Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 369215, 23 pages
doi:10.1155/2009/369215

Research Article

A General Iterative Method for Solving the Variational Inequality Problem and Fixed Point Problem of an Infinite Family of Nonexpansive Mappings in Hilbert Spaces

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Received 3 November 2008; Accepted 16 January 2009

Recommended by Anthony Lau

We introduce an iterative scheme for finding a common element of the set of common fixed points of a family of infinitely nonexpansive mappings, and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space. Under suitable conditions, some strong convergence theorems for approximating a common element of the above two sets are obtained. As applications, at the end of the paper we utilize our results to study the problem of finding a common element of the set of fixed points of a family of infinitely nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings. The results presented in the paper improve some recent results of Qin and Cho (2008).

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, C is a nonempty closed convex subset of H, and P_C is the metric projection of H onto C. In the following, we denote by $\to$ strong convergence and by $\rightharpoonup$ weak convergence. Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in C. \tag{1.1}$$

We denote by $F(T)$ the set of fixed points of T. Recall that a mapping $B : C \to H$ is said to be

(i) monotone if $\langle Bu - Bv, u - v \rangle \geq 0$, for all $u, v \in C$;

(ii) L-Lipschitz if there exists a constant $L > 0$ such that $\|Bu - Bv\| \leq L\|u - v\|$, for all $u, v \in C$;

(iii) α-inverse-strongly monotone [1, 2] if there exists a positive real number α such that

$$\langle Bu - Bv, u - v\rangle \geq \alpha\|Bu - Bv\|^2, \quad \forall u, v \in C. \tag{1.2}$$

Remark 1.1. It is obvious that any α-inverse-strongly monotone mapping B is monotone and $(1/\alpha)$-Lipschitz continuous.

Let $B : C \to H$ be a mapping. The classical variational inequality problem is to find a $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of variational inequality (1.3) is denoted by $\mathrm{VI}(B, C)$. The variational inequality has been extensively studied in the literature; see, for example, [3, 4] and the references therein.

A self-mapping $f : C \to C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(u) - f(v)\| \leq \alpha\|u - v\|, \quad \forall u, v \in C. \tag{1.4}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [5–8] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping T, and b is a given point in H. Let H be a real Hilbert space. Recall that a linear bounded operator B is strongly positive if there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.6}$$

Recently, Marino and Xu [9] introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [10]:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.7}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C, \tag{1.8}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.9}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [11] and is defined as follows:

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary}, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{aligned} \tag{1.10}$$

where the sequence $\{\alpha_n\}$ is in the interval $(0,1)$.

The second iteration process is referred to as Ishikawa's iteration process [12] which is defined recursively by

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary}, \\ y_n &= \beta_n x_n + (1-\beta_n) T x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $(0,1)$. However, both (1.16) and (1.11) have only weak convergence in general (see [13], e.g.). Very recently, Qin and Cho [14] introduced a composite iterative algorithm $\{x_n\}$ defined as follows:

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary}, \\ z_n &= \gamma_n x_n + (1-\gamma_n) T x_n, \\ y_n &= \beta_n x_n + (1-\beta_n) T z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A) y_n, \quad n \geq 1, \end{aligned} \tag{1.12}$$

where f is a contraction, T is a nonexpansive mapping, and A is a strongly positive linear bounded self-adjoint operator, proved that, under certain appropriate assumptions on the parameters, $\{x_n\}$ defined by (1.12) converges strongly to a fixed point of T, which solves some variational inequality and is also the optimality condition for the minimization problem (1.9).

On the other hand, for finding an element of $F(T) \cap \mathrm{VI}(B,C)$, under the assumption that a set $C \subseteq H$ is nonempty, closed, and convex, a mapping $T : C \to C$ is nonexpansive and a mapping $B : C \to H$ is α-inverse-strongly monotone, Takahashi and Toyoda [15] introduced the following iterative scheme:

$$\begin{aligned} x_1 &= x \in C \text{ chosen arbitrary}, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) T P_C(x_n - \eta_n B x_n), \quad n \geq 1, \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{\eta_n\}$ is a sequence in $(0,2\alpha)$. They proved that if $F(T)\cap \mathrm{VI}(B,C)\neq\emptyset$, then the sequence $\{x_n\}$ generated by (1.13) converges weakly to some $z\in F(T)\cap \mathrm{VI}(B,C)$. Recently, Iiduka and Takahashi [16] proposed another iterative scheme as follows

$$\begin{gathered}x_1 = x\in C \text{ chosen arbitrary},\\ x_{n+1} = \alpha_n x + (1-\alpha_n)TP_C(x_n - \eta_n Bx_n),\quad n\geq 1,\end{gathered}\tag{1.14}$$

where B is an α-inverse strongly monotone mapping, $\{\alpha_n\}\subseteq(0,1)$ and $\{\lambda_n\}\subseteq(0,2\alpha)$ satisfy some parameters controlling conditions. They showed that if $F(T)\cap\mathrm{VI}(B,C)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.14) converges strongly to some $z\in F(T)\cap\mathrm{VI}(B,C)$.

The existence of common fixed points for a finite family of nonexpansive mappings has been considered by many authors (see [17–20] and the references therein). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [21, 22]). The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see [18, 23]). A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [23, 24]).

In this paper, we study the mapping W_n defined by

$$\begin{gathered}U_{n,n+1} = I,\\ U_{n,n} = \mu_n T_n U_{n,n+1} + (1-\mu_n)I,\\ U_{n,n-1} = \mu_{n-1}T_{n-1}U_{n,n} + (1-\mu_{n-1})I,\\ \vdots\\ U_{n,k} = \mu_k T_k U_{n,k+1} + (1-\mu_k)I,\\ U_{n,k-1} = \mu_{k-1}T_{k-1}U_{n,k} + (1-\mu_{k-1})I,\\ \vdots\\ U_{n,2} = \mu_2 T_2 U_{n,3} + (1-\mu_2)I,\\ W_n = U_{n,1} = \mu_1 T_1 U_{n,2} + (1-\mu_1)I,\end{gathered}\tag{1.15}$$

where $\{\mu_i\}$ is a nonnegative real sequence with $0\leq\mu_i<1$, for all $i\geq 1$, $T_1,T_2,\ldots,$ form a family of infinitely nonexpansive mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Such a W_n is nonexpansive from C to C and it is called a W-mapping generated by $T_1,T_2,\ldots,T_n$ and $\mu_1,\mu_2,\ldots,\mu_n$.

In this paper, motivated and inspired by Su et al. [25], Marino and Xu [9], Takahashi and Toyoda [15], and Iiduka and Takahashi [16], we will introduce a new iterative scheme:

$$\begin{gathered}x_1 = x\in C \text{ chosen arbitrary},\\ z_n = \gamma_n x_n + (1-\gamma_n)W_n x_n,\\ y_n = \beta_n x_n + (1-\beta_n)W_n z_n,\\ x_{n+1} = \alpha_n\gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)P_C(y_n - \lambda_n B y_n),\end{gathered}\tag{1.16}$$

where W_n is a mapping defined by (1.15), f is a contraction, A is strongly positive linear bounded self-adjoint operator, B is a α-inverse strongly monotone, and we prove that under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$, the sequences $\{x_n\}$ defined by (1.16) converge strongly to a common element of the set of common fixed points of a family of $\{T_n\}$ and the set of solutions of the variational inequality for an inverse-strongly monotone mapping, which solves some variational inequality and is also the optimality condition for the minimization problem (1.9).

2. Preliminaries

Let H be a real Hilbert space. It is well known that for any $\lambda \in [0,1]$

$$\left\|\lambda x + (1-\lambda)y\right\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\left\|x - P_C x\right\| \le \|x-y\|, \quad \forall y \in C. \tag{2.2}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \ge \left\|P_C x - P_C y\right\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{gathered}\langle x - P_C x, y - P_C x\rangle \le 0,\\ \|x-y\|^2 \ge \left\|x - P_C x\right\|^2 + \left\|y - P_C x\right\|^2,\end{gathered} \tag{2.4}$$

for all $x \in H$, $y \in C$. It is easy to see that the following is true:

$$u \in \mathrm{VI}(B,C) \Longleftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{2.5}$$

A Banach space X is said to satisfy the Opial's condition if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty}\left\|x_n - x\right\| < \liminf_{n\to\infty}\left\|x_n - y\right\|, \quad y \ne x. \tag{2.6}$$

It is well known that each Hilbert space satisfies the Opial's condition.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0$, for all $u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{2.7}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(B, C)$; see [26].

Now we collect some useful lemmas for proving the convergence result of this paper.

Lemma 2.1. *In a Hilbert space H. Then the following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y)\rangle, \quad \forall x, y \in H. \tag{2.8}$$

Lemma 2.2 (see [27]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.3 (see [28]). *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 1, \tag{2.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$

(2) $\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 (see [9]). *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Throughout this paper, we will assume that $0 < \mu_n \leq \mu < 1$, for all $n \geq 1$. Concerning W_n defined by (1.15), we have the following lemmas which are important to prove our main result.

Lemma 2.5 (see [29]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $T_i : C \to C$ be a family of infinitely nonexpansive mapping with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for each $n \geq 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*

(3) *the mapping $W : C \to C$ define by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{2.10}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$ and it is called the W-mapping generated by $T_1, T_2, \ldots,$ and $\mu_1, \mu_2, \ldots$.

Lemma 2.6 (see [30]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$, and let $\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.11}$$

3. Main Results

Now we are in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, let f be a contraction of C into itself, let B be an α-inverse strongly monotone mapping of C into H, and let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $F := \bigcap_{n=1}^{\infty} F(T_i) \cap \mathrm{VI}(B, C) \neq \emptyset$. Let A be a strongly positive linear bounded self-adjoint operator with the coefficient $\overline{\gamma} > 0$ such that $\|A\| \leq 1$. Assume that $0 < \gamma \leq \overline{\gamma}/\alpha$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be sequences in $[0,1]$ satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,

(C3) $(1+\beta_n)\gamma_n - 2\beta_n > d$ *for some* $d \in (0,1)$,

(C4) $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then the sequence $\{x_n\}$ defined by (1.16) *converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))q$ which solves the following variational inequality:*

$$\langle \gamma f(q) - Ap, p - q \rangle \leq 0, \quad \forall p \in F. \tag{3.1}$$

Proof. Since $\alpha_n \to 0$ as $n \to \infty$ by the condition (C1), we may assume, without loss of generality that $\alpha_n < (1-\delta_n)\|A\|^{-1}$ for all $n \geq 0$. First, we will show that $I - \lambda_n B$ is nonexpansive. Indeed, for all $x, y \in C$ and $\lambda_n \in [0, 2\alpha]$,

$$\begin{aligned} \left\|(I-\lambda_n B)x - (I-\lambda_n B)y\right\|^2 &= \left\|(x-y) - \lambda_n(Ax - Ay)\right\|^2 \\ &= \|x-y\|^2 - 2\lambda_n\langle x-y, Ax-Ay\rangle + \lambda_n^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2, \end{aligned} \tag{3.2}$$

which implies that $I - \lambda_n B$ is nonexpansive. Noticing that A is a linear bounded self-adjoint operator, one has

$$\|A\| = \sup\left\{\left|\langle Ax, x\rangle\right| : x \in H,\ \|x\| = 1\right\}. \tag{3.3}$$

Observing that

$$\begin{aligned}\langle((1-\delta_n)I-\alpha_n A)x,x\rangle &= 1-\delta_n-\alpha_n\langle Ax,x\rangle\\ &\le 1-\delta_n-\alpha_n\|A\|\\ &\le 0,\end{aligned}$$

we obtain $(1-\delta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\delta_n)I-\alpha_n A\| &= \sup\{\langle((1-\delta_n)I-\alpha_n A)x,x\rangle : x\in H,\ \|x\|=1\}\\ &= \sup\{1-\delta_n-\alpha_n\langle Ax,x\rangle : x\in H,\ \|x\|=1\}\\ &\le 1-\delta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Next, we observe that $\{x_n\}$ is bounded. Indeed, pick $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{VI}(B,C)$ and notice that

$$\begin{aligned}\|z_n-p\| &\le \gamma_n\|x_n-p\|+(1-\gamma_n)\|W_nx_n-p\|\le\|x_n-p\|,\\ \|y_n-p\| &\le \beta_n\|x_n-p\|+(1-\beta_n)\|W_nz_n-p\|\\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|z_n-p\|\\ &\le \|x_n-p\|.\end{aligned}\tag{3.4}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\delta_nx_n+((1-\delta_n)I-\alpha_nA)P_C(y_n-\lambda By_n)-p\|\\ &= \|\alpha_n(\gamma f(x_n)-Ap)+\delta_n(x_n-p)+((1-\delta_n)I-\alpha_nA)P_C(y_n-\lambda By_n)-p\|\\ &\le \alpha_n\|\gamma f(x_n)-Ap\|+\delta_n\|x_n-p\|+(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-p\|\\ &\le \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Ap\|+(1-\alpha_n\overline{\gamma})\|x_n-p\|\\ &= [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|\\ &= [1-\alpha_n(\overline{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(p)-Ap\|}{\overline{\gamma}-\gamma\alpha}.\end{aligned}\tag{3.5}$$

By simple induction, we have

$$\|x_n-p\|\le\max\left\{\|x_0-p\|,\frac{\|Ap-\gamma f(p)\|}{\overline{\gamma}-\gamma\alpha}\right\},\tag{3.6}$$

which gives that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.7}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_n x_n - W_{n-1} x_n\| &= \|U_{n,1} x_n - U_{n-1,1} x_n\| \\
&= \|\mu_1 T_1 U_{n,2} x_n - (1-\mu_1) x_n - \mu_1 T_1 U_{n-1,2} x_n - (1-\mu_1) x_n\| \\
&\le \mu_1 \|U_{n,2} x_n - U_{n-1,2} x_n\| \\
&= \mu_1 \|\mu_2 T_2 U_{n,3} x_n - (1-\mu_2) x_n - \mu_2 T_2 U_{n-1,3} x_n - (1-\mu_2) x_n\| \\
&\le \mu_1 \mu_2 \|U_{n,3} x_n - U_{n-1,3} x_n\| \\
&\quad \vdots \\
&\le \left(\prod_{i=1}^{n} \mu_i\right) \|U_{n,n} x_n - U_{n-1,n} x_n\| \\
&\le M_1 \left(\prod_{i=1}^{n} \mu_i\right),
\end{aligned} \tag{3.8}$$

where $M_1 \ge 0$ is a constant such that $\|U_{n,n} x_n - U_{n-1,n} x_n\| \le M_1$. Similarly, there exists $M_2 \ge 0$ such that $\|U_{n,n} y_n - U_{n-1,n} y_n\| \le M_2$.

Observing that

$$\begin{aligned}
z_n &= \gamma_n x_n + (1-\gamma_n) W_n x_n, \\
z_{n-1} &= \gamma_{n-1} x_{n-1} + (1-\gamma_{n-1}) W_{n-1} x_{n-1},
\end{aligned} \tag{3.9}$$

we obtain that

$$z_n - z_{n-1} = (1-\gamma_n)(W_n x_n - W_{n-1} x_{n-1}) + \gamma_n (x_n - x_{n-1}) + (\gamma_{n-1} - \gamma_n)(W_{n-1} x_{n-1} - x_{n-1}). \tag{3.10}$$

It follows that

$$\begin{aligned}
\|z_n - z_{n-1}\| &\le (1-\gamma_n) \|W_n x_n - W_{n-1} x_{n-1}\| + \gamma_n \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n| \|W_n x_{n-1} - x_{n-1}\| \\
&\le (1-\gamma_n) \|W_n x_n - W_{n-1} x_n\| + (1-\gamma_n) \|W_{n-1} x_n - W_{n-1} x_{n-1}\| \\
&\quad + \gamma_n \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n| \|W_n x_{n-1} - x_{n-1}\| \\
&\le (1-\gamma_n) \|W_n x_n - W_{n-1} x_n\| + (1-\gamma_n) \|x_n - x_{n-1}\| \\
&\quad + \gamma_n \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n| \|W_n x_{n-1} - x_{n-1}\| \\
&= (1-\gamma_n) \|W_n x_n - W_{n-1} x_n\| + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n| \|W_n x_{n-1} - x_{n-1}\| \\
&\le (1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i + \|x_n - x_{n-1}\| + |\gamma_{n-1} - \gamma_n| \|W_n x_{n-1} - x_{n-1}\|.
\end{aligned} \tag{3.11}$$

Noticing that

$$y_n = \beta_n x_n + (1-\beta_n) W_n z_n,$$
$$y_{n-1} = \beta_{n-1} x_{n-1} + (1-\beta_{n-1}) W_{n-1} z_{n-1}, \tag{3.12}$$

we obtain

$$y_n - y_{n-1} = (1-\beta_n)(W_n z_n - W_{n-1} z_{n-1}) + \beta_n (x_n - x_{n-1}) + (W_{n-1} z_{n-1} - x_{n-1})(\beta_{n-1} - \beta_n). \tag{3.13}$$

It follows that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le (1-\beta_n)\|W_n z_n - W_n z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| + \|W_n z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\
&\le (1-\beta_n)\|W_n z_n - W_{n-1} z_n\| + (1-\beta_n)\|W_{n-1} z_n - W_{n-1} z_{n-1}\| \\
&\quad + \beta_n\|x_n - x_{n-1}\| + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\
&\le (1-\beta_n)\|W_n z_n - W_{n-1} z_n\| + (1-\beta_n)\|z_n - z_{n-1}\| \\
&\quad + \beta_n\|x_n - x_{n-1}\| + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n| \\
&\le (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)\|z_n - z_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\
&\quad + \|W_{n-1} z_{n-1} - x_{n-1}\| |\beta_{n-1} - \beta_n|.
\end{aligned} \tag{3.14}$$

Substituting (3.11) into (3.14), we get

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)(1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i \\
&\quad + (1-\beta_n)\|x_n - x_{n-1}\| + (1-\beta_n)|\gamma_{n-1} - \gamma_n| \|W_{n-1} x_{n-1} - x_{n-1}\| \\
&\quad + \beta_n\|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| \|W_{n-1} z_{n-1} - x_{n-1}\| \\
&= (1-\beta_n) M_2 \prod_{i=1}^{n} \mu_i + (1-\beta_n)(1-\gamma_n) M_1 \prod_{i=1}^{n} \mu_i + \|x_n - x_{n-1}\| \\
&\quad + M_3\big((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|\big),
\end{aligned} \tag{3.15}$$

where M_3 is an appropriate constant such that

$$M_3 \ge \max\left\{ \sup_{n\ge1}\|W_{n-1} x_{n-1} - x_{n-1}\|, \sup_{n\ge1}\|W_{n-1} z_{n-1} - x_{n-1}\| \right\}. \tag{3.16}$$

Putting $l_n = (x_{n+1} - \delta_n x_n)/(1-\delta_n)$, we get, $x_{n+1} = (1-\delta_n) l_n + \delta_n x_n$.

Now, we compute $l_{n+1} - l_n$. Observing that

$$\begin{aligned} l_{n+1} - l_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\delta_{n+1})I - \alpha_{n+1}A)P_C(y_{n+1} - \lambda_{n+1}By_{n+1})}{1-\delta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + ((1-\delta_n)I - \alpha_n A)P_C(y_n - \lambda_n By_n)}{1-\delta_n} \\ &= \frac{\alpha_{n+1}}{1-\delta_{n+1}}(\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})) \\ &\quad + \frac{\alpha_n}{1-\delta_n}(AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)) \\ &\quad + P_C(y_{n+1} - \lambda_{n+1}By_{n+1}) - P_C(y_n - \lambda_n By_n). \end{aligned} \tag{3.17}$$

It follows from (3.15) that

$$\begin{aligned} \|l_{n+1} - l_n\| &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| + \|y_{n+1} - y_n\| \\ &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| \\ &\quad + (1-\beta_n)M_2\prod_{i=1}^{n}\mu_i + (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i \\ &\quad + \|x_n - x_{n-1}\| + M_3((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|). \end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned} \|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| &\le \frac{\alpha_{n+1}}{1-\delta_{n+1}}\|\gamma f(x_{n+1}) - AP_C(y_{n+1} - \lambda_{n+1}By_{n+1})\| \\ &\quad + \frac{\alpha_n}{1-\delta_n}\|AP_C(y_n - \lambda_n By_n) - \gamma f(x_n)\| \\ &\quad + (1-\beta_n)M_2\prod_{i=1}^{n}\mu_i + (1-\beta_n)(1-\gamma_n)M_1\prod_{i=1}^{n}\mu_i \\ &\quad + M_3((1-\beta_n)|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|). \end{aligned} \tag{3.19}$$

Observing the conditions (C1) and (C4) and taking the superior limit as $n \to \infty$, we get

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_n - x_{n-1}\|) \le 0. \tag{3.20}$$

We can obtain $\lim_{n\to\infty}\|l_n - x_n\| = 0$ easily by Lemma 2.2 since

$$x_{n+1} - x_n = (1-\delta_n)(l_n - x_n), \tag{3.21}$$

one obtains that (3.7) holds. Setting $t_n = P_C(y_n - \lambda_n y_n)$, we have

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)t_n. \tag{3.22}$$

Observing that

$$\begin{aligned} x_n - t_n &= x_n - x_{n+1} + x_{n+1} - t_n \\ &= x_n - x_{n+1} + \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n)I - \alpha_n A)t_n - t_n \\ &= x_n - x_{n+1} + \alpha_n(\gamma f(x_n) - At_n) + \delta_n(x_n - t_n), \end{aligned} \tag{3.23}$$

we arrive at

$$(1-\delta_n)(x_n - t_n) = x_n - x_{n+1} + \alpha_n(\gamma f(x_n) - At_n). \tag{3.24}$$

This implies

$$(1-\delta_n)\|x_n - t_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - At_n\|. \tag{3.25}$$

From (3.7) and (C1) we obtain that

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.26}$$

Next, we will show that $\|By_n - Bp\| \to 0$ as $n \to \infty$ for any $p \in F$. Observe that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|((1-\delta_n)I - \alpha_n A)(t_n - p) + \delta_n(x_n - p) + \alpha_n(\gamma f(x_n) - Ap)\|^2 \\ &= \|((1-\delta_n)I - \alpha_n A)(t_n - p) + \delta_n(x_n - p)\|^2 + \alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\ &\quad + 2\delta_n\alpha_n\langle x_n - p, \gamma f(x_n) - Ap\rangle + 2\alpha_n\langle((1-\delta_n)I - \alpha_n A)(t_n - p), \gamma f(x_n) - Ap\rangle \\ &\le ((1-\delta_n - \alpha_n\overline{\gamma})\|t_n - p\| + \delta_n\|x_n - p\|)^2 + \alpha_n^2\|\gamma f(x_n) - Ap\|^2 \\ &\quad + 2\delta_n\alpha_n\langle x_n - p, \gamma f(x_n) - Ap\rangle + 2\alpha_n\langle((1-\delta_n)I - \alpha_n A)(t_n - p), \gamma f(x_n) - Ap\rangle \\ &= (1-\delta_n - \alpha_n\overline{\gamma})^2\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + 2(1-\delta_n - \alpha_n\overline{\gamma})\delta_n\|t_n - p\|\|x_n - p\| + c_n \\ &\le (1-\delta_n - \alpha_n\overline{\gamma})^2\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + (1-\delta_n - \alpha_n\overline{\gamma})\delta_n\left(\|t_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\ &= \left[(1-\alpha_n\overline{\gamma})^2 - 2(1-\alpha_n\overline{\gamma})\delta_n + \delta_n^2\right]\|t_n - p\|^2 + \delta_n^2\|x_n - p\|^2 \\ &\quad + ((1-\alpha_n\overline{\gamma})\delta_n - \delta_n^2)\left(\|t_n - p\|^2 + \|x_n - p\|^2\right) + c_n \\ &= (1-\alpha_n\overline{\gamma})^2\|t_n - p\|^2 - (1-\alpha_n\overline{\gamma})\delta_n\|t_n - p\|^2 + (1-\alpha_n\overline{\gamma})\delta_n\|x_n - p\|^2 + c_n \end{aligned}$$

$$\begin{aligned}
&= (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|t_n-p\|^2 + (1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2 + c_n \\
&\leq (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\left[\|(y_n-\lambda_n By_n)-(p-\lambda_n Bp)\|^2\right] \\
&\quad + (1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2 + c_n \\
&\leq (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\left[\|y_n-p\|^2 + \lambda_n(\lambda_n-2\alpha)\|By_n-Bp\|^2\right] \\
&\quad + (1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2 + c_n \\
&\leq \|x_n-p\|^2 + b(b-2\alpha)\|By_n-Bp\|^2 + c_n,
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
c_n = \alpha_n^2\|\gamma f(x_n)-Ap\|^2 + 2\delta_n\alpha_n\langle x_n-p, \gamma f(x_n)-Ap\rangle \\
+ 2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(t_n-p), \gamma f(x_n)-Ap\rangle.
\end{aligned} \tag{3.28}$$

This impies that

$$\begin{aligned}
-b(b-2\alpha)\|By_n-Bp\|^2 &\leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + c_n \\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + c_n.
\end{aligned} \tag{3.29}$$

Since $\lim_{n\to\infty} c_n = 0$ and from (3.7), we obtain

$$\lim_{n\to\infty}\|By_n-Bp\| = 0. \tag{3.30}$$

From (2.3), we have

$$\begin{aligned}
\|t_n-p\|^2 &= \|P_C(y_n-\lambda_n By_n) - P_C(p-\lambda_n Bp)\|^2 \\
&\leq \langle(y_n-\lambda_n By_n)-(p-\lambda_n Bp), t_n-p\rangle \\
&= \frac{1}{2}\Big\{\|(y_n-\lambda_n By_n)-(p-\lambda_n Bp)\|^2 + \|t_n-p\|^2 \\
&\quad - \|(y_n-\lambda_n By_n)-(p-\lambda_n Bp)-(t_n-p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|y_n-p\|^2 + \|t_n-p\|^2 - \|(y_n-t_n)-\lambda_n(By_n-Bp)\|^2\Big\} \\
&= \frac{1}{2}\Big\{\|y_n-p\|^2 + \|t_n-p\|^2 - \|y_n-t_n\|^2 + 2\lambda_n\langle y_n-t_n, By_n-Bp\rangle - \lambda_n^2\|By_n-Bp\|^2\Big\},
\end{aligned} \tag{3.31}$$

so, we obtain

$$\|t_n-p\|^2 \leq \|y_n-p\|^2 - \|y_n-t_n\|^2 + 2\lambda_n\langle y_n-t_n, By_n-Bp\rangle - \lambda_n^2\|By_n-Bp\|^2. \tag{3.32}$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|t_n-p\|^2+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le (1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\\
&\quad\times\left[\|y_n-p\|^2-\|y_n-t_n\|^2+2\lambda_n\langle y_n-t_n,By_n-Bp\rangle-\lambda_n^2\|By_n-Bp\|^2\right]\\
&\quad+(1-\alpha_n\overline{\gamma})\delta_n\|x_n-p\|^2+c_n\\
&\le (1-\alpha_n\overline{\gamma})\|x_n-p\|^2-(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|^2\\
&\quad+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\
&\quad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n,
\end{aligned}
\tag{3.33}
$$

which implies that

$$
\begin{aligned}
&(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|^2\\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\
&\qquad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n\\
&\quad\le \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)\\
&\qquad+2\lambda_n(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|y_n-t_n\|\|By_n-Bp\|\\
&\qquad-\lambda_n^2(1-\alpha_n\overline{\gamma})(1-\delta_n-\alpha_n\overline{\gamma})\|By_n-Bp\|^2+c_n.
\end{aligned}
\tag{3.34}
$$

Applying (3.7), (3.30), and $\lim_{n\to\infty}c_n=0$ to the last inequality, we obtain that

$$
\lim_{n\to\infty}\|y_n-t_n\|=0. \tag{3.35}
$$

It follows from (3.26) and (3.35) that

$$
\|x_n-y_n\|\le\|x_n-t_n\|+\|t_n-y_n\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.36}
$$

On the other hand, one has

$$
\begin{aligned}
\|W_nx_n-x_n\| &\le \|x_n-y_n\|+\|y_n-W_nx_n\|\\
&\le \|x_n-y_n\|+\|y_n-W_nz_n\|+\|W_nz_n-W_nx_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nz_n\|+\|z_n-x_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+\beta_n\|W_nx_n-W_nz_n\|+\|z_n-x_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+(1+\beta_n)\|z_n-x_n\|\\
&\le \|x_n-y_n\|+\beta_n\|x_n-W_nx_n\|+(1+\beta_n)(1-\gamma_n)\|W_nx_n-x_n\|\\
&= \|x_n-y_n\|-\left[(1+\beta_n)\gamma_n-2\beta_n-1\right]\|W_nx_n-x_n\|,
\end{aligned}
\tag{3.37}
$$

which implies

$$\left[(1+\beta_n)\gamma_n - 2\beta_n\right]\left\|W_n x_n - x_n\right\| \leq \left\|x_n - y_n\right\|. \tag{3.38}$$

From the conditions (C3), it follows that

$$\lim_{n\to\infty}\left\|W_n x_n - x_n\right\| = 0. \tag{3.39}$$

Applying Lemma 2.6 and (3.39), we obtain that

$$\begin{aligned}\left\|W x_n - x_n\right\| &\leq \left\|W x_n - W_n x_n\right\| + \left\|W_n x_n - x_n\right\| \\ &\leq \sup_{x\in\{x_n\}}\left\|W x - W_n x\right\| + \left\|W_n x_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.40}$$

It follows from (3.26) and (3.40) that

$$\begin{aligned}\left\|W t_n - t_n\right\| &\leq \left\|W t_n - W x_n\right\| + \left\|W x_n - x_n\right\| + \left\|x_n - t_n\right\| \\ &\leq 2\left\|t_n - x_n\right\| + \left\|W x_n - x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.41}$$

We observe that $P_F(\gamma f + (I - A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}&\left\|P_F(\gamma f + (I - A))(x) - P_F(\gamma f + (I - A))(y)\right\| \\ &\quad\leq \left\|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\right\| \\ &\quad\leq \gamma\left\|f(x) - f(y)\right\| + \|I - A\|\|x - y\| \\ &\quad\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\ &\quad< \gamma\|x - y\|.\end{aligned} \tag{3.42}$$

Banach's Contraction Mapping Principle guarantees that $P_F(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I - A))(q)$.

Next, we claim that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, t_n - q\rangle \leq 0. \tag{3.43}$$

Indeed, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, W t_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Aq, W t_{n_i} - q\rangle. \tag{3.44}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup z$. From $\|Wt_{n_i} - t_{n_i}\| \to 0$, we obtain $Wt_{n_i} \rightharpoonup z$. Therefore, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Aq, Wt_n - q\rangle &= \lim_{i\to\infty}\langle \gamma f(q) - Aq, Wt_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, z - q\rangle.\end{aligned} \tag{3.45}$$

Next we prove that $z \in F := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(B, C)$.

First, we prove that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Suppose the contrary, $z \notin F(W)$, that is, $Wz \neq z$. Since $t_{n_i} \rightharpoonup z$, by the Opial's condition and (3.41), we have

$$\begin{aligned}\liminf_{i\to\infty}\|t_{n_i} - z\| &< \liminf_{i\to\infty}\|t_{n_i} - Wz\| \\ &\le \liminf_{i\to\infty}\{\|t_{n_i} - Wt_{n_i}\| + \|Wt_{n_i} - Wz\|\} \\ &\le \liminf_{i\to\infty}\{\|t_{n_i} - Wt_{n_i}\| + \|t_{n_i} - z\|\} \\ &= \liminf_{i\to\infty}\|t_{n_i} - z\|.\end{aligned} \tag{3.46}$$

This is a contradiction, which shows that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Next, we prove $z \in \mathrm{VI}(B, C)$. For this purpose, let T be the maximal monotone mapping defined by (2.7):

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \varnothing, & v \notin C.\end{cases} \tag{3.47}$$

For any given $(v, w) \in G(T)$, hence $w - Bv \in N_C(v)$. Since $t_n \in C$, we have

$$\langle v - t_n, w - Bv\rangle \ge 0. \tag{3.48}$$

On the other hand, from $t_n = P_C(y_n - \lambda_n B y_n)$, we have

$$\langle v - t_n, t_n - (y_n - \lambda_n B y_n)\rangle \ge 0, \tag{3.49}$$

that is,

$$\left\langle v - t_n, \frac{t_n - y_n}{\lambda_n} + By_n \right\rangle \ge 0. \tag{3.50}$$

Therefore, we obtian

$$\begin{aligned}
\langle v-t_{n_i},w\rangle &\geq \langle v-t_{n_i},Bv\rangle \\
&\geq \langle v-t_{n_i},Bv\rangle - \left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle \\
&= \left\langle v-t_{n_i},Bv-By_{n_i}-\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle v-t_{n_i},Bv-Bt_{n_i}\rangle + \langle v-t_{n_i},Bt_{n_i}-By_{n_i}\rangle - \left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v-t_{n_i},Bt_{n_i}\rangle - \left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle \\
&= \langle v-t_{n_i},Bt_{n_i}-By_{n_i}\rangle - \left\langle v-t_{n_i},\frac{t_{n_i}-y_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned} \tag{3.51}$$

Noting that $\|t_{n_i}-y_{n_i}\| \to 0$ as $n \to \infty$ and B is Lipschitz continuous, hence from (3.18), we obtain

$$\langle v-z,w\rangle \geq 0. \tag{3.52}$$

Since T is maximal monotone, we have $z \in T^{-1}0$, and hence $z \in \mathrm{VI}(B,C)$.

The conclusion $z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{VI}(B,C)$ is proved.

Hence by (3.45), we obtain

$$\limsup_{n\to\infty}\langle \gamma f(q)-Aq,Wt_n-q\rangle = \langle \gamma f(q)-Aq,z-q\rangle \leq 0. \tag{3.53}$$

Since $q = P_F f(q)$, it follows from (3.39), (3.41), and (3.53) that

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(q)-Aq,t_n-q\rangle &= \limsup_{n\to\infty}\langle \gamma f(q)-Aq,(t_n-Wt_n)+(Wt_n-q)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(q)-Aq,Wt_n-q\rangle \\
&\leq 0.
\end{aligned} \tag{3.54}$$

Hence (3.43) holds. Using (3.26) and (3.54), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(q)-Aq,x_n-q\rangle &= \limsup_{n\to\infty}\langle \gamma f(q)-Aq,(x_n-t_n)+(t_n-q)\rangle \\
&\leq \limsup_{n\to\infty}\langle \gamma f(q)-Aq,t_n-q\rangle \\
&\leq 0.
\end{aligned} \tag{3.55}$$

Now, from Lemma 2.1, it follows that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n\gamma f(x_n)+\delta_n x_n+((1-\delta_n)I-\alpha_n A)t_n-q\|^2\\
&= \|((1-\delta_n)I-\alpha_n A)(t_n-q)+\delta_n(x_n-q)+\alpha_n(\gamma f(x_n)-Aq)\|^2\\
&= \|((1-\delta_n)I-\alpha_n A)(t_n-q)+\delta_n(x_n-q)\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Aq\rangle\\
&\quad +2\alpha_n\langle((1-\delta_n)I-\alpha_n A)(t_n-q),\gamma f(x_n)-Aq\rangle\\
&\le ((1-\delta_n-\alpha_n\overline{\gamma})\|t_n-q\|+\delta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\gamma\langle x_n-q,f(x_n)-f(q)\rangle+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-A(q)\rangle\\
&\quad +2(1-\delta_n)\gamma\alpha_n\langle t_n-q,f(x_n)-f(q)\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle\\
&\le ((1-\delta_n-\alpha_n\overline{\gamma})\|x_n-q\|+\delta_n\|x_n-q\|)^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\gamma\alpha\|x_n-q\|^2+2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle\\
&\quad +2(1-\delta_n)\gamma\alpha_n\alpha\|x_n-q\|^2+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle\\
&= \left[(1-\alpha_n\overline{\gamma})^2+2\delta_n\alpha_n\gamma\alpha+2(1-\delta_n)\gamma\alpha_n\alpha\right]\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad -2\alpha_n^2\langle A(t_n-q),\gamma f(q)-Aq\rangle\\
&\le \left[1-2(\overline{\gamma}-\alpha_n\gamma)\alpha_n\right]\|x_n-q\|^2+\overline{\gamma}^2\alpha_n^2\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Aq\|^2\\
&\quad +2\delta_n\alpha_n\langle x_n-q,\gamma f(q)-Aq\rangle+2(1-\delta_n)\alpha_n\langle t_n-q,\gamma f(q)-Aq\rangle\\
&\quad +2\alpha_n^2\|A(t_n-q)\|\|\gamma f(q)-Aq\|\\
&= \left[1-2(\overline{\gamma}-\alpha_n\gamma)\alpha_n\right]\|x_n-q\|^2\\
&\quad +\alpha_n\Big\{\alpha_n(\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2\\
&\qquad +2\|A(t_n-q)\|\|\gamma f(q)-Aq\|)+2\delta_n\langle x_n-q,\gamma f(q)-Aq\rangle\\
&\qquad +2(1-\delta_n)\langle t_n-q,\gamma f(q)-Aq\rangle\Big\}.
\end{aligned} \tag{3.56}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and t_n are bounded, we can take a constant $M_5>0$ such that

$$\overline{\gamma}^2\|x_n-q\|^2+\|\gamma f(x_n)-Aq\|^2+2\|A(t_n-q)\|\|\gamma f(q)-Aq\|\le M_5, \tag{3.57}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq [1 - 2(\overline{\gamma} - \alpha_n \gamma)\alpha_n] \|x_n - q\|^2 + \alpha_n \sigma_n, \tag{3.58}$$

where

$$\sigma_n = 2\delta_n \langle x_n - q, \gamma f(q) - Aq \rangle + 2(1 - \delta_n)\langle t_n - q, \gamma f(q) - Aq \rangle + \alpha_n M_4. \tag{3.59}$$

Using (C1), (3.54), and (3.55), we get $\limsup_{n\to\infty} \delta_n \leq 0$. Now applying Lemma 2.3 to (3.58), we conclude that $x_n \to q$. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improve the results of Qin and Cho [14] from a single nonexpansive mapping to an infinite family of nonexpansive mappings.

4. Applications

In this section, we obtain two results by using a special case of the proposed method.

Theorem 4.1. *Let H be a real Hilbert space, let B be an α-inverse strongly monotone mapping on H, let $\{T_i : H \to H\}$ be a family of infinitely nonexpansive mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \cap B^{-1}(0) \neq \emptyset$. Let $f : H \to H$ a contraction with coefficient $\alpha \in (0,1)$, and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Suppose the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by*

$$\begin{aligned} x_1 &= x \in H \text{ chosen arbitrary}, \\ z_n &= \gamma_n x_n + (1 - \gamma_n) W_n x_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) W_n z_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A)(y_n - \lambda_n B y_n), \end{aligned} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

- (C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (C2) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$,
- (C3) $(1 + \beta_n)\gamma_n - 2\beta_n > a$ *for some* $a \in (0,1)$,
- (C4) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = \lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$,
- (C5) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $q = P_F(\gamma f + (I - A))(q)$ which solves the variational inequality:

$$\langle (A - \gamma f)q, q - z \rangle \leq 0, \quad z \in F. \tag{4.2}$$

Proof. We have $B^{-1}(0) = \mathrm{VI}(B, H)$ and $P_H = I$. Applying Theorem 3.1, we obtain the desired result. □

Next, we will apply the main results to the problem for finding a common element of the set of fixed points of a family of infinitely nonexpansive mappings and the set of fixed points of a finite family of k-strictly pseudocontractive mappings.

Definition 4.2. A mappings $S : C \to H$ is said to be a k-strictly pseudocontractive mapping if there exists $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{4.3}$$

The following lemmas can be obtained from [31, Proposition 2.6] by Acedo and Xu, easily.

Lemma 4.3. *Let H be a Hilbert space, let C be a closed convex subset of H. For any integer $N \ge 1$, assume that, for each $1 \le i \le N$, $S_i : C \to H$ is a k_i-strictly pseudocontractive mapping for some $0 \le k_i < 1$. Assume that $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$. Then $\sum_{i=1}^N \varphi_i S_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \le i \le N\}$.*

Lemma 4.4. *Let $\{S_i\}_{i=1}^N$ and $\{\varphi_i\}_{i=1}^N$ be as in Lemma 4.3. Suppose that $\{S_i\}_{i=1}^N$ has a common fixed point in C. Then $F(\sum_{i=1}^N \varphi_i S_i) = \bigcap_{i=1}^N F(S_i)$.*

Let $S_i : C \to H$ be a k_i-strictly pseudocontractive mapping for some $0 \le k_i < 1$. We define a mapping $A = I - \sum_{i=1}^N \varphi_i S_i : C \to H$, where $\{\varphi_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \varphi_i = 1$, then A is a $((1-k)/2)$-inverse-strongly monotone mapping with $k = \max\{k_i : 1 \le i \le N\}$. In fact, from Lemma 4.3, we have

$$\left\|\sum_{i=1}^N \varphi_i S_i x - \sum_{i=1}^N \varphi_i S_i y\right\|^2 \le \|x - y\|^2 + k\left\|\left(I - \sum_{i=1}^N \varphi_i S_i\right)x - \left(I - \sum_{i=1}^N \varphi_i S_i\right)y\right\|^2, \quad \forall x, y \in C. \tag{4.4}$$

That is,

$$\|(I - A)x - (I - A)y\|^2 \le \|x - y\|^2 + k\|Ax - Ay\|^2. \tag{4.5}$$

On the other hand

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay\rangle + \|Ax - Ay\|^2. \tag{4.6}$$

Hence, we have

$$\langle x - y, Ax - Ay\rangle \ge \frac{1-k}{2}\|Ax - Ay\|^2. \tag{4.7}$$

This shows that A is $((1-k)/2)$-inverse-strongly monotone.

Theorem 4.5. *Let C be a closed convex subset of a real Hilbert space H. For any integer $N > 1$, assume that, for each $1 \le i \le N$, $S_i : C \to H$ is a k_i-strictly pseudocontractive mapping for* *some* $0 \leq k_i < 1$. *Let* $\{T_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with* $F := \bigcap_{i=1}^{\infty} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \neq \emptyset$. *Let* $f : C \to C$ *a contraction with coefficient* $\alpha \in (0,1)$ *and let* A *be a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > 0$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated by*

$$\begin{gathered} x_1 = x \in H \text{ chosen arbitrary}, \\ z_n = \gamma_n x_n + (1-\gamma_n) W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n) W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1-\delta_n) I - \alpha_n A) P_C\left((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n \right), \end{gathered} \tag{4.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *are the sequences in* $[0,1]$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\delta_n \leq \limsup_{n\to\infty}\delta_n < 1$,

(C3) $(1+\beta_n)\gamma_n - 2\beta_n > a$ *for some* $a \in (0,1)$,

(C4) $\lim_{n\to\infty}|\beta_{n+1} - \beta_n| = \lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,

(C5) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 2\alpha)$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to* $q = P_F(\gamma f + (I - A))(q)$ *which solves the variational inequality:*

$$\langle (A - \gamma f) q, q - z \rangle \leq 0, \quad z \in F. \tag{4.9}$$

Proof. Taking $B = I - \sum_{i=1}^{N} \varphi_i S_i : C \to H$, we know that $B : C \to H$ is α-inverse strongly monotone with $\alpha = (1-k)/2$. Hence, B is a monotone L-Lipschitz continuous mapping with $L = 2/(1-k)$. From Lemma 4.4, we know that $\sum_{i=1}^{N} \varphi_i S_i$ is a k-strictly pseudocontractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and then $F(\sum_{i=1}^{N} \varphi_i S_i) = \mathrm{VI}(B, C)$ by Chang [30, Proposition 1.3.5]. Observe that

$$P_C(y_n - \lambda_n B y_n) = P_C\left((1-\lambda_n) y_n - \lambda_n \sum_{i=1}^{N} \varphi_i S_i y_n \right). \tag{4.10}$$

The conclusion of Theorem 4.5 can be obtained from Theorem 3.1. □

Remark 4.6. Theorem 4.5 is a generalization and improvement of the theorems by Qin and Cho [14], Iiduka and Takahashi [16, Thorem 3.1], and Takahashi and Toyoda [15].

Acknowledgments

The authors would like to thank the referee for the comments which improve the manuscript. R. Wangkeeree was supported for CHE-PhD-THA-SUP/2551 from the Commission on Higher Education and the Thailand Research Fund under Grant TRG5280011.

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