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Research Article

Nonlinear Mean Ergodic Theorems for Semigroups in Hilbert Spaces

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Let K be a nonempty subset (not necessarily closed and convex) of a Hilbert space and let $\Gamma = \{T(t);\ t \geq 0\}$ be a semigroup on K and let $\alpha(\cdot) : [0,\infty) \to K$ be an almost orbit of Γ. In this paper, we prove that every almost orbit of Γ is almost weakly and strongly convergent to its asymptotic center.

1. Introduction

Let K be a nonempty subset of a Hilbert space $\mathcal{H}$, where K is not necessarily closed and convex. A family $\Gamma = \{T(t);\ t \geq 0\}$ of mappings $T(t)$ is called a semigroup on K if

(S1) $T(t)$ is a mapping from K into itself for $t \geq 0$,

(S2) $T(0)x = x$ and $T(t+s)x = T(t)T(s)x$ for $x \in K$ and $t,s \geq 0$,

(S3) for each $x \in K$, $T(\cdot)x$ is strongly measurable and bounded on every bounded subinterval of $[0,\infty)$.

Let Γ be a semigroup on K. Then $F = \{x \in K : T(t)x = x,\ t \geq 0\}$ is said to be fixed-points set of Γ. We state a condition introduced by Miyadera [1]. If, for every bounded set $B \subset K$, $v \in K$, and $s \geq 0$, there exists a $\delta_s(B,v) \geq 0$ with $\lim_{s\to\infty} \delta_s(B,v) = 0$ such that

$$\left\|T(s)u - T(s)v\right\| \leq \|u - v\| + \delta_s(B,v) \tag{1.1}$$

for $u \in B$, then Γ is said to be an asymptotically nonexpansive semigroup.

Definition 1.1. A function $a(\cdot):[0,\infty)\to K$ is called almost-orbit of Γ if $a(\cdot):[0,\infty)\to K$ is strongly measurable and bounded on every bounded subinterval of $[0,\infty)$ and if

$$\lim_{t\to\infty}\sup_{s\geq 0}||a(s+t)-T^k(s)a(t)||^p=0. \tag{1.2}$$

Using these conditions, we prove that every almost-orbit of Γ is weakly and strongly convergent to its asymptotic center (see [1]). Xu [2] studied strong asymptotic behavior of almost-orbits of both of nonexpansive and asymptotically nonexpansive semigroups. Takahashi [3] generalized the nonlinear ergodic theorems for general semigroups of nonexpansive mappings. Kada and Takahashi [4] proved a strong ergodic theorem for general semigroups of nonexpansive mappings. Oka [5] proved nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings. All of the abovementioned authors studied, except Miyadera's works, K as a closed and convex subset of a Hilbert space. Miyadera [1] studied almost convergence of almost-orbits of semigroup of non-Lipschitzian mappings in Hilbert spaces. Miyadera [1] proved the following theorem. If Γ is asymptotically nonexpansive in the weak sense and F is nonempty set, then the following conditions holds:

(a1) $a(\cdot)$ is weakly almost convergent to its asymptotic center y,

(a2) if y is an element of K and if $T(t_0):K\to K$ is continuous for some $t_0>0$, then y is a fixed point of Γ, that is, y belongs to F.

There are some conditions in the discrete case in [6–8]. Miyadera [7, 8] showed that the condition in [6] could be replaced by a weaker condition introduced in [7, 8]. Miyadera [7, 8] and Wittmann [6] proved nonlinear ergodic theorems where the closedness and convexity of K and the asymptotically nonexpansivity of T were not assumed. In this paper, in the light of these papers we establish weak ergodic theorem for semigroups of mappings on K satisfying condition (I) given in the statement of Theorem 3.1. We also establish strong ergodic theorem for semigroups of mappings on K satisfying condition (II) given before statement of Theorem 4.1. This paper is organized as follows.

In Section 2, we prove the covering lemmas we need for establishing weakly convergence result. In Section 3, we deal with $a(\cdot)$ almost-orbit weakly almost-convergent to its asymptotic center with respect to condition (I). In the last section, we investigate strong convergence using condition (II). We establish that every almost-orbit of Γ is strongly almost-convergent to its asymptotic center.

2. Lemmas

Let $a(\cdot):[0,\infty)\to\mathcal{H}$ be a function strongly measurable and bounded on every bounded subinterval of $[0,\infty)$ and let $\|a(t)\|$ be convergent as $t\to\infty$.

Lemma 2.1 [1]. *For $r,s,t\geq 0$, the following statements are mutually equivalent:*

(i) $\overline{\lim}_{s\to\infty}\,\overline{\lim}_{t\to\infty}\,\overline{\lim}_{r\to\infty}[(a(t+r),a(t))-(a(s+r),a(s))]\leq 0$;

(ii) $\overline{\lim}_{s\to\infty}\,\overline{\lim}_{t\to\infty}\,\overline{\lim}_{r\to\infty}[\|a(t+r)+a(t)\|^2-\|a(s+r)+a(s)\|^2]\leq 0$;

(iii) $\overline{\lim}_{s\to\infty}\,\overline{\lim}_{t\to\infty}\,\overline{\lim}_{r\to\infty}[\|a(s+r)-a(s)\|^2-\|a(t+r)-a(t)\|^2]\leq 0$.

If $a(\cdot)$ satisfies the equivalent conditions (i), (ii), and (iii), then $a(\cdot)$ is weakly almost-convergent to its asymptotic center y.

Lemma 2.2 [1]. *Let $a(\cdot):[0,\infty)\to\mathcal{H}$ be a function strongly measurable and bounded on every bounded subinterval of $[0,\infty)$ and let $\|a(t)\|$ be convergent as $t\to\infty$. Then, one has that the following statements are mutually equivalent:*

(i) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[(a(t+r),a(t))-(a(s+r),a(s))]\leq 0$;

(ii) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[\|a(t+r)+a(t)\|^2-\|a(s+r)+a(s)\|^2]\leq 0$;

(iii) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[\|a(s+r)-a(s)\|^2-\|a(t+r)-a(t)\|^2]\leq 0$.

$\|a(t)\|$ is convergent as $t\to\infty$. Moreover, if $a(\cdot)$ satisfies the equivalent conditions (i), (ii), and (iii), then $a(\cdot)$ is strongly almost-convergent to its asymptotic center y.

Remark 2.3. We can take the following conditions instead of (ii) and (iii) in Lemma 2.2, for $A,C>0$,

(ii′) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[\|a(t+r)+a(t)\|^2-A\|a(s+r)+a(s)\|^2]\leq 0$;

(iii′) $\overline{\lim}_{s\to\infty}\overline{\lim}_{t\to\infty}\sup_{r\geq 0}[\|a(s+r)-a(s)\|^2-A\|a(t+r)-a(t)\|^2]\leq 0$.

We can obtain

$$\begin{aligned}&\overline{\lim_{s\to\infty}}\ \overline{\lim_{t\to\infty}}\sup_{r\geq 0}[||a(t+r)+a(t)||^2-A||a(s+r)+a(s)||^2]\\&\quad\leq\overline{\lim_{s\to\infty}}\ \overline{\lim_{t\to\infty}}\sup_{r\geq 0}[||a(t+r)+a(t)||^2-||a(s+r)+a(s)||^2]\leq 0,\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\overline{\lim_{s\to\infty}}\ \overline{\lim_{t\to\infty}}\sup_{r\geq 0}[||a(s+r)-a(s)||^2-A||a(t+r)-a(t)||^2]\\&\quad\leq\overline{\lim_{s\to\infty}}\ \overline{\lim_{t\to\infty}}\sup_{r\geq 0}[||a(s+r)-a(s)||^2-||a(t+r)-a(t)||^2]\leq 0.\end{aligned}\tag{2.2}$$

Moreover, we can write

$$\overline{\lim_{s\to\infty}}\ \overline{\lim_{t\to\infty}}\sup_{r\geq 0}[||a(s+r)-a(s)||^2-A||a(t+r)-a(t)||^2-C]\leq 0.\tag{2.3}$$

Note that Lemma 2.2 holds for this condition.

3. Weak ergodic theorems

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and $\|\cdot\|$ norm, and let K be a nonempty subset of $\mathcal{H}$, where K is not necessarily closed and convex. Let $\Gamma=\{T(t);\ t\geq 0\}$ be a semigroup acting on K.

Theorem 3.1. *Suppose that for every bounded set $B\subset K$, $v\in K$, $u\in B$ and $r\geq 0$, there exists $\delta_r(B,v)\geq 0$ with $\lim_{r\to\infty}\delta_r(B,v)=0$ such that*

$$\begin{aligned}&||T^k(r)u-T^k(r)v||^p\\&\quad\leq\lambda_r\|u-v\|^p+c\big[\lambda_r\|u\|^p-||T^k(r)u||^p+\lambda_r\|v\|^p-||T^k(r)v||^p\big]+\delta_r(B,v),\end{aligned}\tag{I}$$

where λ_r, c are nonnegative constants such that $\lim_{r\to\infty}\lambda_r = 1$, *and* $p \geq 1$. *If* $F \neq \emptyset$ *or* $c > 0$, *then* $a(\cdot)$ *is almost weakly convergent to its asymptotic center, which is* y.

Proof. Suppose $F \neq \emptyset$ and $c = 0$ for the semigroup $\Gamma = \{T(t);\ t \in \mathbb{R}^+\}$. Then for $u = x$ and $f \in F$, we can take $B = \{x\}$. If we write $u = x$ and $v = f$ in (I), then we have

$$||T^k(r)x - T^k(r)f||^p = ||T^k(r)x - f||^p \leq \lambda_r \|x - f\|^p + \delta_r(B, f). \tag{3.1}$$

Thus, for every $x \in K$, the sequence $\{T^k(r)x - f + f\} = \{T^k(r)x\}$ is bounded. Let $a(\cdot) : [0,\infty) \to K$ be almost-orbit of Γ.

From Definition 1.1, we have $\lim_{t\to\infty}\sup_{s\geq 0}[\|a(t+s) - T^k(s)a(t)\|^p] = 0$. There is $t_0 = t_0(\varepsilon) > 0$ for $\varepsilon > 0$, $t \geq t_0$, and $s \geq 0$ such that $\|a(t+s) - T^k(s)a(t)\|^p < \varepsilon$.

In particular, for $s \geq 0$, we have $\|a(s+t_0) - T^{t_0}(s)a(t_0)\|^p < \varepsilon$. If we consider both this inequality and boundness of sequence $\{T^k(s)x\}$, we have

$$\begin{aligned} &||a(s+t_0) - T^{t_0}(s)a(t_0) + T^{t_0}(s)a(t_0)||^p \\ &\quad < 2^{p-1}\left[||a(s+t_0) - T^{t_0}(s)a(t_0)||^p + ||T^{t_0}(s)a(t_0)||^p\right] \\ &\quad < 2^{p-1}\varepsilon + 2^{p-1}||T^{t_0}(s)||^p||a(t_0)||^p, \end{aligned} \tag{3.2}$$

then $\{a(s);\ s \in \mathbb{R}^+\}$ is bounded.

If we take in (I), $B = \{a(t);\ t \in \mathbb{R}^+\}$, and $v = f$, then we obtain

$$||T^k(r)a(t) - T^k(r)f||^p \leq \lambda_r||a(t) - f||^p. \tag{3.3}$$

Thus

$$\begin{aligned} ||a(r+t) - f||^p &\leq ||a(r+t) - T^k(r)a(t) + T^k(r)a(t) - f||^p \\ &\leq 2^{p-1}\left[||a(r+t) - T^k(r)a(t)||^p + ||T^k(r)a(t) - f||^p\right] \\ &< 2^{p-1}\left(\varepsilon + \lambda_r||a(t) - f||^p\right) \end{aligned} \tag{3.4}$$

(since $\|T^k(r)a(t) - T^k(r)f\|^p \leq \lambda_r\|a(t) - f\|^p$).

Taking limit as $r \to \infty$, because of $\lim_{r\to\infty}\lambda_r = 1$, for arbitrary ε,

$$\overline{\lim_{r\to\infty}}||a(r+t) - f||^p < 2^{p-1}\underline{\lim_{r\to\infty}}||a(t) - f||^p. \tag{3.5}$$

Therefore $\{\|a(t) - f\|\}$ is convergent.

Let $t > s > 0$. We know that sequence $\{T^k(r);\ r \in \mathbb{R}^+\}$ is bounded. Moreover, since sequence $\{a(s); s \in \mathbb{R}^+\}$ is bounded, $\{T^k(h)a(s);\ h \in \mathbb{R}^+\}$ is also bounded. Then we can take $B = \{T^k(h)a(s);\ h \in \mathbb{R}^+\}$ and $a(s) \in K$. Taking $u = T^k(h)a(s), v = a(s)$, and $r = t - s$, for $h \geq 0$, we have

$$||T^k(t-s)T^k(h)a(s) - T^k(t-s)a(s)||^p \leq \lambda_{t-s}||T^k(h)a(s) - a(s)||^p + \delta_{t-s}(B,a(s)). \tag{3.6}$$

Consequently,

$$\begin{aligned}
||a(t+h) - a(t)||^p &\leq ||a(t+h) - T^k(t+h-s)a(s) + T^k(t+h-s)a(s) - a(t)||^p \\
&\leq 2^{p-1}\left[||a(t+h) - T^k(t+h-s)a(s)||^p + ||T^k(t+h-s)a(s) - a(t)||^p\right] \\
&= 2^{p-1}\big[||a(t+h) - T^k(t+h-s)a(s)||^p \\
&\qquad + ||T^k(t+h-s)a(s) + T^k(t-s)a(s) - T^k(t-s)a(s) - a(t)||^p\big] \\
&\leq 2^{p-1}||a((t+h-s)+s) - T^k(t+h-s)a(s)||^p \\
&\quad + 2^{2(p-1)}||T^k(t-s)T^k(h)a(s) - T^k(t-s)a(s)||^p \\
&\quad + 2^{2(p-1)}||T^k(t-s)a(s) - a(t)||^p \\
&< 2^{p-1}\varepsilon + 2^{2(p-1)}\lambda_{t-s}||T^k(h)a(s) - a(s)||^p \\
&\quad + 2^{2(p-1)}||T^k(t-s)a(s) - a(t-s+s)||^p + \delta_{t-s}(B,a(s)) \\
&< 2^{p-1}\varepsilon(1+2^{p-1}) + 2^{2(p-1)}\lambda_{t-s}\big(||T^k(h)a(s) - a(s+h) \\
&\qquad\qquad + a(s+h) - a(s)||^p\big) + \delta_{t-s}(B,a(s)) \\
&< 2^{p-1}\varepsilon(1+2^{p-1}) + 2^{2(p-1)}\varepsilon\lambda_{t-s} \\
&\quad + 2^{2(p-1)}\lambda_{t-s}||a(s+h) - a(s)||^p + \delta_{t-s}(B,a(s)).
\end{aligned} \tag{3.7}$$

Then

$$||a(t+h)-a(t)||^p - 2^{2(p-1)}\lambda_{t-s}||a(s+h)-a(s)||^p < 2^{p-1}\varepsilon(1+2^{p-1}+2^{p-1}\lambda_{t-s}) + \delta_{t-s}(B,a(s)). \tag{3.8}$$

Taking limit as $t,s \to \infty$, for $h \geq 0$ and arbitrary ε, from the last inequality, we obtain

$$\overline{\lim_{t\to\infty}}\ \overline{\lim_{s\to\infty}}\ \sup_{h\geq 0}\left[||a(t+h) - a(t)||^p - 2^{2(p-1)}||a(s+h) - a(s)||^p\right] \leq 0. \tag{3.9}$$

From Remark 2.3, $a(\cdot)$ is weakly almost convergent to its asymptotic center.

Now, we investigate the case $F \neq \emptyset$ and $c > 0$.

For $x \in K$, if we write $B = \{x\}$ and $v = x$ in (I), then we obtain

$$0 \leq \lambda_r 0 + c\left[2\lambda_r\|x\|^p - 2||T^k(r)x||^p\right] + \delta_r(B,x), \tag{3.10}$$

and from this we can write

$$||T^k(r)x||^p \le \lambda_r \|x\|^p + \frac{\delta_r(B,x)}{2c}. \tag{3.11}$$

Then for every $x \in K$, $\{T^k(t)x;\ t \in \mathbb{R}^+\}$ is bounded. By Definition 1.1, for $\varepsilon > 0$ taking $t \ge t_0$, $s \ge 0$, there exists $t_0 = t_0(\varepsilon)$ such that $\|a(t_0+s) - T^k(s)a(t_0)\|^p < \varepsilon$. Since $\{T^k(t)x;\ t \in \mathbb{R}^+\}$ is bounded,

$$\begin{aligned} &||a(t_0+s) - T^k(s)a(t_0) + T^k(s)a(t_0)||^p \\ &\quad \le 2^{(p-1)}[||a(t_0+s) - T^k(s)a(t_0)||^p + ||T^k(s)||^p||a(t_0)||^p] \end{aligned} \tag{3.12}$$

$\{a(t);\ t \in \mathbb{R}^+\}$ is bounded. We can take $B = \{T^k(h)a(s) : h \ge 0\}$, if we write $v = a(s)$ and $r = t - s$ in (I), then we obtain

$$\begin{aligned} &||T^k(t-s)T^k(h)a(s) - T^k(t-s)a(s)||^p \\ &\quad \le \lambda_{t-s}||T^k(h)a(s) - a(s)||^p + c[\lambda_{t-s}||T^k(h)a(s)||^p \\ &\qquad - ||T^k(t-s)T^k(h)a(s)||^p + \lambda_{t-s}||a(s)||^p \\ &\qquad - ||T^k(t-s)a(s)||^p] + \delta_{t-s}(B,a(s)). \end{aligned} \tag{3.13}$$

Consequently,

$$\begin{aligned} ||a(t+h) - a(t)||^p &= ||a(t+h) - T^k(t+h-s)a(s) + T^k(t+h-s)a(s) - a(t)||^p \\ &\le 2^{p-1}(||a(t+h) - T^k(t+h-s)a(s)||^p + ||T^k(t+h-s)a(s) - a(t)||^p) \\ &\le 2^{p-1}||a(t+h) - T^k(t+h-s)a(s)||^p \\ &\quad + 2^{p-1}||T^k(t+h-s)a(s) - T^k(t-s)a(s)||^p \\ &\quad + ||T^k(t-s)a(s) - a(t)||^p \le 2^{p-1}||a(t+h) - T^k(t+h-s)a(s)||^p \\ &\quad + 2^{2(p-1)}[||T^k(t-s)T^k(h)a(s) - T^k(t-s)a(s)||^p \\ &\qquad\qquad + ||T^k(t-s)a(s) - a(t-s+s)||^p] \\ &< 2^{p-1}\varepsilon + 2^{2(p-1)}[\lambda_{t-s}||T^k(h)a(s) - a(s)||^p \\ &\qquad\qquad + c[\lambda_{t-s}||T^k(h)a(s)||^p - ||T^k(t-s)T^k(h)a(s)||^p \\ &\qquad\qquad\quad + \lambda_{t-s}||a(s)||^p - ||T^k(t-s)a(s)||^p]] \\ &\quad + \delta_{t-s}(B,a(s)) + \varepsilon(2^{2(p-1)}). \end{aligned} \tag{3.14}$$

Taking $\|a(s)\| \leq M$ and $\|T^k(h)\| \leq N$,

$$\begin{aligned}
&< 2^{p-1}\varepsilon(1+2^{p-1}) + 2^{2(p-1)}\lambda_{t-s}\|T^k(h)a(s)-a(s)\|^p \\
&\quad + c(\lambda_{t-s}M^pN^p - (M^2N)^p + \lambda_{t-s}M^p - M^pN^p) + \delta_{t-s}(B,a(s)) \\
&< 2^{p-1}\varepsilon(1+2^{p-1}) + c\lambda_{t-s}M^p(N^p+1) - cM^pN^p(M^p-1) \\
&\quad + 2^{2(p-1)}\lambda_{t-s}\|T^k(h)a(s)-a(s+h)+a(s+h)-a(s)\|^p + \delta_{t-s}(B,a(s)) \\
&< 2^{p-1}\varepsilon(1+2^{p-1}) + c\lambda_{t-s}M^p(N^p+1) - cM^pN^p(M^p-1) \\
&\quad + 2^{2(p-1)}\lambda_{t-s}2^{p-1}\|T^k(h)a(s)-a(s+h)\|^p \\
&\quad + 2^{2(p-1)}\lambda_{t-s}2^{p-1}\|a(s+h)-a(s)\|^p + \delta_{t-s}(B,a(s)) \\
&< 2^{p-1}\varepsilon(1+2^{p-1}) + c\lambda_{t-s}M^p(N^p+1) - cM^pN^p(M^p-1) + 2^{3(p-1)}\lambda_{t-s}\varepsilon \\
&\quad + 2^{3(p-1)}\lambda_{t-s}\|a(s+h)-a(s)\|^p + \delta_{t-s}(B,a(s)).
\end{aligned} \tag{3.15}$$

Taking limit as $t, s \to \infty$, for $h \geq 0$,

$$\overline{\lim_{t\to\infty}}\ \overline{\lim_{s\to\infty}}\ \sup_{h\geq 0}\left[\|a(t+h)-a(t)\|^p - 2^{3(p-1)}\|a(s+h)-a(s)\|^p\right] \leq \acute{A}. \tag{3.16}$$

That is,

$$\overline{\lim_{t\to\infty}}\ \overline{\lim_{s\to\infty}}\ \sup_{h\geq 0}\left[\|a(t+h)-a(t)\|^p - 2^{3(p-1)}\|a(s+h)-a(s)\|^p\right] - \acute{A} \leq 0. \tag{3.17}$$

Then from Remark 2.3, $a(\cdot)$ is weakly almost convergent to its asymptotic center. Thus, the proof is completed. □

4. Strong ergodic theorems

Let $\Gamma = \{T(t);\ t \geq 0\}$ be semigroup on K. Suppose that for every bounded set $B \subset K$ and integer $k \geq 0$, there exists a $\delta_r(B,v) \geq 0$ with $\lim_{r\to\infty}\delta_r(B,v) = 0$ such that

$$\begin{aligned}
&\|T^k(r)u + T^k(r)v\|^p \\
&\qquad \leq \lambda_r\|u+v\|^p + c\left[\lambda_r\|u\|^p - \|T^k(r)u\|^p + \lambda_r\|v\|^p - \|T^k(r)v\|^p\right] + \delta_r(B)
\end{aligned} \tag{II}$$

for $u, v \in B$, where λ_r, c, and p are nonnegative constants such that $\lim_{r\to\infty}\lambda_r = 1$ and $p \geq 1$.

Theorem 4.1. *If $\Gamma = \{T(t);\ t \geq 0\}$ is a semigroup on K satisfying condition (II), then every almost-orbit of Γ is strongly almost convergent to its asymptotic center.*

Proof. Let $a(\cdot) : [0,\infty) \to K$ be almost-orbit of Γ. For $t \geq 0$, we set

$$\varphi(s) = \sup_{t\geq 0}\|a(t+s) - T^k(t)a(s)\|. \tag{4.1}$$

When $s \to \infty$, $\varphi(s) \to 0$ and from condition (II) by taking $B = \{x\}$ and $v = x$, we have

$$\|T^k(r)x\|^p \leq \lambda_r\|x\|^p + \frac{\delta_r(\{x\})}{2^p} + 2c. \tag{4.2}$$

Thus for $x \in K$, $\{T^k(r)x : r \geq 0\}$ is bounded. By Definition 1.1, since $\{T^k(r)x : r \geq 0\}$ is bounded, we have

$$||a(s+t) - T^k(s)a(t)||^p \leq \varepsilon. \tag{4.3}$$

Therefore $\{a(s) : s \geq 0\}$ is bounded. Let $r > h \geq 0$. Since $\{T^k(h)x : h \geq 0\}$ and $\{a(s) : s \geq 0\}$ are bounded then $\{T^k(h)a(s) : h \geq 0\}$ is bounded, by using (II) with $B = \{T^k(h)a(s) : h \geq 0\}$, $v = a(s)$ and $r = t - s$ we have

$$\begin{aligned} ||T^k(t-s)T^k(h)a(s) + T^k(t-s)a(s)||^p \leq{}& \lambda_{t-s}||T^k(h)a(s) + a(s)||^p \\ &+ c[\lambda_{t-s}||T^k(h)a(s)||^p - ||T^k(t-s)T^k(h)a(s)||^p \\ &+ \lambda_{t-s}||a(s)||^p - ||T^k(t-s)a(s)||^p] + \delta_{t-s}(B). \end{aligned} \tag{4.4}$$

For $c = 0$, we have

$$||T^k(t-s)T^k(h)a(s) + T^k(t-s)a(s)||^p \leq \lambda_{t-s}||T^k(h)a(s) + a(s)||^p + \delta_{t-s}(B, a(s)). \tag{4.5}$$

Consequently,

$$\begin{aligned} ||a(t+h) + a(t)||^p \leq{}& 2^{p-1}(||a((t+r)+s-s) - T^k(t+h-s)a(s)||^p) \\ &+ 2^{2(p-1)}(||T^k(t+h-s)a(s) + T^k(t-s)a(s)||^p \\ &\quad + ||a(t-s+s) - T^k(t-s)a(s)||^p) \\ \leq{}& 2^{p-1}\varphi^p(s) + 2^{2(p-1)}[||T^k(t-s)T^k(h)a(s) + T^k(t-s)a(s)||^p + \varphi^p(s)] \\ \leq{}& 2^{p-1}\varphi^p(s) + 2^{2(p-1)}[\lambda_{t-s}||T^k(h)a(s) + a(s)||^p + \varphi^p(s)] + \delta_{t-s}(B, a(s)) \\ \leq{}& 2^{p-1}\varphi^p(s) + 2^{2(p-1)}\lambda_{t-s}\varphi^p(s) \\ &+ 2^{2(p-1)}\lambda_{t-s}||T^k(h)a(s) + a(s) + a(h+s) - a(h+s)||^p + \delta_{t-s}(B, a(s)) \\ \leq{}& 2^{p-1}\varphi^p(s)(1 + 2^{p-1}\lambda_{t-s}) \\ &+ 2^{2(p-1)}2^{p-1}\lambda_{t-s}[||a(h+s) + a(s)||^p + ||T^k(h)a(s) - a(h+s)||^p] \\ &+ \delta_{t-s}(B, a(s)). \end{aligned} \tag{4.6}$$

Taking limit as $s, t \to \infty$, for $h \geq 0$,

$$\begin{aligned} &\overline{\lim_{t\to\infty}} \sup_{h\geq 0} [||a(t+h) + a(t)||^p - 2^{3(p-1)}\lambda_{t-s}||a(h+s) + a(s)||^p] \\ &\qquad \leq 2^{p-1}\varphi^p(s)(1 + 2^{p-1}\lambda_{t-s} + \lambda_{t-s}2^{p-1}). \end{aligned} \tag{4.7}$$

Since $\varphi(s) \to 0$, we have

$$\overline{\lim_{s\to\infty}}\, \overline{\lim_{t\to\infty}} \sup_{h\geq 0} [||a(t+h) + a(t)||^p - 2^{3(p-1)}||a(h+s) + a(s)||^p] \leq 0, \tag{4.8}$$

that is, condition (ii) in Lemma 2.2 is satisfied. Thus, every almost-orbit of Γ is strongly almost convergent to its asymptotic center. $\square$

Remark 4.2. Our results presented in this paper generalize the results of Miyadera [7, 8] to the case of $F \neq \emptyset$ and $c > 0$ for semigroups of asymptotically nonexpansive mappings in Hilbert spaces.

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