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Research Article

Data Dependence for Ishikawa Iteration When Dealing with Contractive-Like Operators

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We prove a convergence result and a data dependence for Ishikawa iteration when applied to contraction-like operators. An example is given, in which instead of computing the fixed point of an operator, we approximate the operator with a contractive-like one. For which it is possible to compute the fixed point, and therefore to approximate the fixed point of the initial operator.

1. Introduction

Let X be a real Banach space; let $B \subset X$ be a nonempty convex closed and bounded set. Let $T, S : B \to B$ be two maps. For a given $x_0, u_0 \in B$, we consider the Ishikawa iteration (see [1]) for T and S:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad y_n = (1-\beta_n)x_n + \beta_n T x_n, \tag{1.1}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n S v_n, \quad v_n = (1-\beta_n)u_n + \beta_n S u_n, \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset 0,1)$, and

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0, \ \sum_{n=1}^{\infty}\alpha_n = \infty. \tag{1.3}$$

Set $\beta_n = 0$, $\forall n \in N$, to obtain the Mann iteration, see [2].

The map T is called Kannan mappings, see [3], if there exists $b \in (0, 1/2)$ such that for all $x, y \in B$,

$$\|Tx - Ty\| \le b\left(\|x - Tx\| + \|y - Ty\|\right). \tag{1.4}$$

Similar mappings are Chatterjea mappings, see [4], for which there exists $c \in (0, 1/2)$ such that for all $x, y \in B$,

$$\|Tx - Ty\| \le c\left(\|x - Ty\| + \|y - Tx\|\right). \tag{1.5}$$

Zamfirescu collected these classes. He introduced the following definition, see [5].

Definition 1.1 (see [5, 6]). The operator $T : X \to X$ satisfies condition Z (Zamfirescu condition) if and only if there exist the real numbers a, b, c satisfying $0 < a < 1$, $0 < b,\ c < 1/2$ such that for each pair x, y in X, at least one condition is true:

(i) (z_1) $\|Tx - Ty\| \le a\|x - y\|$,

(ii) (z_2) $\|Tx - Ty\| \le b(\|x - Tx\| + \|y - Ty\|)$,

(iii) (z_3) $\|Tx - Ty\| \le c(\|x - Ty\| + \|y - Tx\|)$.

It is known, see Rhoades [7], that (z_1), (z_2), and (z_3) are independent conditions. Consider $x, y \in B$. Since T satisfies condition Z, at least one of the conditions from (z_1), (z_2), and (z_3) is satisfied. If (z_2) holds, then

$$\|Tx - Ty\| \le b\left(\|x - Tx\| + \|y - Ty\|\right) \le b\big(\|x - Tx\| + \left(\|y - x\| + \|x - Tx\| + \|Tx - Ty\|\right)\big). \tag{1.6}$$

Thus

$$(1 - b)\|Tx - Ty\| \le b\|x - y\| + 2b\|x - Tx\|. \tag{1.7}$$

From $0 \le b < 1$ one obtains,

$$\|Tx - Ty\| \le \frac{b}{1 - b}\|x - y\| + \frac{2b}{1 - b}\|x - Tx\|. \tag{1.8}$$

If (z_3) holds, then one gets

$$\|Tx - Ty\| \le c\left(\|x - Ty\| + \|y - Tx\|\right) \le c\big(\|x - Tx\| + \|Tx - Ty\| + \|x - y\| + \|x - Tx\|\big) \tag{1.9}$$

Hence,

$$(1 - c)\|Tx - Ty\| \le c\|x - y\| + 2c\|x - Tx\|, \tag{1.10}$$

that is,

$$\|Tx - Ty\| \le \frac{c}{1 - c}\|x - y\| + \frac{2c}{1 - c}\|x - Tx\|. \tag{1.11}$$

Denote

$$\delta := \max\left\{a, \frac{b}{1 - b}, \frac{c}{1 - c}\right\}, \tag{1.12}$$

to obtain

$$0 \le \delta < 1. \tag{1.13}$$

Finally, we get

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|x - Tx\|, \quad \forall x, y \in B. \tag{1.14}$$

Formula (1.14) was obtained as in [8].

Osilike and Udomene introduced in [9] a more general definition of a quasicontractive operator; they considered the operator for which there exists $L \ge 0$ and $q \in (0,1)$ such that

$$\|Tx - Ty\| \le q\|x - y\| + L\|x - Tx\|, \quad \forall x, y \in B. \tag{1.15}$$

Imoru and Olatinwo considered in [10], the following general definition. Because they failed to name them, we will call them here contractive-like operators.

Definition 1.2. One calls *contractive-like* the operator T if there exist a constant $q \in 0,1)$ and a strictly increasing and continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in X$,

$$\|Tx - Ty\| \le q\|x - y\| + \phi(\|x - Tx\|). \tag{1.16}$$

In both papers [9, 10], the T-stability of Picard and Mann iterations was studied.

2. Preliminaries

The data dependence abounds in literature of fixed point theory when dealing with Picard-Banach iteration, but is quasi-inexistent when dealing with Mann-Ishikawa iteration. As far as we know, the only data-dependence result concerning Mann-Ishikawa iteration is in [11]. There, the data dependence of Ishikawa iteration was proven when applied to contractions. In this note, we will prove data-dependence results for Ishikawa iteration when applied to the above contractive-like operators. Usually, Ishikawa iteration is more complicated but nevertheless more stable as Mann iteration. There is a classic example, see [12], in which Mann iteration does not converge while Ishikawa iteration does. This is the main reason for considering Ishikawa iteration in Theorem 3.2.

The following remark is obvious by using the inequality $(1 - x) \le \exp(x)$, $\forall x \ge 0$.

Remark 2.1. Let $\{\theta_n\}$ be a nonnegative sequence such that $\theta_n \in (0,1]$, $\forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\theta_n = \infty$, then $\prod_{n=1}^{\infty}(1 - \theta_n) = 0$.

The following is similar to lemma from [13]. (Note that another proof for this lemma [13] can be found in [11].)

Lemma 2.2. *Let $\{a_n\}$ be a nonnegative sequence for which one supposes there exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ one has satisfied the following inequality:*

$$a_{n+1} \le (1 - \lambda_n)a_n + \lambda_n\sigma_n, \tag{2.1}$$

where $\lambda_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty}\lambda_n = \infty$, and $\sigma_n \ge 0$ $\forall n \in \mathbb{N}$. Then,

$$0 \le \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} \sigma_n. \tag{2.2}$$

Proof. There exists $n_1 \in \mathbb{N}$ such that $\sigma_n \le \limsup \sigma_n$, $\forall n \ge n_1$. Set $n_2 = \max\{n_0, n_1\}$ such that the following inequality holds, for all $n \ge n_2$:

$$a_{n+1} \le (1-\lambda_n)(1-\lambda_{n-1})\cdots(1-\lambda_{n_1})a_{n_1} + \limsup_{n\to\infty} \sigma_n. \tag{2.3}$$

Using the above Remark 2.1 with $\theta_n = \lambda_n$, we get the conclusion. In order to prove (2.3), consider (2.1) and the induction step:

$$\begin{aligned} a_{n+2} &\le (1-\lambda_{n+1})a_{n+1} + \lambda_{n+1}\sigma_{n+1} \le (1-\lambda_{n+1})(1-\lambda_n)(1-\lambda_{n-1})\cdots(1-\lambda_{n_1})a_{n_1} \\ &\quad + (1-\lambda_{n+1})\limsup_{n\to\infty} \sigma_n + \lambda_{n+1}\sigma_{n+1} \\ &= (1-\lambda_{n+1})(1-\lambda_n)(1-\lambda_{n-1})\cdots(1-\lambda_{n_1})a_{n_1} + \limsup_{n\to\infty} \sigma_n. \end{aligned} \tag{2.4}$$

□

3. Main results

Theorem 3.1. *Let X be a real Banach space, $B \subset X$ a nonempty convex and closed set, and $T : B \to B$ a contractive-like map with x^* being the fixed point. Then for all $x_0 \in B$, the iteration* (1.1) *converges to the unique fixed point of T.*

Proof. The uniqueness comes from (1.16); supposing we have two fixed points x^* and y^*, we get

$$\|x^* - y^*\| = \|Tx^* - Ty^*\| \le q\|x^* - y^*\| + \phi(\|x^* - Tx^*\|) = q\|x^* - y^*\|, \tag{3.1}$$

that is, $(1-q)\|x^* - y^*\| = 0$. From (1.1) and (1.16) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|Ty_n - Tx^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n q\|y_n - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n q(1-\beta_n)\|x_n - x^*\| + q\alpha_n\beta_n\|Tx_n - Tx^*\| \\ &\le (1-\alpha_n(1-q))(1-(1-q)\beta_n))\|x_n - x^*\| \\ &\le (1-\alpha_n(1-q))\|x_n - x^*\| \le \cdots \le \left(\prod_{k=1}^{n}(1-\alpha_k q)\right)\|x_0 - x^*\|. \end{aligned} \tag{3.2}$$

Use Remark 2.1 with $\theta_k = \alpha_k q$ to obtain the conclusion. □

This result allows us to formulate the following data dependence theorem.

Theorem 3.2. *Let X be a real Banach space, let $B \subset X$ be a nonempty convex and closed set, and let $\varepsilon > 0$ be a fixed number. If $T : B \to B$ is a contractive-like operator with the fixed point x^* and $S : B \to B$ is an operator with a fixed point u^*, (supposed nearest to x^*) , and if the following relation is satisfied:*

$$\|Tz - Sz\| \le \varepsilon, \quad \forall z \in B, \tag{3.3}$$

then

$$\|x^* - u^*\| \le \frac{\varepsilon}{1-q}. \tag{3.4}$$

Proof. From (1.1) and (1.2), we have

$$x_{n+1} - u_{n+1} = (1-\alpha_n)(x_n - u_n) + \alpha_n(Ty_n - Sv_n). \tag{3.5}$$

Thus

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &= \|(1-\alpha_n)(x_n - u_n) + \alpha_n(Sv_n - Ty_n)\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|Sv_n - Tv_n + Tv_n - Ty_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\|Tv_n - Sv_n\| + \alpha_n\|Tv_n - Ty_n\| \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varepsilon + q\alpha_n\|y_n - v_n\| + \alpha_n\phi(\|y_n - Ty_n\|) \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varepsilon + q\alpha_n(1-\beta_n)\|x_n - u_n\| + q\alpha_n\beta_n\|Tx_n - Su_n\| + \alpha_n\phi(\|y_n - Ty_n\|) \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varepsilon + q\alpha_n(1-\beta_n)\|x_n - u_n\| \\
&\quad + \alpha_n\beta_n q(\|Tx_n - Tu_n\| + \|Tu_n - Su_n\|) + \alpha_n\phi(\|y_n - Ty_n\|) \\
&\le (1-\alpha_n)\|x_n - u_n\| + \alpha_n\varepsilon + q\alpha_n(1-\beta_n)\|x_n - u_n\| \\
&\quad + q^2\alpha_n\beta_n\|x_n - u_n\| + q\alpha_n\beta_n\phi(\|x_n - Tx_n\|) + q\alpha_n\beta_n\varepsilon + \alpha_n\phi(\|y_n - Ty_n\|) \\
&= (1 - \alpha_n(1 - q(1-\beta_n) - \beta_n q^2))\|x_n - u_n\| + \alpha_n\varepsilon + q\alpha_n\beta_n\varepsilon \\
&\quad + q\alpha_n\beta_n\phi(\|x_n - Tx_n\|) + \alpha_n\phi(\|y_n - Ty_n\|) \\
&= (1-\alpha_n(1-q)(1+q\beta_n))\|x_n - u_n\| + \alpha_n(q\beta_n\phi(\|x_n - Tx_n\|) + \phi(\|y_n - Ty_n\|) + q\beta_n\varepsilon + \varepsilon) \\
&\le (1-\alpha_n(1-q))\|x_n - u_n\| + (\alpha_n(1-q))\frac{q\beta_n\phi(\|x_n - Tx_n\|) + \phi(\|y_n - Ty_n\|) + q\beta_n\varepsilon + \varepsilon}{1-q}.
\end{aligned} \tag{3.6}$$

Note that $\lim_{n\to\infty}\phi(\|x_n - Tx_n\|) = \lim_{n\to\infty}\phi(\|y_n - Ty_n\|) = 0$ because ϕ is a continuous map and both $\{x_n\}$, $\{y_n\}$ converge to the fixed point of T. Set

$$\begin{aligned}
\lambda_n &:= \alpha_n(1-q), \\
\sigma_n &:= \frac{q\beta_n\phi(\|x_n - Tx_n\|) + \phi(\|y_n - Ty_n\|) + q\beta_n\varepsilon + \varepsilon}{1-q},
\end{aligned} \tag{3.7}$$

and use Lemma 2.2 to obtain the conclusion

$$\|x^* - u^*\| \leq \frac{\varepsilon}{1-q}. \tag{3.8}$$

□

Remark 3.3. (i) Set $\beta_n = 0$, $\forall n \in \mathbb{N}$, to obtain the data dependence for Mann iteration.

(ii) The Zamfirescu operators and implicitly (Chatterjea and Kannan) are contractive-like operators, therefore our Theorem 3.2 remains true for these classes.

4. Numerical example

The following example follows the example from [8].

Example 4.1. Let $T : \mathbb{R}\to\mathbb{R}$ be given by

$$\begin{aligned} Tx &= 0, \quad \text{if } x \in (-\infty, 2] \\ &= -0.5, \quad \text{if } x \in (2, +\infty). \end{aligned} \tag{4.1}$$

Then T is contractive-like operator with $q = 0.2$ and ϕ = identity.

Note the unique fixed point is 0. Consider now the map $S : R\to R$,

$$\begin{aligned} Sx &= 1, \quad \text{if } x \in (-\infty, 2] \\ &= -1.5, \quad \text{if } x \in (2, +\infty) \end{aligned} \tag{4.2}$$

with the unique fixed point 1. Take ε to be the distance between the two maps as follows:

$$\|Sx - Tx\| \leq 1, \quad \forall x \in \mathbb{R}. \tag{4.3}$$

Set $u_0 = x_0 = 0$, $\alpha_n = \beta_n = 1/(n+1)$. Independently of above theory, the Ishikawa iteration applied to S, leads to

Iteration step	Ishikawa iteration
1	0.5
10	0.9
100	0.99

(4.4)

Note that for $n = 1$,

$$0.5 = \frac{1}{n+1}0 + \frac{1}{n+1}S\left(\frac{1}{2}\right), \tag{4.5}$$

since $y_1 = (1/(n+1))0 + (1/n+1)1 = 1/2$. (The above computations can be obtained also by using a Matlab program.) This leads us to "conclude" that Ishikawa iteration applied to S converges to fixed point, ($x^* = 1$). Eventually, one can see that the distance between the two fixed points is one. Actually, without knowing the fixed point of S (and without computing it), via Theorem 3.2, we can do the following estimate for it:

$$\|x^* - u^*\| \leq \frac{1}{1-q} = \frac{1}{1-0.2} = \frac{10}{8} = 1.2. \tag{4.6}$$

As a conclusion, instead of computing fixed points of S, choose T more closely to S and the distance between the fixed points will shrink too.

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