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Research Article

Best Proximity Sets and Equilibrium Pairs for a Finite Family of Multimaps

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We establish the existence of a best proximity pair for which the best proximity set is nonempty for a finite family of multimaps whose product is either an $\mathfrak{A}_c^\kappa$-multimap or a multimap $T : A \to 2^B$ such that both T and $S \circ T$ are closed and have the **KKM** property for each Kakutani multimap $S : B \to 2^A$. As applications, we obtain existence theorems of equilibrium pairs for free n-person games as well as for free 1-person games. Our results extend and improve several well-known and recent results.

1. Introduction

Let $E := (E, \|\cdot\|)$ be a normed space, A a nonempty subset of E, and $T : A \to E$ a single-valued map. Whenever the equation $T(x) = x$ has no solution in A, it is natural to ask if there exists an approximate solution. Fan [1] provided sufficient conditions for the existence of an approximate solution $a \in A$ (called a best approximant) such that

$$\|a - T(a)\| = d(T(a), A) := \inf\{d(T(a), x) : x \in A\}, \tag{1.1}$$

where A is compact and convex and T is continuous. However, there is no guarantee that such an approximate solution is optimal. For suitable subsets A and B of E and multimap $T : A \to 2^B$, Sadiq Basha and Veeramani [2] provided sufficient conditions for the existence of an optimal solution $(a, T(a))$ (called a best proximity pair) such that

$$d(a, T(a)) = d(A, B) := \inf\{\|x - y\| : x \in A,\ y \in B\}. \tag{1.2}$$

Srinivasan and Veeramani [3, 4] extended these results and obtained existence theorems of equilibrium pairs for constrained generalized games. Kim and Lee [5, 6] generalized

Srinivasan and Veeramani results and obtained existence theorems of equilibrium pairs for free n-person games. Recently, Al-Thagafi and Shahzad [7] generalized and extended the above results to Kakutani multimaps.

In this paper, we establish the existence of a best proximity pair for which the best proximity set is nonempty for a finite family of multimaps whose product is either an $\mathfrak{A}_c^\kappa$-multimap or a multimap $T : A \to 2^B$ such that both T and $S \circ T$ are closed and have the **KKM** property for each Kakutani multimap $S : B \to 2^A$. As applications, we obtain existence theorems of equilibrium pairs for free n-person games as well as free 1-person games. Our results extend and improve several well-known and recent results.

2. Preliminaries

Throughout, $E := (E, \|\cdot\|)$ is a normed space, A and B are nonempty subsets of E, 2^A is the family of all subsets of A, $\mathrm{co}A$ is the convex hull of A in E, $\operatorname{int} A$ is the interior of A in E, $\mathfrak{C}(A, B)$ is the set of all continuous single-valued maps, $d(x, A) := \inf\{d(x, a) : a \in A\}$, and $d(A, B) := \inf\{\|a - b\| : a \in A \text{ and } b \in B\}$. A map $T : A \to 2^B$ is called a multimap (multifunction or correspondence) if $T(x)$ is nonempty for each $x \in A$. A multimap $T : A \to 2^A$ is said to have a fixed point $a \in A$ if $a \in T(a)$; the set of fixed points of T is denoted by $F(T)$. A multimap $T : A \to 2^B$ is said to be (a) upper semicontinuous if $T^{-1}(D) = \{x \in A : T(x) \cap D \neq \emptyset\}$ is closed in A whenever D is closed in B, (b) compact if $\overline{T(A)}$ is compact in B, (c) closed if its graph $\mathrm{Gr}(T) := \{(x, y) : x \in A \text{ and } y \in T(x)\}$ is closed in $A \times B$ and (d) compact-valued (resp., convex) if $T(x)$ is compact (resp., convex) in B for every $x \in A$. A map $f : A \to B$ is proper if $f^{-1}(K)$ is compact in A whenever K is compact in B. A map $f : A \to E$ is quasiaffine if the set $Q(x) := \{a \in A : \|f(a) - x\| \leq r\}$ is convex for every $x \in E$ and $r \in [0, \infty)$.

Lemma 2.1 (see [8]). *Let A and B be nonempty subsets of a normed space E. If $T : A \to 2^B$ is an upper semicontinuous multimap with compact values, then T is closed.*

The set of all $a \in A$ such that $\|a - x\| = d(x, A)$, denoted by $P_A(x)$, is called the set of best approximations in A to $x \in E$. The multimap $P_A : E \to 2^A$ is called the metric projection on A. Whenever A is compact and convex, P_A is upper semicontinuous with compact and convex values (see [8]).

A polytope P in A is any convex hull of a nonempty finite subset D of A. Whenever $\mathfrak{X}$ is a class of maps, denote the set of all finite compositions of maps in $\mathfrak{X}$ by $\mathfrak{X}_c$ and denote the set of all multimaps $T : A \to 2^B$ in $\mathfrak{X}$ by $\mathfrak{X}(A, B)$. Let $\mathfrak{A}$ be an abstract class of maps [9] satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathfrak{C}$ of continuous single-valued maps;

(2) each $T \in \mathfrak{A}_c$ is upper semicontinuous with compact values;

(3) for any polytope P, each $T \in \mathfrak{A}_c(P, P)$ has a fixed point.

Let $T : A \to 2^B$. We say that (e) T is an $\mathfrak{A}_c^\kappa$-multimap [9] if for every compact set K in A, there exists an $\mathfrak{A}_c$-multimap $f : K \to 2^B$ such that $f(x) \subseteq T(x)$ for each $x \in K$, (f) T is a **K**-multimap (or Kakutani multimap) [10] if T is upper semicontinuous with compact and convex values, (g) $S : A \to 2^B$ is a generalized **KKM**-multimap with respect to T [11] if $T(\mathrm{co}D) \subseteq S(D)$ for each finite subset D of A, (h) T has the **KKM** property [11] if, whenever $S : A \to 2^B$ is a generalized **KKM** multimap w.r.t. T, the family $\{\overline{S(x)} : x \in A\}$ has the finite intersection property; (i) T is a **PK**-multimap [12] if there exists a multimap $g : A \to 2^B$ satisfying $A = \bigcup\{\operatorname{int} g^{-1}(y) : y \in B\}$ and $\operatorname{co}(g(x)) \subseteq T(x)$ for every $x \in A$. Note that each $\mathfrak{A}_c^\kappa$-multimap has the **KKM** property and each **K**-multimap (resp., $\mathfrak{A}_c$-multimap, **PK**-multimap) is an $\mathfrak{A}_c^\kappa$-multimap (see [9, 13, 14]).

Let A and B_i be nonempty subsets of a normed space E for each $i \in I_n := \{1,2,\dots,n\}$. Define

$$A_i^0 := \{a \in A : \|a-b\| = d(A,B_i) \text{ for some } b \in B_i\},$$
$$B_i^0 := \{b \in B_i : \|a-b\| = d(A,B_i) \text{ for some } a \in A\}, \tag{2.1}$$

$A^0 := \bigcap_{i\in I_n} A_i^0$. For $n = 1$, let $A_0 := A_1^0 = A^0$ and $B_0 := B_1^0$.

The following result is a part of [7, Theorem 3.1].

Lemma 2.2. *Let A and B_i be nonempty subsets of E for each $i \in I_n$:*

(a) $P_A(B_i^0) = P_{A_i^0}(B_i^0) = A_i^0$;

(b) *if A_i^0 and B_i are compact (resp., convex), then B_i^0 is compact (resp., convex);*

(c) *if A_i^0 is nonempty, compact, and convex and B_i^0 is convex, then $P_{A_i^0}|_{B_i^0}$ is a **K**-multimap.*

Remark 2.3. We note, from part (a) of Lemma 2.2 and the definitions of A^0, A_i^0, and B_i^0, that

(a_1) A_i^0 is nonempty if and only if B_i^0 is nonempty;

(a_2) $P_A(B_i^0) = A^0$ if and only if $A_i^0 = A^0$; so [5, Theorems 1, 2, and 4] by Kim and Lee are valid only whenever $A_i^0 = A^0$;

(a_3) $\bigcap_{i=1}^n P_A(B_i^0) = \bigcap_{i=1}^n P_{A_i^0}(B_i^0) = \bigcap_{i=1}^n A_i^0 = A^0$. So $A^0 \neq \varnothing$ if and only if $\bigcap_{i=1}^n P_{A_i^0}(y_i) \neq \varnothing$ for some $(y_1,\dots,y_n) \in \prod_{i=1}^n B_i^0$.

Lemma 2.4 (see [11, 14]). *Let A be a nonempty convex subset of a normed space E. If $T : A \to 2^A$ is a closed and compact multimap having the **KKM** property, then T has a fixed point.*

Lemma 2.5 (see [15]). *For each $i \in I_n$, let B_i be a nonempty, compact, and convex subset of a normed space E, $P_i : \prod_{j=1}^n B_j \to 2^{B_i}$ a map such that*

(a) *$x_i \notin \operatorname{co}P_i(x)$ for each $x = (x_1,\dots,x_n) \in B := \prod_{j=1}^n B_j$;*

(b) *$P_i^{-1}(y)$ is open in B for each $y \in B_i$.*

Then there exists $b \in B$ such that $P_i(b) = \varnothing$ for each $i \in I_n$.

Lemma 2.6 (see [5, 6, 15, 16]). *Let B be a nonempty, compact, and convex subset of a normed space E and $P : B \to 2^B$ a map such that*

(a) *$x \notin \operatorname{co}P(x)$ for each $x \in B$.*

Assume that one of the following conditions is satisfied:

(b_1) *if $z \in P^{-1}(y)$, then there exists some $y' \in B$ such that $z \in \operatorname{int} P^{-1}(y')$;*

(b_2) *$P^{-1}(y)$ is open in B for each $y \in B$.*

Then there exists $b \in B$ such that $P(b) = \varnothing$.

3. Best proximity results

Lemma 3.1. *Let A and B_i be subsets of a normed space E such that A_i^0 (resp., B_i^0) are nonempty, compact (resp., closed), and convex for each $i \in I_n$. Suppose that $f : A^0 \to A^0$ is a continuous, proper, quasiaffine, and surjective self-map, and $P : Y \to 2^{A^0}$ is a multimap defined by $P(y_1,\dots,y_n) := \bigcap_{i=1}^n P_{A_i^0}(y_i)$ for each $(y_1,\dots,y_n) \in Y := \prod_{i=1}^n B_i^0$. Then $f^{-1}P : Y \to 2^{A^0}$ is a $\mathbf{K}$-multimap.*

Proof. Fix $i \in I_n$. Since A_i^0 is compact and convex, then $P_{A_i^0} : E \to 2^{A_i^0}$ is a $\mathbf{K}$-multimap. As B_i^0 is closed, we conclude, from Lemma 2.2(c), that $P_{A_i^0}|_{B_i^0}$ is a $\mathbf{K}$-multimap and, hence, $P : Y \to 2^{A^0}$ is a $\mathbf{K}$-multimap. Let $S := f^{-1}P$. As f is surjective and

$$S(Y) = f^{-1}P(Y) \subseteq f^{-1}(A^0) = A^0, \tag{3.1}$$

then $S : Y \to 2^{A^0}$ is a multimap. To show that S is upper semicontinuous, let D be a closed subset of A^0 and let $\{y_m\}$ be a sequence in $S^{-1}(D)$ such that $y_m = (y_{m1},\dots,y_{mn}) \to y = (y_1,\dots,y_n) \in Y$ as $m \to \infty$. Choose a sequence $\{x_m\}$ in D such that $x_m \in S(y_m)$. Then $f(x_m) \in P(y_m) \subseteq A^0$ for each $m \geq 1$. As D is compact, we may assume that $x_m \to x \in D$ as $m \to \infty$. The continuity of f and the compactness of A^0 imply that $f(x_m) \to f(x) \in A^0$ as $m \to \infty$. Since $f(x_m) \in P_{A_i^0}(y_{mi})$, it follows that

$$\begin{aligned} \|f(x) - y_i\| &\leq \|f(x) - f(x_m)\| + \|f(x_m) - y_{mi}\| + \|y_{mi} - y_i\| \\ &= \|f(x) - f(x_m)\| + d(y_{mi}, A_i^0) + \|y_{mi} - y_i\| \end{aligned} \tag{3.2}$$

for each m. Letting $m \to \infty$, we obtain $\|f(x) - y_i\| = d(y_i, A_i^0)$. This implies that $f(x) \in P_{A_i^0}(y_i)$ and hence $f(x) \in P(y)$. From this, we conclude that $x \in S(y) \cap D$ and $y \in S^{-1}(D)$. Therefore, $S^{-1}(D)$ is closed and hence S is upper semicontinuous.

Notice, as f is proper and $P(y)$ is compact, that $S(y)$ is compact. Also, as f is quasiaffine, the set

$$Q(y_i) := \{a \in A^0 : \|f(a) - y_i\| = d(y_i, A_i^0)\} \tag{3.3}$$

is convex. For $a_1, a_2 \in S(y)$, we have $f(a_1), f(a_2) \in P(y)$ and hence $f(a_1), f(a_2) \in P_{A_i^0}(y_i)$. This implies that $a_1, a_2 \in Q(y_i)$ and, by the convexity of $Q(y_i)$, $y_\lambda := \lambda a_1 + (1-\lambda)a_2 \in Q(y_i)$ for each $\lambda \in [0,1]$. Thus $f(y_\lambda) \in P_{A_i^0}(y_i)$ and hence $f(y_\lambda) \in P(y)$. From this, we conclude that $y_\lambda \in S(y)$ and hence $S(y)$ is convex. Therefore, $S : Y \to 2^{A^0}$ is a $\mathbf{K}$-multimap. □

Definition 3.2. Let A and B_i be nonempty subsets of a normed space E, $T_i : A \to 2^{B_i}$ a multimap for each $i \in I_n$, $f : A' \to A'$ a self-map of a nonempty subset A' of A, and $a \in A$. If $d(f(a), T_i(a)) = d(A, B_i)$, one says that $(f(a), T_i(a))$ is a best proximity pair. The best proximity set for the pair $(f(a), T_i(a))$ is given by

$$\mathfrak{T}_a^i(f) := \{b \in T_i(a) : d(f(a), T_i(a)) = \|f(a) - b\| = d(A, B_i)\}. \tag{3.4}$$

For $n = 1$, let $\mathfrak{T}_a(f) := \mathfrak{T}_a^1(f)$. Whenever f is the identity map, we write $\mathfrak{T}_a^i$ instead of $\mathfrak{T}_a^i(f)$.

Definition 3.3. Let $T : A \to 2^B$ be a multimap. One says that T is a $\mathbf{KKM}_0$-multimap if T and $S \circ T : A \to 2^A$ are closed and have the **KKM** property for each **K**-multimap $S : B \to 2^A$.

Theorem 3.4. *Let A and B_i be subsets of a normed space E, A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, and $T_i : A \to 2^{B_i}$ a multimap for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1,\ldots,y_n) \in Y$ and $T : A^0 \to 2^Y$ is a $\mathbf{KKM}_0$-multimap (resp., $\mathfrak{A}_{\mathbf{c}}^{\kappa}$-multimap) where $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$ and $Y := \prod_{i=1}^n B_i^0$. Then, for each continuous, proper, quasiaffine, and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed.*

Proof. Fix $i \in I_n$. Define $P : Y \to 2^{A^0}$ by $P(y_1,\ldots,y_n) := \bigcap_{i=1}^n P_{A_i^0}(y_i)$ for each $(y_1,\ldots,y_n) \in Y$. Let $f : A^0 \to A^0$ be a continuous, proper, and quasiaffine self-map. As $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1,\ldots,y_n) \in \prod_{i=1}^n B_i^0$, it follows from Lemma 3.1 that $f^{-1}P : Y \to 2^{A^0}$ is a **K**-multimap. Now, assume that $T : A^0 \to 2^Y$ is a $\mathbf{KKM}_0$-multimap. It follows from the definition of a $\mathbf{KKM}_0$-multimap that $f^{-1}P \circ T : A^0 \to 2^{A^0}$ is a closed multimap having the **KKM** property. As A^0 is a compact set, $f^{-1}P \circ T$ is a compact multimap. By Lemma 2.4, there exists $a \in A^0$ such that $a \in (f^{-1}P \circ T)(a)$ and hence $f(a) \in P(T(a))$. Thus, there exists $(b_1,\ldots,b_n) \in T(a) = \prod_{i=1}^n T_i(a)$ such that $f(a) \in P(b_1,\ldots,b_n) = \bigcap_{i=1}^n P_{A_i^0}(b_i) \subseteq A^0$. Hence, $f(a) \in P_{A_i^0}(b_i) \subseteq A_i^0$ and $b_i \in T_i(a) \subseteq B_i^0$. This implies that there exists $a_i' \in A_i^0$ such that $\|a_i' - b_i\| = d(A, B_i)$ and hence

$$d(A,B_i) \leq d(f(a),T_i(a)) \leq \|f(a) - b_i\| = d(b_i, A_i^0) \leq \|a_i' - b_i\| = d(A,B_i). \tag{3.5}$$

Thus $d(f(a),T_i(a)) = \|f(a) - b_i\| = d(A,B_i)$.

Next, assume that $T : A^0 \to 2^Y$ is an $\mathfrak{A}_{\mathbf{c}}^{\kappa}$-multimap. Then, there exists an $\mathfrak{A}_{\mathbf{c}}$-multimap $T' : A^0 \to 2^Y$ such that T' is upper semicontinuous with compact values and $T'(x) := \prod_{i=1}^n T_i'(x) \subseteq T(x)$ for each $x \in A^0$ for every $x \in A^0$. Since $f^{-1}P \circ T' : A^0 \to 2^{A^0}$ is an $\mathfrak{A}_{\mathbf{c}}^{\kappa}$-multimap (hence, a multimap having the **KKM** property) and $f^{-1}P \circ T'$ is closed, then $T' : A^0 \to 2^Y$ is a $\mathbf{KKM}_0$-multimap. It follows from the previous paragraph that there exists $(a,b) \in A^0 \times Y$ such that $b = (b_1,\ldots,b_n)$, $b_i \in T_i'(a)$, and

$$d(f(a),T_i'(a)) = \|f(a) - b_i\| = d(A,B_i). \tag{3.6}$$

As $d(A,B_i) \leq d(f(a),T_i(a)) \leq d(f(a),T_i'(a))$, we conclude that

$$d(f(a),T_i(a)) = \|f(a) - b_i\| = d(A,B_i). \tag{3.7}$$

Therefore, in both cases, the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and its closedness follows from the continuity of the norm. □

Corollary 3.5. *Let A and B_i be subsets of a normed space E such that A_i^0 (resp., B_i^0) is nonempty, compact (resp., closed), and convex. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1,\ldots,y_n) \in Y := \prod_{i=1}^n B_i^0$ and $T_i : A^0 \to 2^{B_i^0}$ is an $\mathfrak{A}_{\mathbf{c}}^{\kappa}$-multimap for each $i \in I_n$. Then, for each continuous, proper, quasiaffine, and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed.*

Proof. Define $T : A^0 \multimap 2^Y$ by $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$. As $T : A^0 \multimap 2^Y$ is an $\mathfrak{A}_c^\kappa$-multimap, the result follows from Theorem 3.4. □

Remark 3.6. Since each **PK**-multimap is an $\mathfrak{A}_c^\kappa$-multimap, Theorem 4.1 of [12] is a special case of Corollary 3.5.

Corollary 3.7. *Let A and B_i be subsets of a normed space E, A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, and $T_i : A \multimap 2^{B_i}$ a multimap for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in Y$ and $T : A^0 \multimap 2^Y$ is a* **KKM**$_0$*-multimap (resp., $\mathfrak{A}_c^\kappa$-multimap) where $T(x) := \prod_{i=1}^n T_i(x)$ for each $x \in A^0$ and $Y := \prod_{i=1}^n B_i^0$. Then, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i$ is nonempty and closed.*

Theorem 3.8. *Let A and B_i be subsets of a normed space E, A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, $T_i : A^0 \multimap 2^{B_i}$ an upper semicontinuous multimap with compact values, and $T_i(x) \cap B_i^0$ nonempty for each $x \in A^0$ for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in Y := \prod_{i=1}^n B_i^0$. Then, for each continuous, proper, quasiaffine and, surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed.*

Proof. Fix $i \in I_n$. Define $T_i' : A^0 \multimap 2^{B_i^0}$ by $T_i'(x) := T_i(x) \cap B_i^0$ for each $x \in A^0$. Thus $T_i' : A^0 \multimap 2^{B_i^0}$ is an upper semicontinuous multimap with compact values. Define $T : A^0 \multimap 2^Y$ by $T(x) := \prod_{i=1}^n T_i'(x)$ for each $x \in A^0$. As A^0 is compact and $T : A^0 \multimap 2^Y$ is an upper semicontinuous multimap with compact values, then T is an $\mathfrak{A}_c^\kappa$-multimap. It follows from Corollary 3.5 that there exists $(a, b) \in A^0 \times Y$ such that $b = (b_1, \dots, b_n) \in \prod_{i=1}^n T_i'(a)$ and

$$d\big(f(a), T_i'(a)\big) = \big\| f(a) - b_i \big\| = d\big(A, B_i\big). \tag{3.8}$$

As $d(A, B_i) \le d(f(a), T_i(a)) \le d(f(a), T_i'(a))$, the result follows as in Theorem 3.4. □

Corollary 3.9. *Let A and B_i be subsets of a normed space E, A_i^0 (resp., B_i^0) nonempty, compact (resp., closed), and convex, $T_i : A^0 \multimap 2^{B_i}$ an upper semicontinuous multimap with compact values, and $T_i(x) \cap B_i^0$ nonempty for each $x \in A^0$ for each $i \in I_n$. Suppose that $\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \dots, y_n) \in Y := \prod_{i=1}^n B_i^0$. Then, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i$ is nonempty and closed.*

Remark 3.10. Corollary 3.9 extends and improves [7, Theorems 3.2 and 3.4] by Al-Thagafi and Shahzad, [5, Theorems 1 and 2] by Kim and Lee, [3, Theorem 3.4] by Srinivasan and Veeramani, and [4, Theorem 3.2] by Srinivasan and Veeramani.

4. Equilibrium pair results for free n-person games

A free n-person game is a family of ordered quadruples $(A, B_i, T_i, P_i)_{i \in I_n}$ such that A and B_i are nonempty subsets of a normed space E, $T_i : A \multimap 2^{B_i}$ is a constraint multimap, and $P_i : B \multimap 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$ (see [5]). An equilibrium pair for $(A, B_i, T_i, P_i)_{i \in I_n}$ is a point $(a, b) \in A \times B$ such that $T_i(a) \cap P_i(b) = \varnothing$. For details on economic terminology (see [5, 16]).

Theorem 4.1. *Let $(A, B_i, T_i, P_i)_{i \in I_n}$ be a free n-person game such that A and B_i are nonempty subsets of a normed space E, $T_i : A \multimap 2^{B_i}$ is a constraint multimap, and $P_i : B \multimap 2^{B_i}$ is a preference map* *where* $B := \prod_{j=1}^{n} B_j$. *Assume that* A^0 *is nonempty,* $T(x) := \prod_{i=1}^{n} T_i(x)$ *for each* $x \in A^0$, $Y := \prod_{i=1}^{n} B_i^0$, *and for each* $i \in I_n$,

(a) A_i^0 *and* B_i *are nonempty, compact, and convex;*

(b) $\bigcap_{i=1}^{n} P_{A_i^0}(y_i)$ *is nonempty for each* $(y_1, \ldots, y_n) \in \prod_{i=1}^{n} B_i^0$;

(c) $T : A^0 \to 2^Y$ *is a* $\mathbf{KKM}_0$*-multimap (resp.,* $\mathfrak{A}_c^\kappa$*-multimap);*

(d) $x_i \notin \mathrm{co} P_i(x)$ *for each* $x = (x_1, \ldots, x_n) \in B$;

(e) $P_i^{-1}(y)$ *is open for each* $y \in B_i$.

Then, there exists $b \in B$ *such that* $P_i(b) = \varnothing$ *and, for each continuous, proper, quasiaffine, and surjective self-map* $f : A^0 \to A^0$, *there exists* $a \in A^0$ *such that the best proximity set* $\mathfrak{T}_a^i(f)$ *is nonempty and compact. If, in addition,* $P_i(z)$ *is nonempty for each* $z \notin \prod_{i=1}^{n} \mathfrak{T}_a^i(f)$, *then* (a, b) *is an equilibrium pair in* $A^0 \times \prod_{i=1}^{n} \mathfrak{T}_a^i(f)$.

Proof. Fix $i \in I_n$. As A_i^0 and B_i are compact and convex, it follows from Lemma 2.2(b) that B_i^0 is compact and convex. By Theorem 3.4, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and closed. By Lemma 2.5, there exists $b = (b_1, \ldots, b_n) \in Y$ such that $P_i(b) = \varnothing$. As $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^{n} \mathfrak{T}_a^i(f)$, we conclude that $b = (b_1, \ldots, b_n) \in \prod_{i=1}^{n} \mathfrak{T}_a^i(f)$. Thus $(a, b) \in A^0 \times Y$, $b = (b_1, \ldots, b_n) \in \prod_{i=1}^{n} T_i(a)$, $T_i(a) \cap P_i(b) = \varnothing$ and $d(f(a), T_i(a)) = \|f(a) - b_i\| = d(A, B_i)$. Thus (a, b) is an equilibrium pair in $A^0 \times \prod_{i=1}^{n} \mathfrak{T}_a^i(f)$. □

Corollary 4.2. *Let* $(A, B_i, T_i, P_i)_{i \in I_n}$ *be a free* n*-person game such that* A *and* B_i *are nonempty subsets of a normed space* E, $T_i : A \to 2^{B_i}$ *is a constraint multimap, and* $P_i : B \to 2^{B_i}$ *is a preference map where* $B := \prod_{j=1}^{n} B_j$. *Assume that* A^0 *is nonempty,* $T(x) := \prod_{i=1}^{n} T_i(x)$ *for each* $x \in A^0$, $Y := \prod_{i=1}^{n} B_i^0$, *and for each* $i \in I_n$,

(a) A_i^0 *and* B_i *are nonempty, compact, and convex;*

(b) $\bigcap_{i=1}^{n} P_{A_i^0}(y_i)$ *is nonempty for each* $(y_1, \ldots, y_n) \in \prod_{i=1}^{n} B_i^0$;

(c) $T : A^0 \to 2^Y$ *is a* $\mathbf{KKM}_0$*-multimap (resp.,* $\mathfrak{A}_c^\kappa$*-multimap);*

(d) $x_i \notin \mathrm{co} P_i(x)$ *for each* $x = (x_1, \ldots, x_n) \in B$;

(e) $P_i^{-1}(y)$ *is open for each* $y \in B_i$.

Then, there exists $b \in B$ *such that* $P_i(b) = \varnothing$ *and there exists* $a \in A^0$ *such that the best proximity set* $\mathfrak{T}_a^i$ *is nonempty and compact. If, in addition,* $P_i(z)$ *is nonempty for each* $z \notin \prod_{i=1}^{n} \mathfrak{T}_a^i$, *then* (a, b) *is an equilibrium pair in* $A^0 \times \prod_{i=1}^{n} \mathfrak{T}_a^i$.

Theorem 4.3. *Let* $(A, B_i, T_i, P_i)_{i \in I_n}$ *be a free* n*-person game such that* A *and* B_i *are subsets of a normed space* E, $T_i : A \to 2^{B_i}$ *is a constraint multimap, and* $P_i : B \to 2^{B_i}$ *is a preference map where* $B := \prod_{j=1}^{n} B_j$. *Assume that* A^0 *is nonempty,* $Y := \prod_{i=1}^{n} B_i^0$, *and for each* $i \in I_n$,

(a) A_i^0 *and* B_i *are nonempty, compact, and convex;*

(b) $\bigcap_{i=1}^{n} P_{A_i^0}(y_i)$ *is nonempty for each* $(y_1, \ldots, y_n) \in \prod_{i=1}^{n} B_i^0$;

(c) $T_i \mid A^0$ *is an upper semicontinuous multimap with compact values and* $T_i(x) \cap B_i^0$ *is nonempty for each* $x \in A^0$;

(d) $x_i \notin \mathrm{co} P_i(x)$ *for each* $x = (x_1, \ldots, x_n) \in B$;

(e) $P_i^{-1}(y)$ *is open for each* $y \in B_i$.

Then, there exists $b \in B$ such that $P_i(b) = \varnothing$ and, for each continuous, proper, quasiaffine, and surjective self-map $f : A^0 \to A^0$, there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i(f)$ is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{T}_a^i(f)$, then (a,b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{T}_a^i(f)$.

Proof. Use Theorem 3.8 instead of Theorem 3.4 in the proof of Theorem 4.1. □

Corollary 4.4. *Let $(A, B_i, T_i, P_i)_{i\in I_n}$ be a free n-person game such that A and B_i are subsets of a normed space E, $T_i : A \to 2^{B_i}$ is a constraint multimap, and $P_i : B \to 2^{B_i}$ is a preference map where $B := \prod_{j=1}^n B_j$. Assume that A^0 is nonempty, $Y := \prod_{i=1}^n B_i^0$, and for each $i \in I_n$,*

(a) *A_i^0 and B_i are nonempty, compact, and convex;*

(b) *$\bigcap_{i=1}^n P_{A_i^0}(y_i)$ is nonempty for each $(y_1, \ldots, y_n) \in \prod_{i=1}^n B_i^0$;*

(c) *$T_i \mid A^0$ is an upper semicontinuous multimap with compact values and $T_i(x) \cap B_i^0$ is nonempty for each $x \in A^0$;*

(d) *$x_i \notin \mathrm{co} P_i(x)$ for each $x = (x_1, \ldots, x_n) \in B$;*

(e) *$P_i^{-1}(y)$ is open for each $y \in B_i$.*

Then, there exists $b \in B$ such that $P_i(b) = \varnothing$, and there exists $a \in A^0$ such that the best proximity set $\mathfrak{T}_a^i$ is nonempty and compact. If, in addition, $P_i(z)$ is nonempty for each $z \notin \prod_{i=1}^n \mathfrak{T}_a^i$, then (a,b) is an equilibrium pair in $A^0 \times \prod_{i=1}^n \mathfrak{T}_a^i$.

Remark 4.5. Corollary 4.4 extends and improves [7, Theorem 4.1] by Al-Thagafi and Shahzad and [5, Theorem 4] by Kim and Lee.

Theorem 4.6. *Let (A, B, T, P) be a free 1-person game such that A and B are subsets of a normed space E, $T : A \to 2^B$ is a constraint multimap, and $P : B \to 2^B$ is a preference map. Assume that*

(a) *A_0 and B are nonempty, compact, and convex;*

(b) *$T : A_0 \to 2^{B_0}$ is a $\mathbf{KKM}_0$-multimap (resp., $\mathfrak{A}_{\mathbf{c}}^{\kappa}$-multimap);*

(c) *$x \notin \mathrm{co} P(x)$ for each $x \in B$;*

(d) *one of the following conditions is satisfied:*

(d $_1$) *if $z \in P^{-1}(y)$ for some $y \in B$, then there exists some $y' \in B$ such that $z \in \mathrm{int}\, P^{-1}(y')$;*

(d $_2$) *for each $y \in B$, $P^{-1}(y)$ is open in B.*

Then, there exists $b \in B$ such that $P(b) = \varnothing$ and, for each continuous, proper, quasiaffine, and surjective self-map $f : A_0 \to A_0$, there exists $a \in A_0$ such that the best proximity set $\mathfrak{T}_a(f)$ is nonempty and compact. If, in addition, $P(z)$ is nonempty for each $z \notin \mathfrak{T}_a(f)$, then (a,b) is an equilibrium pair in $A_0 \times \mathfrak{T}_a(f)$.

Proof. Since A_0 and B_0 are nonempty, compact, and convex, it follows from Theorem 3.4 that there exists $(a,c) \in A_0 \times B_0$ such that $c \in T(a)$ and $d(f(a), T(a)) = \|f(a) - c\| = d(A,B)$ and so $\mathfrak{T}_a(f)$ is nonempty. By Lemma 2.6, there exists $b \in B_0$ such that $P(b) = \varnothing$. As $P(z)$ is nonempty whenever $z \in B \setminus \mathfrak{T}_a(f)$, we conclude that $b \in \mathfrak{T}_a(f)$. So $(a,b) \in A_0 \times B_0$, $b \in T(a)$ and $d(f(a), T(a)) = \|f(a) - b\| = d(A,B)$. Thus (a,b) is an equilibrium pair in $A_0 \times \mathfrak{T}_a(f)$. □

Corollary 4.7. *Let (A, T, P) be a free 1-person game such that A is a nonempty, compact, and convex subset of a normed space E, $T : A \to 2^A$ is a constraint multimap, and $P : A \to 2^A$ is a preference map. Assume that*

(a) $T : A \to 2^A$ *is a* $\mathbf{KKM}_0$*-multimap (resp.,* $\mathfrak{A}_{\mathfrak{c}}^{\kappa}$*-multimap);*

(b) $x \notin \operatorname{co}P(x)$ *for each* $x \in A$*;*

(c) *one of the following conditions is satisfied:*

(c $_1$) *if* $z \in P^{-1}(y)$ *for some* $y \in A$*, then there exists some* $y' \in A$ *such that* $z \in \operatorname{int} P^{-1}(y')$*;*
(c $_2$) *for each* $y \in A$*,* $P^{-1}(y)$ *is open in* A*.*

Then, there exists $b \in A$ *such that* $P(b) = \varnothing$ *and, for each continuous, proper, quasiaffine, and surjective self-map* $f : A \to A$*, there exists* $a \in A$ *such that* $f(a) = b$*. If, in addition,* $P(z)$ *is nonempty for each* $z \notin \{x \in A : f(x) \in T(x)\}$*, then* $f(a) \in T(a)$*.*

Remark 4.8. Corollary 4.7 extends and improves [7, Theorem 4.3] by Al-Thagafi and Shahzad and [5, Theorem 3] by Kim and Lee.

Corollary 4.7 follows also from the following result.

Theorem 4.9. *Let (A, B, T, P) be a free 1-person game such that A and B are subsets of a normed space E, $T : A \to 2^B$ is a constraint multimap, and $P : B \to 2^B$ is a preference map. Assume that*

(a) A_0 *and* B *are nonempty, compact, and convex;*

(b) $T \mid A_0$ *is an upper semicontinuous multimap with compact values and* $T(x) \cap B_0$ *is nonempty for each* $x \in A_0$*;*

(c) $x \notin \operatorname{co}P(x)$ *for each* $x \in B$*;*

(d) *one of the following conditions is satisfied:*

(d $_1$) *if* $z \in P^{-1}(y)$ *for some* $y \in B$*, then there exists some* $y' \in B$ *such that* $z \in \operatorname{int} P^{-1}(y')$*;*
(d $_2$) *for each* $y \in B$*,* $P^{-1}(y)$ *is open in* B*.*

Then, there exists $b \in B$ *such that* $P(b) = \varnothing$ *and, for each continuous, proper, quasiaffine, and surjective self-map* $f : A_0 \to A_0$*, there exists* $a \in A_0$ *such that the best proximity set* $\mathfrak{T}_a(f)$ *is nonempty and compact. If, in addition,* $P(z)$ *is nonempty for each* $z \notin \mathfrak{T}_a(f)$*, then* (a, b) *is an equilibrium pair in* $A_0 \times \mathfrak{T}_a(f)$*.*

Proof. Use Theorem 3.8 instead of Theorem 3.4 in the proof of Theorem 4.3. □

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