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Research Article

A Hybrid Iterative Scheme for Equilibrium Problems, Variational Inequality Problems, and Fixed Point Problems in Banach Spaces

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The purpose of this paper is to introduce a new hybrid projection algorithm for finding a common element of the set of solutions of the equilibrium problem and the set of the variational inequality for an inverse-strongly monotone operator and the set of fixed points of relatively quasi-nonexpansive mappings in a Banach space. Then we show a strong convergence theorem. Using this result, we obtain some applications in a Banach space.

1. Introduction

Let E be a real Banach space and let E^* be the dual of E. Let C be a closed convex subset of E. Let $A : C \to E^*$ be an operator. The classical variational inequality problem for A is to find $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(A, C)$. Such a problem is connected with the convex minimization problem, the complementarity, the problem of finding a point $\hat{x} \in E$ satisfying $0 = A\hat{x}$, and so on. First, we recall that

(1) an operator A is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{1.2}$$

(2) an operator A is called *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ with

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.3}$$

Assume that

(C1) A is α-inverse-strongly monotone,

(C2) $VI(A, C) \neq \emptyset$,

(C3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$.

Iiduka and Takahashi [1] introduced the following algorithm for finding a solution of the variational inequality for an operator A that satisfies conditions (C1)–(C3) in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_1 = x \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \quad \forall n \geq 1, \tag{1.4}$$

where J is the duality mapping on E, and Π_C is the generalized projection from E onto C. Assume that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$ where $1/c$ is the p-uniformly convexity constant of E. They proved that if J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to some element z in $VI(A, C)$ where $z = \lim_{n\to\infty}\Pi_{VI(A,C)}(x_n)$.

The problem of finding a common element of the set of the variational inequalities for monotone mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors; see, for instance, [2–4] and the references cited therein.

Let $f : C \times C \to R$ be a bifunction. The equilibrium problem for f is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $EP(f)$.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y); \tag{1.6}$$

(A4) for all $x \in C$, $\quad f(x, \cdot)$ is convex and lower semicontinuous.

Recently, Takahashi and Zembayashi [5], introduced the following iterative scheme which is called the shrinking projection method:

$$
\begin{gathered}
x_0 = x \in C, \qquad C_0 = C, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{1.7}
$$

where J is the duality mapping on E and Π_C is the generalized projection from E onto C. They proved that the sequence $\{x_n\}$ converges strongly to $q = \Pi_{F(T)\cap EP(f)} x_0$ under appropriate conditions.

Very recently, Qin et al. [6] extend the iteration process (1.7) from a single relatively nonexpansive mapping to two relatively quasi-nonexpansive mappings:

$$
\begin{gathered}
x_0 \in E, \text{ chosen arbitrarily}, \\
C_1 = C, \qquad x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0.
\end{gathered} \tag{1.8}
$$

Under suitable conditions over $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$, they obtain that the sequence $\{x_n\}$ generated by (1.8) converges strongly to $q = \Pi_{F(T)\cap F(S)\cap EP(f)} x_0$.

The problem of finding a common element of the set of fixed points and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces has been studied by many authors; see [5, 7–16].

Motivated by Iiduka and Takahashi [1], Takahashi and Zembayashi [5], and Qin et al. [6], we introduce a new general process for finding common elements of the set of the equilibrium problem and the set of the variational inequality problem for an inverse-strongly monotone operator and the set of the fixed points for relatively quasi-nonexpansive mappings.

2. Preliminaries

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if for any $x, y \in U$,

$$x \neq y \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| < 1 - \delta. \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex; and we define a function $\delta : [0,2] \to [0,1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \quad x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| \geq \varepsilon \right\}. \tag{2.3}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$; see [17–19] for more details. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in U$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [19]. It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1} \right\} \tag{2.5}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. See [20, 21] for more details.

Lemma 2.1 (See [18, 22]). *Let p be a given real number with $p \geq 2$ and E a p-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p, \tag{2.6}$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.7}$$

for all $x, y \in E$. In a Hilbert space H, we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

Recall that a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive if T satisfies the following conditions:

(1) $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T;

(2) $\phi(p, Tx) \le \phi(p, x)$ for all $p \in F(T)$ and $x \in C$;

(3) $F(\widehat{T}) = F(T)$, where $F(\widehat{T})$ is the set of all asymptotic fixed points of T;

see [10, 23, 24] for more details.

T is said to be relatively quasi-nonexpansive if T satisfies the conditions (1) and (2). It is easy to see that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [9, 25, 26].

We give some examples which are closed relatively quasi-nonexpansive; see [6].

Example 2.2. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0 \neq \emptyset$. Then, $J_r = (J + rA)^{-1}J$ is a closed relatively quasi-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.3. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed relatively quasi-nonexpansive mapping with $F(\Pi_C) = C$.

Lemma 2.4 (Kamimura and Takahashi [27]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Let C be a nonempty closed convex subset of E. If E is reflexive, strictly convex and smooth, then there exists $x_0 \in C$ such that $\phi(x_0, x) = \min\phi(y, x)$ for $x \in E$ and $y \in C$. The generalized projection $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$. The existence and uniqueness of the operator Π_C follows from the properties of the functional ϕ and strict monotonicity of the duality mapping J; for instance, see [20, 27–30]. In a Hilbert space, Π_C is coincident with the metric projection.

Lemma 2.5 (Alber [28]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \ge 0$ for all $y \in C$.*

Lemma 2.6 (Alber [28]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x), \quad \forall y \in C. \tag{2.8}$$

Lemma 2.7 (Qin et al. [6]). *Let E be a uniformly convex, smooth Banach space, let C be a closed convex subset of E, let T be a closed and relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.8 (Cho et al. [31]). *Let E be a uniformly convex Banach space and let $B_r(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g(\|x - y\|), \tag{2.9}$$

for all $x, y, z \in B_r(0)$, and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.9 (Blum and Oettli [7]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

Lemma 2.10 (Qin et al. [6]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\right\}. \tag{2.11}$$

Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping [32], that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.12}$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

Lemma 2.11 (Takahashi and Zembayashi [14]). *Let C be a closed convex subset of a smooth, strictly, and reflexive Banach space E, let f be a bifucntion from $C \times C$ to R satisfying (A1)–(A4), let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.13}$$

We make use of the following mapping V studied in Alber [28]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.14}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.12 (Alber [28]). *Let E be a reflexive, strictly convex, smooth Banach space and let V be as in* (2.14). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.15}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

An operator A of C into E^* is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \ \forall y \in C\}. \tag{2.16}$$

Theorem 2.13 (Rockafellar [33]). *Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T_e \subset E \times E^*$ be an operator defined as follows:*

$$T_e v = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.17}$$

Then T_e is maximal monotone and $T_e^{-1}0 = VI(A, C)$.

3. Strong Convergence Theorems

Theorem 3.1. *Let E be a 2-uniformly convex, uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), let A be an operator of C into E^* satisfying (C1)–(C3), and let T, S be two closed relatively quasi-nonexpansive mappings from C into itself such that $F := F(T) \cap F(S) \cap EP(f) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:*

$$\begin{gathered} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S z_n), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{gathered} \tag{3.1}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the restrictions:

(B1) $\alpha_n + \beta_n + \gamma_n = 1$;

(B2) $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;

(B3) $\{r_n\} \subset [s, \infty)$ *for some* $s > 0$;

(B4) $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, *where* $1/c$ *is the* 2*-uniformly convexity constant of* E.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = \Pi_F x_0$.

Proof. We divide the proof into eight steps.

Step 1. Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well defined.

It is obvious that $VI(A, C)$ is a closed convex subset of C. By Lemma 2.7, we know that $F(T) \cap F(S)$ is closed and convex. From Lemma 2.10 (4), we also have $EP(f)$ is closed and convex. Hence $F := F(T) \cap F(S) \cap EP(f) \cap VI(A, C)$ is a nonempty, closed, and convex subset of C; consequently, $\Pi_F x_0$ is well defined.

Clearly, $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for $k \in N$. For all $z \in C_k$, we know $\phi(z, y_k) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k - Jy_k \rangle \leq \|x_k\|^2 - \|y_k\|^2. \tag{3.2}$$

So, C_{k+1} is closed and convex. By induction, C_n is closed and convex for all $n \geq 1$. This shows that $\Pi_{C_{n+1}} x_0$ is well-defined.

Step 2. Show that $F \subset C_n$ for all $n \in N$.

Put $v_n = J^{-1}(Jx_n - \lambda_n A x_n)$. First, we observe that $u_n = T_{r_n} y_n$ for all $n \geq 1$ and $F \subset C_1 = C$. Suppose $F \subset C_k$ for $k \in N$. Then, for all $u \in F$, we know from Lemma 2.6 and Lemma 2.12 that

$$\begin{aligned}
\phi(u, z_k) &= \phi(u, \Pi_C v_k) \\
&\leq \phi(u, v_k) \\
&= \phi\Big(u, J^{-1}(Jx_k - \lambda_k A x_k)\Big) \\
&= V(u, Jx_k - \lambda_k A x_k) \\
&\leq V(u, (Jx_k - \lambda_k A x_k) + \lambda_k A x_k) - 2\Big\langle J^{-1}(Jx_k - \lambda_k A x_k) - u, \lambda_k A x_k \Big\rangle \\
&= V(u, Jx_k) - 2\lambda_k \langle v_k - u, A x_k \rangle \\
&= \phi(u, x_k) - 2\lambda_k \langle x_k - u, A x_k \rangle + 2\langle v_k - x_k, -\lambda_k A x_k \rangle.
\end{aligned} \tag{3.3}$$

Since $u \in VI(A, C)$ and from (C1), we have

$$\begin{aligned}
-2\lambda_k \langle x_k - u, A x_k \rangle &= -2\lambda_k \langle x_k - u, A x_k - Au \rangle - 2\lambda_k \langle x_k - u, Au \rangle \\
&\leq -2\alpha\lambda_k \|A x_k - Au\|^2.
\end{aligned} \tag{3.4}$$

From Lemma 2.1 and (C3), we obtain

$$\begin{aligned}
2\langle v_k - x_k, -\lambda_k A x_k \rangle &= 2\langle J^{-1}(Jx_k - \lambda_k A x_k) - J^{-1}(Jx_k), -\lambda_k A x_k \rangle \\
&\leq 2\left\| J^{-1}(Jx_k - \lambda_k A x_k) - J^{-1}(Jx_k) \right\| \|\lambda_k A x_k\| \\
&\leq \frac{4}{c^2}\left\| JJ^{-1}(Jx_k - \lambda_k A x_k) - JJ^{-1}(Jx_k) \right\| \|\lambda_k A x_k\| \\
&= \frac{4}{c^2}\|(Jx_k - \lambda_k A x_k) - (Jx_k)\| \|\lambda_k A x_k\| \\
&= \frac{4}{c^2}\lambda_k^2 \|Ax_k\|^2 \\
&\leq \frac{4}{c^2}\lambda_k^2 \|Ax_k - Au\|^2.
\end{aligned} \tag{3.5}$$

Replacing (3.4) and (3.5) into (3.3), we get

$$\phi(u, z_k) \leq \phi(u, x_k) + 2\lambda_k\left(\frac{2}{c^2}\lambda_k - \alpha\right)\|Ax_k - Au\|^2 \leq \phi(u, x_k). \tag{3.6}$$

By the convexity of $\|\cdot\|^2$, for each $u \in F \subset C_k$, we obtain

$$\begin{aligned}
\phi(u, u_k) &= \phi(u, T_{r_k} y_k) \\
&\leq \phi(u, y_k) \\
&= \phi\left(u, J^{-1}(\alpha_k J x_k + \beta_k J T x_k + \gamma_k J S z_k)\right) \\
&= \|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2\beta_k\langle u, JTx_k\rangle - 2\gamma_k\langle u, JSz_k\rangle \\
&\quad + \left\|\alpha_k J x_k + \beta_k J T x_k + \gamma_k J S z_k\right\|^2 \\
&\leq \|u\|^2 - 2\alpha_k\langle u, Jx_k\rangle - 2\beta_k\langle u, JTx_k\rangle - 2\gamma_k\langle u, JSz_k\rangle \\
&\quad + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\|JSz_k\|^2 \\
&= \alpha_k\phi(u, x_k) + \beta_k\phi(u, Tx_k) + \gamma_k\phi(u, Sz_k) \\
&\leq \alpha_k\phi(u, x_k) + \beta_k\phi(u, x_k) + \gamma_k\phi(u, z_k) \\
&\leq \phi(u, x_k).
\end{aligned} \tag{3.7}$$

This shows that $u \in C_{k+1}$; consequently, $F \subset C_{k+1}$. Hence $F \subset C_n$ for all $n \geq 1$.

Step 3. Show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1. \tag{3.8}$$

From Lemma 2.6, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \le \phi(u, x_0) - \phi(u, x_n) \le \phi(u, x_0). \tag{3.9}$$

Combining (3.8) and (3.9), we obtain that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

Step 4. Show that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for $m > n$, by Lemma 2.6, we also have

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\le \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.10}$$

Taking $m, n \to \infty$, we obtain that $\phi(x_m, x_n) \to 0$. From Lemma 2.4, we have $\|x_m - x_n\| \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of E and the closedness of C, one can assume that $x_n \to q \in C$ as $n \to \infty$. Further, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) \longrightarrow 0, \tag{3.12}$$

as $n \to \infty$. Applying Lemma 2.4 to (3.11) and (3.12), we get

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.13}$$

This implies that $u_n \to q$ as $n \to \infty$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also obtain

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0. \tag{3.14}$$

Step 5. Show that $x_n \to q \in F(T) \cap F(S)$.

Let $r = \sup_{n\geq 1}\{\|x_n\|, \|Tx_n\|, \|Sz_n\|\}$. From (3.6) and Lemma 2.8, we know that there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, T_{r_n} y_n) \\
&\leq \phi(u, y_n) \\
&= \phi\Big(u, J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S z_n)\Big) \\
&= \|u\|^2 - 2\alpha_n \langle u, Jx_n\rangle - 2\beta_n\langle u, JTx_n\rangle - 2\gamma_n\langle u, JSz_n\rangle \\
&\quad + \big\|\alpha_n J x_n + \beta_n JTx_n + \gamma_n JSz_n\big\|^2 \\
&\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n\rangle - 2\beta_n\langle u, JTx_n\rangle - 2\gamma_n\langle u, JSz_n\rangle \\
&\quad + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 + \gamma_n\|JSz_n\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\
&= \alpha_n\phi(u, x_n) + \beta_n\phi(u, Tx_n) + \gamma_n\phi(u, Sz_n) \\
&\quad - \alpha_n\beta_n g(\|Jx_n - JTx_n\|) \\
&\leq \phi(u, x_n) + 2\gamma_n\lambda_n\left(\frac{2}{c^2}\lambda_n - \alpha\right)\|Ax_n - Au\|^2 \\
&\quad - \alpha_n\beta_n g(\|Jx_n - JTx_n\|).
\end{aligned} \tag{3.15}$$

This implies that

$$\begin{aligned}
\alpha_n\beta_n g(\|Jx_n - JTx_n\|) &\leq \phi(u, x_n) - \phi(u, u_n) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|.
\end{aligned} \tag{3.16}$$

It follows from (3.13), (3.14), and (B2) that

$$\lim_{n\to\infty} g(\|Jx_n - JTx_n\|) = 0. \tag{3.17}$$

By the property of g, we also obtain that

$$\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0. \tag{3.18}$$

Since J is uniformly norm-to-norm continuous on bounded sets, so is J^{-1}. Then

$$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\left\|J^{-1}(Jx_n) - J^{-1}(JTx_n)\right\| = 0. \tag{3.19}$$

In the same manner, we can show that

$$\lim_{n\to\infty}\|x_n - Sz_n\| = 0. \tag{3.20}$$

Again, by (3.15), we have

$$2a\left(\alpha - \frac{2}{c^2}b\right)\|Ax_n - Au\|^2 \leq \frac{1}{\gamma_n}(\phi(u,x_n) - \phi(u,u_n)), \tag{3.21}$$

which yields that

$$\lim_{n\to\infty}\|Ax_n - Au\| = 0. \tag{3.22}$$

From Lemma 2.6, Lemma 2.12, and (3.5), we have

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, \Pi_C v_n) \\
&\leq \phi(x_n, v_n) \\
&= \phi\left(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)\right) \\
&= V(x_n, Jx_n - \lambda_n Ax_n) \\
&\leq V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) \\
&\quad - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\
&= \phi(x_n, x_n) + 2\langle v_n - x_n, -\lambda_n Ax_n\rangle \\
&= 2\langle v_n - x_n, -\lambda_n Ax_n\rangle \leq \frac{4}{c^2}b^2\|Ax_n - Au\|^2.
\end{aligned} \tag{3.23}$$

It follows from Lemma 2.4 and (3.22) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.24}$$

Hence $z_n \to q$ as $n \to \infty$ and

$$\lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \tag{3.25}$$

Combining (3.20) and (3.24), we also obtain

$$\lim_{n\to\infty}\|Sz_n - z_n\| = 0. \tag{3.26}$$

From (3.19), (3.26) and by the closedness of T and S, we get $q \in F(T) \cap F(S)$.

Step 6. Show that $x_n \to q \in EP(f)$.

From (3.15), we see

$$\phi(u, y_n) \leq \phi(u, x_n). \tag{3.27}$$

From (3.16), we observe

$$\lim_{n\to\infty} \phi(u, x_n) - \phi(u, u_n) = 0. \tag{3.28}$$

Note that $u_n = T_{r_n} y_n$. From (3.27) and Lemma 2.11, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, T_{r_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, T_{r_n} y_n) \\ &= \phi(u, x_n) - \phi(u, u_n). \end{aligned} \tag{3.29}$$

From (3.28), we get $\lim_{n\to\infty}\phi(u_n, y_n) = 0$. By Lemma 2.4, we obtain

$$\|u_n - y_n\| \longrightarrow 0 \tag{3.30}$$

as $n \to \infty$. Since $r_n \geq s$, we have

$$\frac{\|Ju_n - Jy_n\|}{r_n} \longrightarrow 0 \tag{3.31}$$

as $n \to \infty$. From $u_n = T_{r_n} y_n$ we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C. \tag{3.32}$$

By (A2), we have

$$\begin{aligned} \|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -f(u_n, y) \\ &\geq f(y, u_n), \quad \forall y \in C. \end{aligned} \tag{3.33}$$

From (A4) and $u_n \to q$, we get $f(y, q) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$. Define $y_t = ty + (1-t)q$, then $y_t \in C$, which implies that $f(y_t, q) \leq 0$. From (A1), we obtain that $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, q) \leq tf(y_t, y)$. Thus, $f(y_t, y) \geq 0$. From (A3), we have $f(q, y) \geq 0$ for all $y \in C$. Hence $q \in EP(f)$.

Step 7. Show that $x_n \to q \in VI(A,C)$.

Define $T_e \subset E \times E^*$ be as in (2.17). By Theorem 2.13, T_e is maximal monotone and $T_e^{-1}0 = VI(A,C)$. Let $(v,w) \in G(T_e)$. Since $w \in T_e v = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \geq 0. \tag{3.34}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, by Lemma 2.5, we have $\langle v - z_n, Jz_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0$ and thus

$$\left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.35}$$

It follows from (3.34) and (3.35) that

$$\begin{aligned}
\langle v - z_n, w \rangle &\geq \langle v - z_n, Av \rangle \\
&\geq \langle v - z_n, Av \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n \right\rangle \\
&= \langle v - z_n, Av - Ax_n \rangle + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\
&= \langle v - z_n, Av - Az_n \rangle + \langle v - z_n, Az_n - Ax_n \rangle \\
&\quad + \left\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} \right\rangle \\
&\geq -\|v - z_n\| \frac{\|z_n - x_n\|}{\alpha} - \|v - z_n\| \frac{\|Jx_n - Jz_n\|}{a} \\
&\geq -M\left(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jz_n\|}{a} \right),
\end{aligned} \tag{3.36}$$

where $M = \sup_{n\geq 1}\{\|v - z_n\|\}$. By taking the limit as $n \to \infty$ and from (3.24) and (3.25), we obtain $\langle v - q, w \rangle \geq 0$. By the maximality of T_e, we have $q \in T_e^{-1}0$ and hence $q \in VI(A,C)$.

Step 8. Show that $q = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0, \quad \forall z \in C_n. \tag{3.37}$$

Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - u \rangle \geq 0, \quad \forall u \in F. \tag{3.38}$$

By taking limit in (3.38), we obtain that

$$\langle Jx_0 - Jq, q - u \rangle \geq 0, \quad \forall u \in F. \tag{3.39}$$

By Lemma 2.5, we can conclude that $q = \Pi_F x_0$. Furthermore, it is easy to see that $u_n \to q$ as $n \to \infty$. This completes the proof. □

As a direct consequence of Theorem 3.1, we obtain the following results.

Corollary 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let T be a closed relatively quasi-nonexpansive mapping from C into itself such that $F(T) \cap EP(f) \neq \emptyset$. Assume that $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [s,\infty)$ for some $s > 0$. Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to $q = \Pi_{F(T)\cap EP(f)} x_0$.*

Proof. Putting $S = T$ and $A \equiv 0$ in Theorem 3.1, we obtain the result. □

Remark 3.3. If $A \equiv 0$ in Theorem 3.1, then Theorem 3.1 reduces to Theorem 3.1 of Qin et al. [6].

Remark 3.4. Corollary 3.2 improves Theorem 3.1 of Takahashi and Zembayashi [5] from the class of relatively nonexpansive mappings to the class of relatively quasi-nonexpansive mappings, that is, we relax the strong restriction: $F(\widehat{T}) = F(T)$. Further, the algorithm in Corollary 3.2 is also simpler to compute than the one given in [14].

4. Applications

Next, we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*. Assume that A satisfies the conditions:

(D1) A is α-inverse-strongly monotone,

(D2) $A^{-1}0 = \{u \in E : \ Au = 0\} \neq \emptyset$.

Theorem 4.1. *Let E be a 2-uniformly convex, uniformly smooth Banach space. Let f be a bifunction from $E \times E$ to R satisfying (A1)–(A4), let A be an operator of E into E^* satisfying (D1) and (D2), and let T, S be two closed relatively quasi-nonexpansive mappings from E into itself such that $F := F(T) \cap F(S) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = E$, define a sequence $\{x_n\}$ as follows:*

$$\begin{gathered}
z_n = J^{-1}(Jx_n - \lambda_n A x_n), \\
y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S z_n), \\
u_n \in E \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in E, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{4.1}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the conditions (B1)–(B4) of Theorem 3.1.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. Putting $C = E$ in Theorem 3.1, we have $\Pi_E = I$. We also have $VI(A, E) = A^{-1}0$ and then the condition (C3) of Theorem 3.1 holds for all $y \in E$ and $u \in A^{-1}0$. So, we obtain the result. □

Let K be a nonempty, closed convex cone in E, A an operator of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \ \forall x \in K\}. \tag{4.2}$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \qquad \langle u, Au \rangle = 0. \tag{4.3}$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Assume that A is an operator satisfying the conditions:

(E1) A is α-inverse-strongly monotone,

(E2) $C(K, A) \neq \emptyset$,

(E3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in C(K, A)$.

Theorem 4.2. *Let E be a 2-uniformly convex, uniformly smooth Banach space, and K a nonempty, closed convex cone in E. Let f be a bifunction from $K \times K$ to R satisfying (A1)–(A4), let A be an operator of K into E^* satisfying (E1)–(E3), and let T, S be two closed relatively quasi-nonexpansive mappings from K into itself such that $F := F(T) \cap F(S) \cap EP(f) \cap C(K, A) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = K$, define a sequence $\{x_n\}$ as follows:*

$$\begin{gathered}
z_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\
y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSz_n), \\
u_n \in K \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in K, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{gathered} \tag{4.4}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the conditions (B1)–(B4) of Theorem 3.1.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. From [20, Lemma 7.1.1], we have $VI(K, A) = C(K, A)$. Hence, we obtain the result. □

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