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Research Article

Coincidence Theorems, Generalized Variational Inequality Theorems, and Minimax Inequality Theorems for the Φ-Mapping on *G*-Convex Spaces

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We establish some coincidence theorems, generalized variational inequality theorems, and minimax inequality theorems for the family G-KKM(X,Y) and the Φ-mapping on G-convex spaces.

1. Introduction and preliminaries

In 1929, Knaster et al. [1] had proved the well-known KKM theorem on n-simplex. In 1961, Fan [2] had generalized the KKM theorem in the infinite-dimensional topological vector space. Later, the KKM theorem and related topics, for example, matching theorem, fixed point theorem, coincidence theorem, variational inequalities, minimax inequalities, and so on had been presented in grand occasions. Recently, Chang and Yen [3] introduced the family, KKM(X,Y), and got some results about fixed point theorems, coincidence theorems, and some applications on this family. Later, Ansari et al. [4] and Lin and Chen [5] studied the coincidence theorems for two families of set-valued mappings, and they also gave some applications of the existence of minimax inequality and equilibrium problems. In this paper, we establish some coincidence theorems, generalized variational inequality theorems, and minimax inequality theorems for the family G-KKM(X,Y) and the Φ-mapping on G-convex spaces.

Let X and Y be two sets, and let $T : X \to 2^Y$ be a set-valued mapping. We will use the following notations in the sequel:

(i) $T(x) = \{y \in Y : y \in T(x)\}$,
(ii) $T(A) = \bigcup_{x\in A} T(x)$,
(iii) $T^{-1}(y) = \{x \in X : y \in T(x)\}$,
(iv) $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \phi\}$,

(v) $T^*(y) = \{x \in X : y \notin T(x)\}$,

(vi) if D is a nonempty subset of X, then $\langle D\rangle$ denotes the class of all nonempty finite subsets of D.

For the case that X and Y are two topological spaces, then $T : X \to 2^Y$ is said to be closed if its graph $\mathcal{G}_T = \{(x,y) \in X \times Y : y \in T(x)\}$ is closed. T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y.

Let X be a topological space. A subset D of X is said to be compactly closed (resp., compactly open) in X if for any compact subset K of X, the set $D \cap K$ is closed (resp., closed) in K. Obviously, D is compactly open in X if and only if its complement D^c is compactly closed in X.

The compact closure of D is defined by

$$\operatorname{ccl}(D) = \cap\{B \subset X : D \subset B,\ B \text{ is compactly closed in } X\}, \tag{1.1}$$

and the compact interior of D is defined by

$$\operatorname{cint}(D) = \cup\{B \subset X : B \subset D,\ B \text{ is compactly open in } X\}. \tag{1.2}$$

Remark 1.1. It is easy to see that $\operatorname{ccl}(X\backslash D) = X\backslash \operatorname{cint}(D)$, D is compactly open in X if and only if $D = \operatorname{cint}(D)$, and for each nonempty compact subset K of X, we have $\operatorname{cint}(D) \cap K = \operatorname{int}_K(D \cap K)$, where $\operatorname{int}_K(D \cap K)$ denotes the interior of $D \cap K$ in K.

Definition 1.2 [6, 7]. Let X and Y be two topological spaces, and let $T : X \to 2^Y$.

(i) T is said to be transfer compactly closed (resp., transfer closed) on X if for any $x \in X$ and any $y \notin T(x)$, there exists $\overline{x} \in X$ such that $y \notin \operatorname{ccl}T(\overline{x})$ (resp., $y \notin \operatorname{cl}T(\overline{x})$).

(ii) T is said to be transfer compactly open (resp., transfer open) on X if for any $x \in X$ and any $y \in T(x)$, there exists $\overline{x} \in X$ such that $y \in \operatorname{cint}T(\overline{x})$ (resp., $y \in \operatorname{int}T(\overline{x})$).

(iii) T is said to have the compactly local intersection property on X if for each nonempty compact subset K of X and for each $x \in X$ with $T(x) \neq \phi$, there exists an open neighborhood $N(x)$ of x in X such that $\cap_{z\in N(x)\cap K} T(z) \neq \phi$.

Remark 1.3. If $T : X \to 2^Y$ is transfer compactly open (resp., transfer compactly closed) and Y is compact, then T is transfer open (resp., transfer closed).

We denote by Δ_n the standard n-simplex with vectors $e_0, e_1, \ldots, e_n$, where e_i is the $(i+1)$th unit vector in $\mathcal{R}^{n+1}$.

A generalized convex space [8] or a G-convex space $(X,D;\Gamma)$ consists of a topological space X, a nonempty subset D of X, and a function $\Gamma : \langle D\rangle \to 2^X$ with nonempty values (in the sequal, we write $\Gamma(A)$ by ΓA for each $A \in \langle D\rangle$) such that

(i) for each $A, B \in \langle D\rangle$, $A \subset B$ implies that $\Gamma A \subset \Gamma B$,

(ii) for each $A \in \langle D\rangle$ with $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma A$ such that $J \in \langle A\rangle$ implies that $\phi_A(\Delta_{|J|-1}) \subset \Gamma J$, where $\Delta_{|J|-1}$ denotes the faces of Δ_n corresponding to $J \in \langle A\rangle$.

Particular forms of G-convex spaces can be found in [8] and references therein. For a G-convex space $(X,D;\Gamma)$ and $K \subset X$,

(i) K is G-convex if for each $A \in \langle D\rangle$, $A \subset K$ implies $\Gamma A \subset K$,

(ii) the G-convex hull of K, denoted by G-Co(K), is the set $\cap\{B \subset X \mid B \text{ is a } G\text{-convex subset of } X \text{ containing } K\}$.

Definition 1.4 [9]. A G-convex space X is said to be a locally G-convex space if X is a uniform topological space with uniformity $\mathcal{U}$ which has an open base $\mathcal{N} = \{V_i \mid i \in I\}$ of symmetric encourages such that for each $V \in \mathcal{N}$, the set $V[x] = \{y \in X \mid (x,y) \in V\}$ is a G-convex set, for each $x \in X$.

Let $(X,D;\Gamma)$ be a G-convex space which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. Then a nonempty subset K of X is said to be almost G-convex if for any finite subset B of K and for any $V \in \mathcal{N}$, there is a mapping $h_{B,V} : B \to X$ such that $x \in V[h_{B,V}(x)]$ for all $x \in B$ and G-Co$(h_{B,V}(B)) \subset K$. subset of K. We call the mapping $h_{B,V} : B \to X$ a G-convex-inducing mapping.

Remark 1.5. (i) In general, the G-convex-inducing mapping $h_{B,V}$ is not unique. If $U \subset V$, then it is clear that any $h_{B,U}$ can be regarded as an $h_{B,V}$.

(ii) It is clear that the G-convex set is almost G-convex, but the inverse is not true, for a counterexample.

Let $E = \Re^2$ be the Euclidean topological space. Then the set $B = \{x = (x_1,x_2) \in E : x_1^{2/3} + x_2^{2/3} < 1\}$ is a G-convex set, but the set $B' = \{x = (x_1,x_2) \in E : 0 < x_1^{2/3} + x_2^{2/3} < 1\}$ is an almost G-convex set, not a G-convex set.

Applying Ding [10, Proposition 1] and Lin [11, Lemma 2.2], we have the following lemma.

Lemma 1.6. *Let X and Y be two topological spaces, and let $F : X \to 2^Y$ be a set-valued mapping. Then the following conditions are equivalent:*

(i) *F has the compactly local intersection property,*

(ii) *for each compact subset K of X and for each $y \in Y$, there exists an open subset O_y of X such that $O_y \cap K \subset F^{-1}(y)$ and $K = \bigcup_{y \in Y}(O_y \cap K)$,*

(iii) *for any compact subset K of X, there exists a set-valued mapping $P : X \to 2^Y$ such that $P(x) \subset F(x)$ for each $x \in X$, $P^{-1}(y)$ is open in X and $P^{-1}(y) \cap K \subset F^{-1}(y)$ for each $y \in Y$ and $K = \bigcup_{y \in Y}(P^{-1}(y) \cap K)$,*

(iv) *for each compact subset K of X and for each $x \in K$, there exists $y \in Y$ such that $x \in \operatorname{cint} F^{-1}(y) \cap K$ and $K = \bigcup_{y \in Y}(\operatorname{cint} F^{-1}(y) \cap K)$,*

(v) *F^{-1} is transfer compactly open valued on Y,*

(vi) *$X = \bigcup_{y \in Y} \operatorname{cint} F^{-1}(y)$.*

Definition 1.7. Let Y be a topological space and let X be a G-convex space. A set-valued mapping $T : Y \to 2^X$ is called a Φ-mapping if there exists a set-valued mapping $F : Y \to 2^X$ such that

(i) for each $y \in Y$, $A \in \langle F(y) \rangle$ implies that G-Co$(A) \subset T(y)$,

(ii) F satisfies one of the conditions (i)–(vi) in Lemma 1.6.

Moreover, the mapping F is called a companion mapping of T.

Remark 1.8. If $T : Y \to 2^X$ is a Φ-mapping, then for each nonempty subset Y_1 of Y, $T|_{Y_1} : Y_1 \to 2^X$ is also a Φ-mapping.

Let X be a G-convex space. A real-valued function $f : X \to \mathfrak{R}$ is said to be G-quasiconvex if for each $\xi \in \mathfrak{R}$, the set $\{x \in X : f(x) \le \xi\}$ is G-convex, and f is said to be G-quasiconcave if $-f$ is G-quasiconvex.

Definition 1.9. Let X be a nonempty almost G-convex subset of a G-convex space. A real-valued function $f : X \to \mathfrak{R}$ is said to be almost G-quasiconvex if for each $\xi \in \mathfrak{R}$, the set $\{x \in X : f(x) \le \xi\}$ is almost G-convex, and f is said to be almost G-quasiconcave if $-f$ is almost G-quasiconvex.

Definition 1.10. Let X be a G-convex space, Y a nonempty set, and let $f, g : X \times Y \to \mathfrak{R}$ be two real-valued functions. For any $y \in Y$, g is said to be f-G-quasiconcave in x if for each $A = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle'$,

$$\min_{1 \le i \le n} f(x_i, y) \le g(x, y), \quad \forall x \in G\text{-Co}(A). \tag{1.3}$$

Definition 1.11. Let X be a nonempty almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, Y a nonempty set, and let $f, g : X \times Y \to \mathfrak{R}$ be two real-valued functions. For any $y \in Y$, g is said to be almost f-G-quasiconcave in x if for each $A = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ and for every $V \in \mathcal{N}$, there exists a G-convex-inducing mapping $h_{A,V} : A \to X$ such that

$$\min_{1 \le i \le n} f(x_i, y) \le g(x, y), \quad \forall x \in G\text{-Co}(h_{A,V}(A)). \tag{1.4}$$

Remark 1.12. It is clear that if $f(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$, and if for each $y \in Y$, the mapping $x \to f(x, y)$ is almost G-quasiconcave (G-quasiconcave), then g is almost f-G-quasiconcave in x (f-G-quasiconcave).

Definition 1.13. Let X be a G-convex space, Y a topological space, and let $T, F : X \to 2^Y$ be two set-valued functions satisfying

$$T(G\text{-Co}(A)) \subset F(A) \quad \text{for any } A \in \langle X \rangle. \tag{1.5}$$

Then F is called a generalized G-KKM mapping with respect to T. If the set-valued function $T : X \to 2^Y$ satisfies the requirement that for any generalized G-KKM mapping F with respect to T the family $\{\overline{F(x)} \mid x \in X\}$ has the finite intersection property, then T is said to have the G-KKM property. The class G-KKM(X, Y) is defined to be the set $\{T : X \to 2^Y \mid T \text{ has the } G\text{-KKM property}\}$.

We now generalize the G-KKM property on a G-convex space to the G-KKM* property on an almost G-convex subset of a G-convex space.

Definition 1.14. Let X be a nonempty almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, and Y a topological space. Let $T, F : X \to 2^Y$ be two set-valued functions satisfying that for each finite subset A of X and for any $V \in \mathcal{N}$, there exists a G-convex-inducing mapping $h_{A,V} : A \to X$ such that

$$T(G\text{-Co}(h_{A,V}(A))) \subset F(A). \tag{1.6}$$

Then F is called a generalized G-KKM* mapping with respect to T. If the set-valued function $T : X \to 2^Y$ satisfies the requirement that for any generalized G-KKM* mapping F with respect to T the family $\{\overline{F(x)} \mid x \in X\}$ has the finite intersection property, then T is said to have the G-KKM* property. The class $G\text{-KKM}^*(X,Y)$ is defined to be the set $\{T : X \to 2^Y \mid T \text{ has the } G\text{-KKM}^* \text{ property}\}$.

2. Coincidence theorems for the Φ-mapping and the G-KKM family

Throughout this paper, we assume that the set $G\text{-Co}(A)$ is compact whenever A is a compact set.

The following lemma will play important roles for this paper.

Lemma 2.1. *Let Y be a compact set, X a G-convex space. Let $T : Y \to 2^X$ be a Φ-mapping. Then there exists a continuous function $f : Y \to X$ such that for each $y \in Y$, $f(y) \in T(y)$, that is, T has a continuous selection.*

Proof. Since Y is compact, there exists $A = \{x_0, x_1, \dots, x_n\} \subset X$ such that $Y = \bigcup_{i=0}^n \operatorname{int} F^{-1}(x_i)$. Since X is a G-convex space and $A \in \langle X \rangle$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $\phi_A(\Delta_{|J|-1}) \subset \Gamma_J$ for each $J \in \langle A \rangle$.

Let $\{\lambda_i\}_{i=0}^n$ be the partition of the unity subordinated to the cover $\{\operatorname{int} F^{-1}(x_i)\}_{i=0}^n$ of Y. Define a continuous mapping $g : Y \to \Delta_n$ by

$$g(y) = \sum_{i=0}^n \lambda_i(y) e_i = \sum_{i \in I(y)} \lambda_i(y) e_i, \quad \text{for each } y \in Y, \tag{2.1}$$

where $I(y) = \{i \in \{0,1,2,\dots,n\} : \lambda_i \neq 0\}$. Note that $i \in I(y)$ if and only if $y \in F^{-1}(x_i)$, that is, $x_i \in F(y)$. Since T is a Φ-mapping, we conclude that $\phi_A \circ g(y) \in \phi_A(\Delta_{I(y)}) \subset G\text{-Co}\{x_i : i \in I(y)\} \subset T(y)$, for each $y \in Y$. This completes the proof. □

Let X be a G-convex space. A polytope in X is denoted by $\Delta = G\text{-Co}(A)$ for each $A \in \langle X \rangle$. By the conception of the $G\text{-KKM}(X,Y)$ family we immediately have the following proposition.

Proposition 2.2 [12]. *Let X be a G-convex space, and let Y and Z be two topological spaces. Then*

(i) *$T \in G\text{-KKM}(X,Y)$ if and only if $T \in G\text{-KKM}(\Delta,Y)$ for every polytopy Δ in X,*

(ii) *if Y is a normal space, Δ a polytope in X, and if $T : X \to 2^Y$ satisfies the requirement that fT has a fixed point in Δ for all $f \in \mathscr{C}(Y,\Delta)$, then $T \in G\text{-KKM}(\Delta,Y)$.*

Following Lemma 2.1 and Proposition 2.2, we prove the following important lemma for this paper.

Lemma 2.3. *Let X be a G-convex space and let Y be a compact G-convex space. If $T : X \to 2^Y$ is a Φ-mapping, then $T \in G\text{-KKM}(X,Y)$.*

Proof. Since T is a Φ-mapping, we have that for any $A \in \langle X \rangle$, let $\Delta = G\text{-Co}(A)$, $T|_\Delta : \Delta \to Y$ is also a Φ-mapping. Since Δ is compact and by Lemma 2.1, $T|_\Delta$ has a continuous selection function, that is, there is a continuous function $f : \Delta \to Y$ such that for each

$x \in \Delta$, $f(x) \in T(x)$. So we conclude that $f^{-1}T$ has a fixed point in Δ. By Proposition 2.2, $T \in G\text{-KKM}(\Delta, Y)$, and so we conclude that $T \in G\text{-KKM}(X, Y)$. □

The following lemma is an extension of Chang et al. [13, Proposition 2.3].

Lemma 2.4. *Let X be a nonempty almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, and let Y, Z be two topological spaces. If $T \in G\text{-KKM}^*(X,Y)$, then $fT \in G\text{-KKM}^*(X,Z)$ for all $f \in \mathcal{C}(Y,Z)$.*

Proof. Let F be a generalized G-KKM* mapping with respect to fT such that $F(x)$ is closed for all $x \in X$, and let $A \in \langle X\rangle$. Then for any $V \in \mathcal{N}$, there exists a G-convex-inducing mapping $h_{A,V} : A \to X$ such that $fT(G\text{-Co}(h_{A,V}(A))) \subset F(A)$. So $T(G\text{-Co}(h_{A,V}(A))) \subset f^{-1}F(A)$. Therefore, $f^{-1}F$ is a generalized G-KKM* mapping with respect to T. Since $T \in \text{KKM}^*(X,Y)$ and $f^{-1}F(x)$ is closed for all $x \in X$, so the family $\{f^{-1}F(x) : x \in X\}$ has the finite intersection property, and so does the family $\{F(x) : x \in X\}$. Hence $fT \in G\text{-KKM}^*(X,Z)$. □

Theorem 2.5. *Let X be a nonempty almost G-convex subset of a locally G-convex space E, and let $T \in G\text{-KKM}^*(X,X)$ be compact and closed. Then T has a fixed point.*

Proof. Since E is a locally G-convex space, there exists a uniform structure $\mathcal{U}$, let $\mathcal{N} = \{V_i \mid i \in I\}$ be an open symmetric base family for the uniform structure $\mathcal{U}$ such that for any $U \in \mathcal{N}$, the set $U[x] = \{y \in X \mid (x,y) \in U\}$ is G-convex for each $x \in X$, and let $U \in \mathcal{N}$.

We now claim that for any $V \in \mathcal{N}$, there exists $x_V \in X$ such that $V[x_V] \cap T(x_V) \neq \phi$. Suppose it is not the case, then there is a $V \in \mathcal{N}$ such that $V[x_V] \cap T(x_V) = \phi$, for all $x_V \in X$. Let $V_1 \in \mathcal{N}$ such that $V_1 \circ V_1 \subset V$. Since T is compact, hence $K = \overline{TX}$ is a compact subset of X. Define $F : X \to 2^X$ by

$$F(x) = K \setminus V_1[x] \quad \text{for each } x \in X. \tag{2.2}$$

We will show that

(1) $F(x)$ is nonempty and closed for each $x \in X$,

(2) F is a generalized G-KKM* mapping with respect to T.

(1) is obvious. To prove (2), we use the contradiction. Let $A = \{x_1, x_2, \ldots, x_n\} \in \langle X\rangle$. Suppose F is not a generalized G-KKM* mapping with respect to T. Then there exists $V_2 \in \mathcal{N}$ such that for any G-convex-inducing mapping $h_{A,V_2} : A \to X$, one has $T(G\text{-Co}(h_{A,V_2}(A))) \nsubseteq F(A)$. Let $V_3 \in \mathcal{N}$ such that $V_3 \subset V_1 \cap V_2$. Then $T(G\text{-Co}(h_{A,V_3}(A))) \nsubseteq F(A)$. So there exist $\mu \in G\text{-Co}(h_{A,V_3}(A))$ and $\nu \in T(\mu)$ such that $\nu \notin \bigcup_{i=1}^n Fx_i$. From the definition of F, it follows that $\nu \in V_1[x_i]$ for each $i \in \{1,2,\ldots,n\}$. Hence, $\nu \in V_1 \circ V_3[h_{A,V_3}(x_i)] \subset V[h_{A,V_3}(x_i)]$ for each $i \in \{1,2,\ldots,n\}$, since X is almost G-convex. Thus, $h_{A,V_3}(x_i) \in V[\nu]$, for each $i \in \{1,2,\ldots,n\}$, and hence $\mu \in G\text{-Co}(h_{A,V_3}(A)) \subset V[\nu]$, that is, $\nu \in V[\mu]$. Therefore, $\nu \in T(\mu) \cap V[\mu]$. This contradicts $V[x] \cap T(x) = \phi$, for all $x \in X$. Hence, F is a generalized G-KKM* mapping with respect to T.

Since $T \in G\text{-KKM}^*(X,X)$, the family $\{F(x) : x \in X\}$ has finite intersection property, and so we conclude that $\bigcap_{x\in X} F(x) \neq \phi$. Let $\eta \in \bigcap_{x\in X} F(x) \subset K \subset X$. Then $\eta \in K \setminus V_1[x]$, for all $x \in X$. This implies that $\eta \in K \setminus V_1[\eta]$. So we have reached a contradiction. Therefore, we have proved that for each $V_i \in \mathcal{N}$, there is an $x_{V_i} \in X$ such that $V[x_{V_i}] \cap T(x_{V_i}) \neq \phi$.

Let $y_{V_i} \in V_i[x_{V_i}] \cap T(x_{V_i})$, then $(x_{V_i}, y_{V_i}) \in \mathcal{G}_T$ and $(x_{V_i}, y_{V_i}) \in V_i$. Since T is compact, without loss of generality, we may assume that $\{y_{V_i}\}_{i\in I}$ converges to y_0, that is, there exists $V_0 \in \mathcal{N}$ such that $(y_{V_j}, y_0) \in V_j$ for all $V_j \in \mathcal{N}$ with $V_j \subset V_0$. Let $V_U \in \mathcal{N}$ with $V_U \circ V_U \subset V_j \subset V_0$, then we have $(x_{V_U}, y_{V_U}) \in V_U$ and $(y_{V_U}, y_0) \in V_U$, so $(x_{V_U}, y_{V_U}) \circ (y_{V_U}, y_0) = (x_{V_U}, y_0) \in V_U \circ V_U \subset V_j$, that is, $x_{V_U} \to y_0$. The closedness of T implies that $(y_0, y_0) \in \mathcal{G}_T$, that is, $y_0 \in T(y_0)$. This completes the proof. □

Corollary 2.6. *Let X be a nonempty G-convex subset of a locally G-convex space E, and let $T \in$ G-KKM(X,X) be compact and closed. Then T has a fixed point.*

We now establish the main coincidence theorem for the Φ-mapping and the family G-KKM(X,Y).

Theorem 2.7. *Let X be a nonempty G-convex subset of a locally G-convex space E, and let Y be a topological space. Assume that*

(i) *$T \in$ G-KKM(X,Y) is compact and closed,*

(ii) *$F : Y \to 2^X$ is Φ-mapping.*

Then there exists $(\overline{x}, \overline{y}) \in X \times Y$ such that $\overline{y} \in T(\overline{x})$ and $\overline{x} \in F(\overline{y})$.

Proof. Since T is compact, we have that $K = \overline{T(X)}$ is compact in Y. By (ii), we have that $F|_K$ is also a Φ-mapping. By Lemma 2.1, $F|_K$ has a continuous selection $f : K \to X$. So, by Lemma 2.4, we have $fT \in \mathrm{KKM}(X,X)$, and so by Corollary 2.6, there exists $x \in X$ such that $x \in fT(x) \subset FT(x)$, that is, there exists $y \in T(x)$ such that $x \in F(y)$. □

Applying Lemma 2.3, Theorem 2.7, and Corollary 2.6, we immediately have the following coincidence theorem for two Φ-mappings.

Theorem 2.8. *Let X be a nonempty G-convex subset of a locally G-convex space E, and Y a topological space. If $T : X \to 2^Y$, $F : Y \to 2^X$ are two Φ-mappings, and if T is compact and closed, then there exists $(x, y) \in X \times Y$ such that $y \in T(x)$ and $x \in F(y)$.*

3. Generalized variational theorems and minimax inequality theorems

Lemma 3.1 [14]. *Let X and Y be two topological spaces, and let $F : X \to 2^Y$ be a set-valued mapping. Then F is transfer closed if and only if $\bigcap_{x\in X} F(x) = \bigcap_{x\in X} \overline{F(x)}$.*

Definition 3.2 [15]. Let X and Y be two topological spaces, and let $f : X \times Y \to \Re \cup \{-\infty, \infty\}$ be a function. For some $\gamma \in \Re$, $f(x,y)$ is said to be γ-transfer compactly lower semicontinuous in y if for each $y \in \{u \in Y : f(x,u) > \gamma\}$, there exists an $\overline{x} \in X$ such that $y \in \mathrm{cint}\{u \in Y : f(\overline{x},u) > \gamma\}$. f is said to be γ-transfer compactly upper semicontinuous in y if for each $y \in \{u \in Y : f(x,u) < \gamma\}$, there exists an $\overline{x} \in X$ such that $y \in \mathrm{cint}\{u \in Y : f(\overline{x},u) < \gamma\}$.

Definition 3.3. Let X and Y be two topological spaces, and let $f : X \times Y \to \Re \cup \{-\infty, \infty\}$ be a function. Then f is said to be transfer compactly lower semicontinuous (resp., transfer lower semicontinuous) in y if for each $y \in Y$ and $\gamma \in \Re$ with $y \in \{u \in Y : f(x,u) > \gamma\}$, there exists an $\overline{x} \in X$ such that $y \in \mathrm{cint}\{u \in Y : f(\overline{x},u) > \gamma\}$ (resp., $y \in \mathrm{int}\{u \in Y : f(\overline{x},u) > \gamma\}$).

f is said to be transfer compactly upper semicontinuous in y if $-f$ is transfer compactly lower semicontinuous in y.

Lemma 3.4 [15]. *Let X and Y be two topological spaces, and let $f : X \times Y \to \mathfrak{R} \cup \{-\infty, \infty\}$ be a function. For some $\gamma \in \mathfrak{R}$, $f : X \times Y \to \mathfrak{R}$ is said to be γ-transfer compactly lower (resp., upper) semicontinuous in y if and only if the set-valued mapping $F : X \to 2^Y$ defined by $F(x) = \{y \in Y : f(x,y) \le \gamma\}$ (resp., $F(x) = \{y \in Y : f(x,y) \ge \gamma\}$) for each $x \in X$ is transfer compactly closed.*

Applying Lemmas 3.1, 3.4, and Remark 1.3, we immediately obtain the following theorem.

Theorem 3.5. *Let X be a nonempty almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, Y a topological space, and let $F \in G\text{-KKM}^*(X,Y)$ be compact. If $f,g : X \times Y \to \mathfrak{R}$ are two real-valued functions satisfying the following conditions:*

(i) *for each $x \in X$, the mapping $y \mapsto f(x,y)$ is transfer compactly lower semicontinuous on Y,*

(ii) *for each $y \in Y$, g is almost f-G-quasiconave in x,*

then for each $\xi \in \mathfrak{R}$, one of the following properties holds:

(1) *there exists $(\overline{x},\overline{y}) \in \mathcal{G}_F$ such that*

$$g(\overline{x},\overline{y}) > \xi, \tag{3.1}$$

(2) *or there exists $y' \in Y$ such that*

$$f(x,y') \le \xi, \quad \forall x \in X. \tag{3.2}$$

Proof. Let $\xi \in \mathfrak{R}$. Since F is compact, $\overline{F(X)}$ is compact in Y. Define $T,S : X \to 2^Y$ by

$$\begin{aligned} T(x) &= \{y \in \overline{F(X)} : g(x,y) \le \xi\}, \quad \forall x \in X, \\ S(x) &= \{y \in \overline{F(X)} : f(x,y) \le \xi\}, \quad \forall x \in X. \end{aligned} \tag{3.3}$$

Suppose the conclusion (1) is false. Then for each $(x,y) \in \mathcal{G}_F$, $g(x,y) \le \xi$. This implies that $\mathcal{G}_F \subset \mathcal{G}_T$.

Let $A = \{x_1,x_2,\dots,x_n\} \in \langle X \rangle$. By the condition (ii), we claim that S is a generalized G-KKM* mapping with respect to T. If the above statement is not true, then there exists $V \in \mathcal{N}$ such that for any G-convex-inducing mapping $h_{A,V} : A \to X$, one has $T(G\text{-Co}(h_{A,V}(A))) \not\subseteq S(A)$. So there exist $x_0 \in G\text{-Co}(h_{A,V}(A))$ and $y_0 \in T(x_0)$ such that $y_0 \notin S(A)$. From the definitions of T and S, it follows that $g(x_0,y_0) \le \xi$ and $f(x_i,y_0) > \xi$ for all $i = 1,2,\dots,n$. This contradicts the condition (ii). Therefore, S is a generalized G-KKM* mapping with respect to T, and so we get that S is a generalized G-KKM* mapping with respect to F. Since $F \in G\text{-KKM}^*(X,Y)$, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property, and since $\overline{S(x)}$ is compact for each $x \in X$, so we have $\bigcap_{x\in X}\overline{S(x)} \ne \phi$. From Lemmas 3.1 and 3.4, Remark 1.3, and the condition (i), we have that $\cap_{x\in X}S(x) \ne \phi$. Take $y_0 \in \bigcap_{x\in X}S(x)$, then $f(x,y_0) \le \xi$ for all $x \in X$. □

THEOREM 3.6. *If all of the assumptions of Theorem 3.5 hold, then one immediately concludes the following inequality:*

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{(x,y)\in\mathcal{G}_F} g(x,y). \tag{3.4}$$

Proof. Let $\xi = \sup_{(x,y)\in\mathcal{G}_F} g(x,y)$. Then the conclusion (1) of Theorem 3.5 is false. So there exists $y_0 \in Y$ such that $f(x,y_0) \le \xi$ for all $x \in X$. This implies that $\sup_{x\in X} f(x,y_0) \le \xi$, ad so we have $\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{(x,y)\in\mathcal{G}_F} g(x,y)$. □

COROLLARY 3.7. *Let X be a G-convex space, Y a topological space, and let $F \in G\text{-}KKM(X,Y)$ be compact. If $f,g : X\times Y \to \Re$ are two real-valued functions satisfying the following conditions:*

(i) *for each $x \in X$, the mapping $y \mapsto f(x,y)$ is transfer compactly lower semicontinuous on Y,*

(ii) *for each $y \in Y$, g is f-G-quasiconave in x,*

then for each $\xi \in \Re$, one of the following properties holds:

(1) *there exists $(\overline{x},\overline{y}) \in \mathcal{G}_F$ such that*

$$g(\overline{x},\overline{y}) > \xi, \tag{3.5}$$

(2) *or there exists $y' \in Y$ such that*

$$f(x,y') \le \xi, \quad \forall x \in X. \tag{3.6}$$

COROLLARY 3.8. *If all of the assumptions of Corollary 3.7 hold, then one immediately concludes the following inequality:*

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{(x,y)\in\mathcal{G}_F} g(x,y). \tag{3.7}$$

PROPOSITION 3.9. *Let X and Y be two G-convex spaces, and let $T,F : X \to 2^Y$ be two set-valued mappings. Then the following two statements are equivalent:*

(i) *for each $y \in Y$, if $A \in \langle T^*(y)\rangle$, then G-$Co(A) \subset F^*(y)$.*

(ii) *T is a generalized G-KKM mapping with respect to F.*

Applying Proposition 3.9, we conclude the following variational theorems and minimax inequality theorems for the Φ-mapping.

THEOREM 3.10. *Let X be a nonempty G-convex space, Y a nonempty compact G-convex space, and let $S,F : X \to 2^Y$ be two set-valued mappings satisfying the following conditions:*

(i) *F is a Φ-mapping,*

(ii) *S is transfer compactly closed valued on X,*

(iii) *for each $y \in Y$, $F^*(y)$ is G-convex,*

(iv) *for each $x \in X$, $F(x) \subset S(x)$.*

Then there exists $\overline{y} \in Y$ such that $S^*(\overline{y}) = \phi$.

Proof. By Lemma 2.3, $F \in G$-KKM(X,Y). By conditions (iii) and (iv), we have that G-Co$(S^*(y)) \subset F^*(y)$ for each $y \in Y$. So, by Proposition 3.9, S is a generalized G-KKM mapping with respect to F. Therefore, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. Since Y is compact, $\bigcap_{x\in X} \overline{S(x)} \neq \phi$. By Lemma 3.1, we have $\bigcap_{x\in X} S(x) \neq \phi$. Let $\overline{y} \in \bigcap_{x\in X} S(x)$. Then $S^*(\overline{y}) = \phi$. □

Theorem 3.11. *Let X and Y be two G-convex spaces, and let $S,T,G,H : X \to 2^Y$ be four set-valued mappings satisfying the following conditions:*

(i) *for each $x \in X$, $T(x) \subset G(x) \subset H(x) \subset S(x)$,*
(ii) *for each $y \in Y$, $H^*(y)$ is G-convex,*
(iii) *for each $x \in X$, $G(x)$ is G-convex,*
(iv) *T^{-1} is transfer compactly open valued on Y,*
(v) *S is transfer compactly closed valued on X.*

Then one has the following two properties.

(1) *If Y is compact, then there exists $\overline{y} \in Y$ such that $S^*(\overline{y}) = \phi$.*
(2) *If X is compact, then there exists $\overline{x} \in X$ such that $T(\overline{x}) = \phi$.*

Proof. Case (1). Suppose Y is compact. We define $F : X \to 2^Y$ by

$$F(x) = G\text{-Co}(T(x)), \quad \text{for each } x \in X. \tag{3.8}$$

Then F is a Φ-mapping and F^{-1} is transfer compactly open valued on Y, and so $F \in G\text{-KKM}(X,Y)$. By conditions (i), (ii), and (iii), we have $G\text{-Co}(S^*(y)) \subset H^*(y) \subset G^*(y) \subset F^*(y)$ for each $y \in Y$. Applying Proposition 3.9 and Theorem 3.10, we could conclude that there exists $\overline{y} \in Y$ such that $S^*(\overline{y}) = \phi$.

Case (2). Suppose X is compact. Conditions (i)–(v) are equivalent to the following statements:

(i) for each $y \in Y$, $S^*(y) \subset H^*(y) \subset G^*(y) \subset T^*(y)$,
(ii) for each $y \in Y$, $H^*(y)$ is G-convex,
(iii) for each $x \in X$, $(G^*)^*(x)$ is G-convex,
(iv) T^* is transfer compactly closed valued on Y,
(v) $(S^*)^{-1}$ is transfer compactly open valued on X.

We now consider the four set-valued mappings $S^*,H^*,G^*,T^* : Y \to 2^X$, then by the same process of the proof of Case (1), we also conclude that there exists $\overline{x} \in X$ such that $T(\overline{x}) = \phi$. □

Theorem 3.12. *Let X and Y be two G-convex spaces, and let $f,g,p,q : X \times Y \to \mathfrak{R}$ be four real-valued functions satisfying the following conditions:*

(i) *for each $(x,y) \in X \times Y$, $f(x,y) \le g(x,y) \le p(x,y) \le q(x,y)$,*
(ii) *for each $y \in Y$, $x \mapsto g(x,y)$ is G-quasiconcave,*
(iii) *for each $x \in X$, $y \mapsto p(x,y)$ is G-quasiconvex,*
(iv) *for each $y \in Y$, $x \mapsto q(x,y)$ is transfer compactly upper semicontinuous,*
(v) *for each $x \in X$, $y \mapsto f(x,y)$ is transfer compactly lower semicontinuous.*

Then for any $\lambda \in \mathfrak{R}$, one has the following two properties.

(1) *If Y is compact, then there exists $\overline{y} \in Y$ such that $f(x,\overline{y}) \le \lambda$ for all $x \in X$.*
(2) *If X is compact, then there exists $\overline{x} \in X$ such that $q(\overline{x},y) \ge \lambda$ for all $y \in Y$.*

Proof. Let $\lambda \in \Re$. We define $S, T, G, H : X \to 2^Y$ by

$$\begin{aligned} T(x) &= \{y \in Y : q(x,y) < \lambda\}, \\ G(x) &= \{y \in Y : p(x,y) < \lambda\}, \\ H(x) &= \{y \in Y : g(x,y) \le \lambda\}, \\ S(x) &= \{y \in Y : f(x,y) < \lambda\} \quad \text{for each } x \in X. \end{aligned} \tag{3.9}$$

Then by condition (i), $T(x) \subset G(x) \subset H(x) \subset S(x)$ for each $x \in X$. Conditions (ii) and (iii) imply that $G(x)$ is G-convex for all $x \in X$ and $H^*(y)$ is G-convex for all $y \in Y$. Conditions (iv) and (v) imply that T^{-1} is transfer compactly open valued on Y and S is transfer compactly closed valued on X. So all the conditions of Theorem 3.10 are satisfied. Therefore, we have the following properties.

(1) If Y is compact, then there exists $\overline{y} \in Y$ such that $S^*(\overline{y}) = \phi$, that is, there exists $\overline{y} \in Y$ such that $f(x,\overline{y}) \le \lambda$ for all $x \in X$.

(2) If X is compact, then there exists $\overline{x} \in X$ such that $T(\overline{x}) = \phi$, that is, there exists $\overline{x} \in X$ such that $q(\overline{x}, y) \ge \lambda$ for all $y \in Y$. □

Following Theorem 2.8, we also have the variational inequality theorem and minimax inequality theorem.

THEOREM 3.13. *Let X be a nonempty G-convex subset of a locally G-convex space E, and Y a compact topological space. If $f, g, p, q : X \times Y \to \Re$ are four real-valued functions, and a, b are two real numbers, suppose the following conditions hold:*

(i) *$g(x,y) \le f(x,y)$ and $p(x,y) \le q(x,y)$ for all $x \in X$, $y \in Y$,*

(ii) *for each $x \in X$, $y \mapsto f(x,y)$ is G-quasiconcave on Y and for each $y \in Y$, $x \mapsto p(x,y)$ is G-quasiconvex on X,*

(iii) *for each $y \in Y$, $x \mapsto g(x,y)$ is transfer compactly lower semicontinuous and for each $x \in X$, $y \mapsto q(x,y)$ is transfer compactly upper semicontinuous in Y,*

(iv) *f is upper semicontinuous on $X \times Y$.*

Then one of the following statesment holds:

(1) *there exists $\mu \in X$ such that $g(\mu, y) \le a$ for each $y \in Y$,*

(2) *there exists $\nu \in Y$ such that $q(x,\nu) \ge b$ for each $x \in X$,*

(3) *there exists $(\mu,\nu) \in X \times Y$ such that $f(\mu,\nu) \ge a$ and $p(\mu,\nu) < b$.*

Proof. Let $S, T : X \to 2^Y$ and $H, F : Y \to 2^X$ be defined by

$$\begin{aligned} Sx &= \{y \in Y : g(x,y) - a > 0\}, \quad \text{for each } x \in X, \\ Tx &= \{y \in Y : f(x,y) - a \ge 0\}, \quad \text{for each } x \in X, \\ Hy &= \{x \in X : q(x,y) - b < 0\}, \quad \text{for each } y \in Y, \\ Fy &= \{x \in X : p(x,y) - b \le 0\}, \quad \text{for each } y \in Y. \end{aligned} \tag{3.10}$$

By the assumption (i), we have that $Sx \subset Tx$ for each $x \in X$, and by (ii), Tx is G-convex for each $x \in X$, and so $G\text{-Co}(Sx) \subset Tx$ for each $x \in X$. By the assumption (iii), S^{-1} is transfer compactly open valued on Y. Similarly, by (ii) and (iii), we have $G\text{-Co}(Hy) \subset Fy$ for each $y \in Y$ and H^{-1} is transfer compactly open valued on X.

Suppose that the conditions (1) and (2) are false. Then $Sx \neq \phi$ for each $x \in X$ and $Hy \neq \phi$ for each $y \in Y$. So, we conclude that T is a Φ-mapping with a companion mapping S and F is a Φ-mapping with a companion mapping H. By the assumption (iv), T is closed. Hence, all of the assumptions of Theorem 2.8 hold, and so there exists $(\mu, \nu) \in X \times Y$ such that $\nu \in T(\mu)$ and $\mu \in F(\nu)$, that is, $f(\mu, \nu) \geq a$ and $p(\mu, \nu) < b$. □

Theorem 3.14. *Let X be a nonempty G-convex subset of a locally G-convex space E, Y a compact topological space. If $f, g, p, q : X \times Y \to \Re$ are four real-valued functions, and a, b are two real numbers, suppose the following conditions hold:*

(i) *$g(x, y) \leq f(x, y) \leq p(x, y) \leq q(x, y)$ for all $x \in X$, $y \in Y$,*

(ii) *for each $x \in X$, $y \mapsto f(x, y)$ is G-quasiconcave on Y and for each $y \in Y$, $x \mapsto P(x, y)$ is G-quasiconvex on X,*

(iii) *for each $y \in Y$, $x \mapsto g(x, y)$ is transfer compactly lower semicontinuous and for each $x \in X$, $y \mapsto q(x, y)$ is transfer compactly upper semicontinuous in Y,*

(iv) *f is upper semicontinuous on $X \times Y$.*

Then

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \inf_{x \in X} q(x, y). \tag{3.11}$$

Proof. Let $\varepsilon > 0$ and let

$$a = \inf_{x \in X} \sup_{y \in Y} g(x, y) - \varepsilon, \qquad b = \sup_{y \in Y} \inf_{x \in X} q(x, y) + \varepsilon. \tag{3.12}$$

Then for each $x \in X$, there exists $y \in Y$ such that $g(x, y) > a$, and for each $y \in Y$, there exist $x \in X$ such that $q(x, y) < b$. Therefore, the conclusions (1) and (2) of Theorem 3.13 are false. So there exist $\mu \in X$ and $\nu \in Y$ such that $f(\mu, \nu) \geq a$ and $p(\mu, \nu) < b$, that is

$$f(\mu, \nu) \geq \inf_{x \in X} \sup_{y \in Y} g(x, y) - \varepsilon, \qquad p(\mu, \nu) < \sup_{y \in Y} \inf_{x \in X} q(x, y) + \varepsilon. \tag{3.13}$$

So by (i), we have

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) - \varepsilon < \sup_{y \in Y} \inf_{x \in X} q(x, y) + \varepsilon. \tag{3.14}$$

Since ε is an arbitrary positive number, by letting $\varepsilon \downarrow 0$, we get

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \inf_{x \in X} q(x, y). \tag{3.15}$$

□

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