$||x - P_C x|| = \min_{u\in C} \|u - x\|. \tag{2.3}$$

It is known that P_C is nonexpansive and characterized by the following inequality: given $x \in H$ and $v \in H$, then $v = P_C x$ if and only if

$$\langle x - v, v - y\rangle \geq 0, \quad y \in C. \tag{2.4}$$

Now, we introduce several lemmas for our main result in this paper. The first lemma can be found in [4, 5, 10].

Lemma 2.1. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert H, and let S,T be nonexpansive mappings of C into itself. For $x \in C$ and $n \in \mathbb{N} \cup \{0\}$, put*

$$G_n(x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} S^i T^j x,$$
$$\overline{G}_n(x) = \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} T^i S^j x. \tag{2.6}$$

Then

$$\lim_{n\to\infty} \sup_{x\in C} ||G_n(x) - SG_n(x)|| = 0,$$
$$\lim_{n\to\infty} \sup_{x\in C} ||\overline{G}_n(x) - T\overline{G}_n(x)|| = 0. \tag{2.7}$$

Proof. We first prove $\lim_{n\to\infty}\sup_{x\in C}\|G_n(x)-SG_n(x)\|=0$.

By an idea in [2], for $\{x_{i,j}\}_{i,j=0}^{\infty}$, $\{\overline{x}_{i,j}\}_{i,j=0}^{\infty}\subseteq C$ and $z_n=(1/l_n)\sum_{k=0}^{n}\sum_{i+j=k}x_{i,j}$, $\overline{z}_n=(1/l_n)\sum_{k=0}^{n}\sum_{i+j=k}\overline{x}_{i,j}\in C$, with $l_n=(n+1)(n+2)/2$, we have

$$||z_n-v||^2=\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||x_{i,j}-v||^2-\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||x_{i,j}-z_n||^2 \tag{2.8}$$

for each $v\in H$. For $x\in C$, put $x_{i,j}=S^iT^jx,\overline{x}_{i,j}=T^iS^jx$ and $v=Sz_n,\overline{v}=T\overline{z}_n$. Then, we have

$$\begin{aligned}
||G_n(x)-SG_n(x)||^2&=\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||S^iT^jx-Sz_n||^2-\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||S^iT^jx-z_n||^2\\
&=\frac{1}{l_n}\sum_{k=0}^{n}||T^kx-Sz_n||^2+\frac{1}{l_n}\sum_{k=1}^{n}\sum_{i+j=k,i\geq 1}||S^iT^jx-Sz_n||^2\\
&\quad-\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||S^iT^jx-z_n||^2\\
&\leq\frac{1}{l_n}\sum_{k=0}^{n}||T^kx-Sz_n||^2+\frac{1}{l_n}\sum_{k=1}^{n}\sum_{i+j=k,i\geq 1}||S^{i-1}T^jx-z_n||^2\\
&\quad-\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||S^iT^jx-z_n||^2\\
&=\frac{1}{l_n}\sum_{k=0}^{n}||T^kx-Sz_n||^2+\frac{1}{l_n}\sum_{k=0}^{n-1}\sum_{i+j=k}||S^iT^jx-z_n||^2\\
&\quad-\frac{1}{l_n}\sum_{k=0}^{n}\sum_{i+j=k}||S^iT^jx-z_n||^2\\
&=\frac{1}{l_n}\sum_{k=0}^{n}||T^kx-Sz_n||^2-\frac{1}{l_n}\sum_{i+j=n}||S^iT^jx-z_n||^2\\
&\leq\frac{1}{l_n}\sum_{k=0}^{n}||T^kx-Sz_n||^2\leq\frac{2}{n+2}\{\operatorname{diam}(C)\}^2,
\end{aligned} \tag{2.9}$$

where $\operatorname{diam}(C)$ is the diameter of C. So, we have, for each $n\in\mathbb{N}\cup\{0\}$,

$$\sup_{x\in C}||G_n(x)-SG_n(x)||^2\leq\frac{2}{n+2}\{\operatorname{diam}(C)\}^2, \tag{2.10}$$

and hence

$$\lim_{n\to\infty}\sup_{x\in C}||G_n(x)-SG_n(x)||=0. \tag{2.11}$$

Similarly, we have

$$\lim_{n\to\infty}\sup_{x\in C}\|\overline{G}_n(x)-T\overline{G}_n(x)\|=0. \tag{2.12}$$

□

3. Convergence theorem

Now we can prove a strong convergence theorem in a Hilbert space.

THEOREM 3.1. *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let S and T be nonexpansive mappings of C into itself such that $F(S)\bigcap F(T)$ is nonempty. Suppose that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$, *and*

(ii) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

For an arbitrary $x\in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ is generated by $x_0=x$ and

$$\begin{aligned} x_{n+1}&=\alpha_n x+(1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}S^iT^jy_n,\\ y_n&=\beta_n x_n+(1-\beta_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}T^iS^jx_n, \quad n\ge 0. \end{aligned} \tag{3.1}$$

Let

$$z_n=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}S^iT^jy_n, \qquad \overline{z}_n=\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k}T^iS^jx_n, \tag{3.2}$$

for each $n\in\mathbb{N}\cup\{0\}$. If there exist subsequences $\{z_{n_i}\}_{i=0}^{\infty}$ of $\{z_n\}_{n=0}^{\infty}$ and $\{\overline{z}_{n_j}\}_{j=0}^{\infty}$ of $\{\overline{z}_n\}_{n=0}^{\infty}$, respectively, which converge weakly to some common point z in some bounded subset D of C, then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.1) converges strongly to $P_{F(S)\cap F(T)}x$.

Proof. Let $x\in C$ and $w\in F(S)\bigcap F(T)$. Putting $r=\|x-w\|$, then the set

$$D=\{y\in H:\|y-w\|\le r\}\cap C \tag{3.3}$$

is a nonempty bounded closed convex subset of C which is S- and T-invariant and contains $x_0=x$. So we may assume, without loss of generality, that S and T are the mappings of D into itself. Since P is the metric projection of H onto $F(S)\cap F(T)$, we have

$$\langle y-Px,x-Px\rangle\le 0 \tag{3.4}$$

for each $y\in F(S)\bigcap F(T)$.

From (3.4), we have

$$\limsup_{n\to\infty}\langle z_n - Px, x - Px\rangle \le 0, \qquad \limsup_{n\to\infty}\langle \overline{z}_n - Px, x - Px\rangle \le 0. \tag{3.5}$$

In fact, assume that, there exist two positive real numbers r_0 and r_1 such that

$$\limsup_{n\to\infty}\langle z_n - Px, x - Px\rangle > r_0, \qquad \limsup_{n\to\infty}\langle \overline{z}_n - Px, x - Px\rangle > r_1. \tag{3.6}$$

Since $\{z_n\}_{n=0}^{\infty}$ and $\{\overline{z}_n\}_{n=0}^{\infty} \subseteq D$ are bounded, from (3.6), there exist subsequences $\{z_{n_i}\}_{i=0}^{\infty}$ of $\{z_n\}_{n=0}^{\infty}$ and $\{\overline{z}_{n_j}\}_{j=0}^{\infty}$ of $\{\overline{z}_n\}_{n=0}^{\infty}$, respectively, such that

$$\begin{aligned}
\limsup_{n\to\infty}\langle z_n - Px, x - Px\rangle &= \lim_{i\to\infty}\langle z_{n_i} - Px, x - Px\rangle > r_0,\\
\limsup_{n\to\infty}\langle \overline{z}_n - Px, x - Px\rangle &= \lim_{j\to\infty}\langle \overline{z}_{n_j} - Px, x - Px\rangle > r_1.
\end{aligned} \tag{3.7}$$

By the assumption, we know that $\{z_{n_i}\}_{i=0}^{\infty}$ and $\{\overline{z}_{n_j}\}_{j=0}^{\infty}$ converge weakly to some common point $z \in D$. Thus from Lemma 2.2 and Opial's condition, we have $z \in F(S)\bigcap F(T)$. In fact, if $z \ne Sz$, we have

$$\begin{aligned}
\liminf_{i\to\infty}||z_{n_i} - z|| < \liminf_{i\to\infty}||z_{n_i} - Sz||
&\le \liminf_{i\to\infty}\left(||z_{n_i} - Sz_{n_i}|| + ||Sz_{n_i} - Sz||\right)\\
&\le \liminf_{i\to\infty}||z_{n_i} - z||.
\end{aligned} \tag{3.8}$$

This is a contradiction. Therefore, we have $z = Sz$.

Similarly, we have $z = Tz$. So, we have

$$\langle z - Px, x - Px\rangle \le 0. \tag{3.9}$$

On the other hand, since $\{z_{n_i}\}$ converges weakly to z, we obtain

$$\langle z - Px, x - Px\rangle \ge r_0. \tag{3.10}$$

This is a contradiction. Hence, we have

$$\limsup_{n\to\infty}\langle z_n - Px, x - Px\rangle \le 0, \qquad \limsup_{n\to\infty}\langle \overline{z}_n - Px, x - Px\rangle \le 0. \tag{3.11}$$

Since

$$\begin{aligned}
\|\overline{z}_n - Px\| &\leq \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} \|T^i S^j x_n - Px\| \right\}^2 \\
&\leq \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} \|x_n - Px\| \right\}^2 = \|x_n - Px\|^2, \\
\|y_n - Px\|^2 &= \|\beta_n x_n + (1-\beta_n)\overline{z}_n - Px\|^2 \\
&= \|\beta_n (x_n - Px) + (1-\beta_n)(\overline{z}_n - Px)\|^2 \\
&= \beta_n^2 \|x_n - Px\|^2 + 2\beta_n(1-\beta_n)(x_n - Px, \overline{z}_n - Px) + (1-\beta_n)^2 \|\overline{z}_n - Px\|^2 \\
&\leq \beta_n^2 \|x_n - Px\|^2 + 2\beta_n(1-\beta_n) \frac{\|x_n - Px\|^2 + \|\overline{z}_n - Px\|^2}{2} \\
&\quad + (1-\beta_n)^2 \|\overline{z}_n - Px\|^2 \leq \|x_n - Px\|^2.
\end{aligned} \tag{3.12}$$

Then, we have

$$\begin{aligned}
\|x_{n+1} - Px\|^2 &= \|\alpha_n x + (1-\alpha_n) z_n - Px\|^2 \\
&= \alpha_n^2 \|x - Px\|^2 + (1-\alpha_n)^2 \|z_n - Px\|^2 + 2\alpha_n(1-\alpha_n)\langle z_n - Px, x - Px\rangle \\
&\leq (1-\alpha_n)^2 \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} \|S^i T^j y_n - Px\| \right\}^2 \\
&\quad + \alpha_n^2 \|x - Px\|^2 + 2\alpha_n(1-\alpha_n)\langle z_n - Px, x - Px\rangle \\
&\leq (1-\alpha_n)^2 \left\{ \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} \|y_n - Px\| \right\}^2 \\
&\quad + \alpha_n^2 \|x - Px\|^2 + 2\alpha_n(1-\alpha_n)\langle z_n - Px, x - Px\rangle \\
&= (1-\alpha_n)^2 \|y_n - Px\|^2 + \alpha_n^2 \|x - Px\|^2 + 2\alpha_n(1-\alpha_n)\langle z_n - Px, x - Px\rangle \\
&\leq (1-\alpha_n)\|x_n - Px\|^2 + \alpha_n \{\alpha_n \|x - Px\|^2 + 2(1-\alpha_n)\langle z_n - Px, x - Px\rangle\}.
\end{aligned} \tag{3.13}$$

Putting $a_n = \|x_n - Px\|^2$, from (3.13), we have

$$a_{n+1} \leq (1-\alpha_n) a_n + \delta_n, \tag{3.14}$$

where $\delta_n = \alpha_n \{\alpha_n \|x - Px\|^2 + 2(1-\alpha_n)\langle z_n - Px,\ x - Px\rangle\}$.

It is easily seen that

$$\limsup_{n\to\infty} \delta_n/\alpha_n = \limsup_{n\to\infty} \{\alpha_n \|x - Px\|^2 + 2(1-\alpha_n)\langle z_n - Px,\ x - Px\rangle\} \le 0. \qquad (3.15)$$

Now applying Lemma 2.1 with (3.15) to (3.14) concludes that $\|x_n - Px\| \to 0$ as $n \to \infty$. This completes the proof. □

References

[1] J.-B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, Comptes Rendus de l'Académie des Sciences de Paris, Série. A-B **280** (1975), no. 22, A1511–A1514.

[2] H. Brézis and F. E. Browder, *Nonlinear ergodic theorems*, Bulletin of the American Mathematical Society **82** (1976), no. 6, 959–961.

[3] B. Halpern, *Fixed points of nonexpanding maps*, Bulletin of the American Mathematical Society **73** (1967), 957–961.

[4] J. S. Jung, *Viscosity approximation methods for a family of finite nonexpansive mappings in Banach spaces*, Nonlinear Analysis **64** (2006), no. 11, 2536–2552.

[5] P.-E. Maingé, *Viscosity methods for zeroes of accretive operators*, Journal of Approximation Theory **140** (2006), no. 2, 127–140.

[6] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bulletin of the American Mathematical Society **73** (1967), 591–597.

[7] S. Reich, *Some problems and results in fixed point theory*, Topological Methods in Nonlinear Functional Analysis (Toronto, Ont., 1982), Contemp. Math., vol. 21, American Mathematical Society, Rhode Island, 1983, pp. 179–187.

[8] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, Journal of Mathematical Analysis and Applications **211** (1997), no. 1, 71–83.

[9] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Archiv der Mathematik **58** (1992), no. 5, 486–491.

[10] H.-K. Xu, *Viscosity approximation methods for nonexpansive mappings*, Journal of Mathematical Analysis and Applications **298** (2004), no. 1, 279–291.

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