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Research Article

The Theory of Reich's Fixed Point Theorem for Multivalued Operators

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The purpose of this paper is to present a theory of Reich's fixed point theorem for multivalued operators in terms of fixed points, strict fixed points, multivalued weakly Picard operators, multivalued Picard operators, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, and the generated fractal operator.

1. Introduction

Let (X, d) be a metric space and consider the following family of subsets $P_{cl}(X) := \{Y \subseteq X \mid Y \text{ is nonempty and closed}\}$. We also consider the following (generalized) functionals:

$$D : P(X) \times P(X) \longrightarrow \mathbb{R}_+, \qquad D(A, B) := \inf\{d(a, b) \mid a \in A, b \in B\}, \tag{1.1}$$

D is called the gap functional between A and B. In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$:

$$\rho : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}, \qquad \rho(A, B) := \sup\{D(a, B) \mid a \in A\}, \tag{1.2}$$

ρ is called the (generalized) excess functional:

$$H : P(X) \times P(X) \longrightarrow \mathbb{R}_+ \cup \{+\infty\}, \qquad H(A, B) := \max\{\rho(A, B), \rho(B, A)\}, \tag{1.3}$$

H is the (generalized) Pompeiu-Hausdorff functional.

It is well known that if (X, d) is a complete metric space, then the pair $(P_{cl}(X), H)$ is a complete generalized metric space. (See [1, 2]).

Definition 1.1. If (X, d) is a metric space, then a multivalued operator $T : X \to P_{cl}(X)$ is said to be a Reich-type multivalued (a, b, c)-contraction if and only if there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$H(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)), \quad \text{for each } x, y \in X. \tag{1.4}$$

Reich proved that any Reich-type multivalued (a, b, c)-contraction on a complete metric space has at least one fixed point (see [3]).

In a recent paper Petruşel and Rus introduced the concept of "theory of a metric fixed point theorem" and used this theory for the case of multivalued contraction (see [4]). For the singlevalued case, see [5].

The purpose of this paper is to extend this approach to the case of Reich-type multivalued (a, b, c)-contraction. We will discuss Reich's fixed point theorem in terms of

(i) fixed points and strict fixed points,

(ii) multivalued weakly Picard operators,

(iii) multivalued Picard operators,

(iv) data dependence of the fixed point set,

(v) sequence of multivalued operators and fixed points,

(vi) Ulam-Hyers stability of a multivaled fixed point equation,

(vii) well-posedness of the fixed point problem;

(viii) fractal operators.

Notice also that the theory of fixed points and strict fixed points for multivalued operators is closely related to some important models in mathematical economics, such as optimal preferences, game theory, and equilibrium of an abstract economy. See [6] for a nice survey.

2. Notations and Basic Concepts

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used (see the papers by Kirk and Sims [7], Granas and Dugundji [8], Hu and Papageorgiou [2], Rus et al. [9], Petruşel [10], and Rus [11]).

Let X be a nonempty set. Then we denote.

$$\mathcal{P}(X) = \{Y \mid Y \text{ is a subset of } X\}, \qquad P(X) = \{Y \in \mathcal{P}(X) \mid Y \text{ is nonempty}\}. \tag{2.1}$$

Let (X, d) be a metric space. Then $\delta(Y) = \sup\{d(a, b) \mid a, b \in Y\}$ and

$$P_b(X) = \{Y \in P(X) \mid \delta(Y) < +\infty\}, \qquad P_{cp}(X) = \{Y \in P(X) \mid Y \text{ is compact}\}. \tag{2.2}$$

Let $T : X \to P(X)$ be a multivalued operator. Then the operator $\widehat{T} : P(X) \to P(X)$, which is defined by

$$\widehat{T}(Y) := \bigcup_{x \in Y} T(x), \quad \text{for } Y \in P(X), \tag{2.3}$$

is called the fractal operator generated by T. For a well-written introduction on the theory of fractals see the papers of Barnsley [12], Hutchinson [13], Yamaguti et al. [14].

It is known that if (X, d) is a metric space and $T : X \to P_{cp}(X)$, then the following statements hold:

(a) if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;

(b) the continuity of T implies the continuity of $\widehat{T} : P_{cp}(X) \to P_{cp}(X)$.

The set of all nonempty invariant subsets of T is denoted by $I(T)$, that is,

$$I(T) := \{Y \in P(X) \mid T(Y) \subset Y\}. \tag{2.4}$$

A sequence of successive approximations of T starting from $x \in X$ is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in X with $x_0 = x$, $x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$.

If $T : Y \subseteq X \to P(X)$, then $F_T := \{x \in Y \mid x \in T(x)\}$ denotes the fixed point set of T and $(\mathrm{SF})_T := \{x \in Y \mid \{x\} = T(x)\}$ denotes the strict fixed point set of T. By

$$\operatorname{Graph}(T) := \{(x, y) \in Y \times X : y \in T(x)\} \tag{2.5}$$

we denote the graph of the multivalued operator T.

If $T : X \to P(X)$, then $T^0 := 1_X$, $T^1 := T, \ldots, T^{n+1} = T \circ T^n$, $n \in \mathbb{N}$, denote the iterate operators of T.

Definition 2.1 (see [15]). Let (X, d) be a metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

(i) $x_0 = x$ and $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

For the following concepts see the papers by Rus et al. [15], Petruşel [10], Petruşel and Rus [16], and Rus et al. [9].

Definition 2.2. Let (X, d) be a metric space, and let $T : X \to P(X)$ be an MWP operator. The multivalued operator $T^\infty : \operatorname{Graph}(T) \to P(F_T)$ is defined by the formula $T^\infty(x, y) = \{z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z\}$.

Definition 2.3. Let (X, d) be a metric space and $T : X \to P(X)$ an MWP operator. Then T is said to be a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x, y)) \le c\, d(x, y)$ for all $(x, y) \in \operatorname{Graph}(T)$.

We recall now the notion of multivalued Picard operator.

Definition 2.4. Let (X,d) be a metric space and $T : X \to P(X)$. By definition, T is called a multivalued Picard operator (briefly MP operator) if and only if

(i) $(\mathrm{SF})_T = F_T = \{x^*\}$;

(ii) $T^n(x) \overset{H}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

In [10] other results on MWP operators are presented. For related concepts and results see, for example, [1, 17–23].

3. A Theory of Reich's Fixed Point Principle

We recall the fixed point theorem for a single-valued Reich-type operator, which is needed for the proof of our first main result.

Theorem 3.1 (see [3]). *Let (X,d) be a complete metric space, and let $f : X \to X$ be a Reich-type single-valued (a,b,c)-contraction, that is, there exist $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$ such that*

$$d(f(x),f(y)) \le ad(x,y) + bd(x,f(x)) + cd(y,f(y)), \quad \text{for each } x,y \in X. \tag{3.1}$$

Then f is a Picard operator, that is, we have:

(i) *$F_f = \{x^*\}$;*

(ii) *for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges in (X,d) to x^*.*

Our main result concerning Reich's fixed point theorem is the following.

Theorem 3.2. *Let (X,d) be a complete metric space, and let $T : X \to P_{cl}(X)$ be a Reich-type multivalued (a,b,c)-contraction. Let $\alpha := (a+b)/(1-c)$. Then one has the following*

(i) *$F_T \neq \emptyset$;*

(ii) *T is a $1/(1-\alpha)$-multivalued weakly Picard operator;*

(iii) *let $S : X \to P_{cl}(X)$ be a Reich-type multivalued (a,b,c)-contraction and $\eta > 0$ such that $H(S(x),T(x)) \le \eta$ for each $x \in X$, then $H(F_S,F_T) \le \eta/(1-\alpha)$;*

(iv) *let $T_n : X \to P_{cl}(X)$ $(n \in \mathbb{N})$ be a sequence of Reich-type multivalued (a,b,c)-contraction, such that $T_n(x) \overset{H}{\to} T(x)$ uniformly as $n \to +\infty$. Then, $F_{T_n} \overset{H}{\to} F_T$ as $n \to +\infty$.*

If, moreover $T(x) \in P_{cp}(X)$ for each $x \in X$, then one additionally has:

(v) *(Ulam-Hyers stability of the inclusion $x \in T(x)$) Let $\epsilon > 0$ and $x \in X$ be such that $D(x,T(x)) \le \epsilon$, then there exists $x^* \in F_T$ such that $d(x,x^*) \le \epsilon/(1-\alpha)$;*

(vi) *$\widehat{T} : (P_{cp}(X),H) \to (P_{cp}(X),H)$, $\widehat{T}(Y) := \bigcup_{x\in Y} T(x)$ is a set-to-set (a,b,c)-contraction and (thus) $F_{\widehat{T}} = \{A_T^*\}$;*

(vii) *$T^n(x) \overset{H}{\to} A_T^*$ as $n \to +\infty$, for each $x \in X$;*

(viii) *$F_T \subset A_T^*$ and F_T are compact;*

(ix) *$A_T^* = \bigcup_{n\in\mathbb{N}\setminus\{0\}} T^n(x)$ for each $x \in F_T$.*

Proof. (i) Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrarily chosen. Then, for each arbitrary $q > 1$ there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \leq qH(T(x_0), T(x_1))$. Hence

$$\begin{aligned} d(x_1, x_2) &\leq q[ad(x_0, x_1) + bD(x_0, T(x_0)) + cD(x_1, T(x_1))] \\ &\leq q[ad(x_0, x_1) + bd(x_0, x_1) + cd(x_1, x_2)]. \end{aligned} \tag{3.2}$$

Thus

$$d(x_1, x_2) \leq \frac{q(a+b)}{1-qc} d(x_0, x_1). \tag{3.3}$$

Denote $\beta := q(a+b)/(1-qc)$. By an inductive procedure, we obtain a sequence of successive approximations for T starting from $(x_0, x_1) \in \text{Graph}(T)$ such that, for each $n \in \mathbb{N}$, we have $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$. Then

$$d(x_n, x_{n+p}) \leq \beta^n \frac{1-\beta^p}{1-\beta} d(x_0, x_1), \quad \text{for each } n, p \in \mathbb{N} \setminus \{0\}. \tag{3.4}$$

If we choose $1 < q < 1/(a+b+c)$, then by (3.4) we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and hence convergent in (X, d) to some $x^* \in X$

Notice that, by $D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*)) \leq d(x_{n+1}, x^*) + H(T(x_n), T(x^*)) \leq d(x_{n+1}, x^*) + ad(x_n, x^*) + bD(x_n, T(x_n)) + cD(x^*, \mathrm{T}(x^*)) \leq d(x_{n+1}, x^*) + ad(x_n, x^*) + bd(x_n, x_{n+1})) + cD(x^*, T(x^*))$, we obtain that

$$D(x^*, T(x^*)) \leq \frac{1}{1-c}[d(x_{n+1}, x^*) + ad(x_n, x^*) + bd(x_n, x_{n+1})] \longrightarrow \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.5}$$

Hence $x^* \in F_T$.

(ii) Let $p \to +\infty$ in (3.4). Then we get that

$$d(x_n, x^*) \leq \beta^n \frac{1}{1-\beta} d(x_0, x_1) \quad \text{for each } n \in \mathbb{N} \setminus \{0\}. \tag{3.6}$$

For $n = 1$, we get

$$d(x_1, x^*) \leq \frac{\beta}{1-\beta} d(x_0, x_1). \tag{3.7}$$

Then

$$d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq \frac{1}{1-\beta} d(x_0, x_1). \tag{3.8}$$

Let $q \searrow 1$ in (3.8), then

$$d(x_0, x^*) \leq \frac{1}{1-\alpha} d(x_0, x_1). \tag{3.9}$$

Hence T is a $1/(1-\alpha)$-multivalued weakly Picard operator.

(iii) Let $x_0 \in F_S$ be arbitrarily chosen. Then, by (ii), we have that

$$d(x_0, t^{\infty}(x_0, x_1)) \leq \frac{1}{1-\alpha} d(x_0, x_1), \quad \text{for each } x_1 \in T(x_0). \tag{3.10}$$

Let $q > 1$ be an arbitrary. Then, there exists $x_1 \in T(x_0)$ such that

$$d(x_0, t^{\infty}(x_0, x_1)) \leq \frac{1}{1-\alpha} qH(S(x_0), T(x_0)) \leq \frac{q\eta}{1-\alpha}. \tag{3.11}$$

In a similar way, we can prove that for each $y_0 \in F_T$ there exists $y_1 \in S(y_0)$ such that

$$d(y_0, s^{\infty}(y_0, y_1)) \leq \frac{q\eta}{1-\alpha}. \tag{3.12}$$

Thus, (3.11) and (3.12) together imply that $H(F_S, F_T) \leq q\eta/(1-\alpha)$ for every $q > 1$. Let $q \searrow 1$ and we get the desired conclusion.

(iv) follows immediately from (iii).

(v) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \epsilon$. Then, since $T(x)$ is compact, there exists $y \in T(x)$ such that $d(x, y) \leq \epsilon$. From the proof of (i), we have that

$$d(x, t^{\infty}(x, y)) \leq \frac{1}{1-\alpha} d(x, y). \tag{3.13}$$

Since $x^* := t^{\infty}(x, y) \in F_T$, we get that $d(x, x^*) \leq \epsilon/(1-\alpha)$.

(vi) We will prove for any $A, B \in P_{cp}(X)$ that

$$H(T(A), T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \tag{3.14}$$

For this purpose, let $A, B \in P_{cp}(X)$ and let $u \in T(A)$. Then, there exists $x \in A$ such that $u \in T(x)$. Since the sets A, B are compact, there exists $y \in B$ such that

$$d(x, y) \leq H(A, B). \tag{3.15}$$

From (3.15) we get that $D(u, T(B)) \leq D(u, T(y)) \leq H(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)) \leq ad(x, y) + b\rho(A, T(x)) + c\rho(B, T(y)) \leq aH(A, B) + b\rho(A, T(A)) + c\rho(B, T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B))$. Hence

$$\rho(T(A), T(B)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \tag{3.16}$$

In a similar way we obtain that

$$\rho(T(B), T(A)) \leq aH(A, B) + bH(A, T(A)) + cH(B, T(B)). \tag{3.17}$$

Thus, (3.16) and (3.17) together imply that

$$H(T(A),T(B)) \leq aH(A,B) + bH(A,T(A)) + cH(B,T(B)). \quad (3.18)$$

Hence, $\widehat{T}$ is a Reich-type single-valued (a,b,c)-contraction on the complete metric space $(P_{cp}(X),H)$. From Theorem 3.1 we obtain that

(a) $F_{\widehat{T}} = \{A_T^*\}$ and

(b) $\widehat{T}^n(A) \xrightarrow{H} A_T^*$ as $n \to +\infty$, for each $A \in P_{cp}(X)$. □

(vii) From (vi)-(b) we get that $T^n(\{x\}) = \widehat{T}^n(\{x\}) \xrightarrow{H} A_T^*$ as $n \to +\infty$, for each $x \in X$.

(viii)-(ix) Let $x \in F_T$ be an arbitrary. Then $x \in T(x) \subset T^2(x) \subset \cdots \subset T^n(x) \subset \cdots$. Hence $x \in T^n(x)$, for each $n \in \mathbb{N}^*$. Moreover, $\lim_{n\to+\infty} T^n(x) = \bigcup_{n\in\mathbb{N}^*} T^n(x)$. From (vii), we immediately get that $A_T^* = \bigcup_{n\in\mathbb{N}^*} T^n(x)$. Hence $x \in \bigcup_{n\in\mathbb{N}^*} T^n(x) = A_T^*$. The proof is complete.

A second result for Reich-type multivalued (a,b,c)-contractions formulates as follows.

Theorem 3.3. *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ a Reich-type multivalued (a,b,c)-contraction with $(SF)_T \neq \emptyset$. Then, the following assertions hold:*

(x) $F_T = (SF)_T = \{x^*\}$;

(xi) (Well-posedness of the fixed point problem with respect to D [24]) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n,T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$ as $n \to \infty$;

(xii) (Well-posedness of the fixed point problem with respect to H [24]) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n,T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$ as $n \to \infty$.

Proof. (x) Let $x^* \in (\mathrm{SF})_T$. Note that $(\mathrm{SF})_T = \{x^*\}$. Indeed, if $y \in (\mathrm{SF})_T$, then $d(x^*,y) = H(T(x^*),T(y)) \leq ad(x^*,y) + bD(x^*,T(x^*)) + cD(y,T(y)) = ad(x^*,y)$. Thus $y = x^*$.

Let us show now that $F_T = \{x^*\}$. Suppose that $y \in F_T$. Then, $d(x^*,y) = D(T(x^*),y) \leq H(T(x^*),T(y)) \leq ad(x^*,y) + bD(x^*,T(x^*)) + cD(y,T(y)) = ad(x^*,y)$. Thus $y = x^*$. Hence $F_T \subset (\mathrm{SF})_T = \{x^*\}$. Since $(\mathrm{SF})_T \subset F_T$, we get that $(\mathrm{SF})_T = F_T = \{x^*\}$.

(xi) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(x_n,T(x_n)) \to 0$ as $n \to \infty$. Then, $d(x_n,x^*) \leq D(x_n,T(x_n)) + H(T(x_n),T(x^*)) \leq D(x_n,T(x_n)) + ad(x_n,x^*) + bD(x_n,T(x_n)) + cD(x^*,T(x^*)) = (1+b)D(x_n,T(x_n)) + ad(x_n,x^*)$. Then $d(x_n,x^*) \leq ((1+b)/(1-a))D(x_n,T(x_n)) \to 0$ as $n \to +\infty$.

(xii) follows by (xi) since $D(x_n,T(x_n)) \leq H(x_n,T(x_n)) \to 0$ as $n \to +\infty$. □

A third result for the case of (a,b,c)-contraction is the following.

Theorem 3.4. *Let (X,d) be a complete metric space, and let $T : X \to P_{cp}(X)$ be a Reich-type multivalued (a,b,c)-contraction such that $T(F_T) = F_T$. Then one has*

(xiii)$T^n(x) \xrightarrow{H} F_T$ *as $n \to +\infty$, for each $x \in X$;*

(xiv)$T(x) = F_T$ *for each $x \in F_T$;*

(xv)*If $(x_n)_{n\in\mathbb{N}} \subset X$ is a sequence such that $x_n \xrightarrow{d} x^* \in F_T$ as $n \to \infty$ and T is H-continuous, then $T(x_n) \xrightarrow{H} F_T$ as $n \to +\infty$.*

Proof. (xiii) From the fact that $T(F_T) = F_T$ and Theorem 3.2 (vi) we have that $F_T = A_T^*$. The conclusion follows by Theorem 3.2 (vii).

(xiv) Let $x \in F_T$ be an arbitrary. Then $x \in T(x)$, and thus $F_T \subset T(x)$. On the other hand $T(x) \subset T(F_T) \subset F_T$. Thus $T(x) = F_T$, for each $x \in F_T$.

(xv) Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence such that $x_n \xrightarrow{d} x^* \in F_T$ as $n \to \infty$. Then, we have $T(x_n) \xrightarrow{H} T(x^*) = F_T$ as $n \to \infty$. The proof is complete. □

For compact metric spaces we have the following result.

Theorem 3.5. *Let (X,d) be a compact metric space, and let $T : X \to P_{cl}(X)$ be a H-continuous Reich-type multivalued (a,b,c)-contraction. Then*

(xvi) if $(x_n)_{n\in\mathbb{N}}$ is such that $D(x_n, T(x_n)) \to 0$ as $n \to \infty$, then there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_i} \xrightarrow{d} x^* \in F_T$ as $i \to \infty$ (generalized well-posedness of the fixed point problem with respect to D [24, 25]).

Proof. (xvi) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(x_n, T(x_n)) \to 0$ as $n \to \infty$. Let $(x_{n_i})_{i\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_i} \xrightarrow{d} x^*$ as $i \to \infty$. Then, there exists $y_{n_i} \in T(x_{n_i})$, such that $y_{n_i} \xrightarrow{d} x^*$ as $i \to \infty$. Then $D(x^*, T(x^*)) \le d(x^*, y_{n_i}) + D(y_{n_i}, T(x_{n_i})) + H(T(x_{n_i}), T(x^*)) \le d(x^*, y_{n_i}) + ad(x^*, x_{n_i}) + bD(x_{n_i}, T(x_{n_i})) + cD(x^*, T(x^*))$. Hence

$$D(x^*, T(x^*)) \le \frac{1}{1-c}\left[d(x^*, y_{n_i}) + ad(x^*, x_{n_i}) + bD(x_{n_i}, T(x_{n_i}))\right] \longrightarrow 0 \qquad (3.19)$$

as $n \to +\infty$. Hence $x^* \in F_T$. □

Remark 3.6. For $b = c = 0$ we obtain the results given in [4]. On the other hand, our results unify and generalize some results given in [12, 13, 17, 26–34]. Notice that, if the operator T is singlevalued, then we obtain the well-posedness concept introduced in [35].

Remark 3.7. An open question is to present a theory of the Ćirić-type multivalued contraction theorem (see [36]). For some problems for other classes of generalized contractions, see for example, [17, 21, 27, 34, 37].

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References

[1] H. Covitz and S. B. Nadler, Jr., "Multi-valued contraction mappings in generalized metric spaces," *Israel Journal of Mathematics*, vol. 8, pp. 5–11, 1970.

[2] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[3] S. Reich, "Fixed points of contractive functions," *Bollettino della Unione Matematica Italiana*, vol. 5, pp. 26–42, 1972.

[4] A. Petruşel and I. A. Rus, "The theory of a metric fixed point theorem for multivalued operators," in *Proceedings of the 9th International Conference on Fixed Point Theory and Its Applications*, Yokohama Publishing, to appear.

[5] I. A. Rus, "The theory of a metrical fixed point theoremml: theoretical and applicative relevances," *Fixed Point Theory*, vol. 9, no. 2, pp. 541–559, 2008.

[6] K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, UK, 1985.

[7] W. A. Kirk and B. Sims, Eds., *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[8] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2003.

[9] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, Romania, 2008.

[10] A. Petruşel, "Multivalued weakly Picard operators and applications," *Scientiae Mathematicae Japonicae*, vol. 59, no. 1, pp. 169–202, 2004.

[11] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, Romania, 2001.

[12] M. F. Barnsley, *Fractals Everywhere*, Academic Press, Boston, Mass, USA, 2nd edition, 1988.

[13] J. E. Hutchinson, "Fractals and self-similarity," *Indiana University Mathematics Journal*, vol. 30, no. 5, pp. 713–747, 1981.

[14] M. Yamaguti, M. Hata, and J. Kigami, *Mathematics of Fractals*, vol. 167 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1997.

[15] I. A. Rus, A. Petruşel, and Alina Sîntămărian, "Data dependence of the fixed point set of some multivalued weakly Picard operators," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 52, no. 8, pp. 1947–1959, 2003.

[16] A. Petruşel and I. A. Rus, "Dynamics on $(P_{cp}(X), H_d)$ generated by a finite family of multi-valued operators on (X, d)," *Mathematica Moravica*, vol. 5, pp. 103–110, 2001.

[17] R. P. Agarwal, J. H. Dshalalow, and D. O'Regan, "Fixed point and homotopy results for generalized contractive maps of Reich type," *Applicable Analysis*, vol. 82, no. 4, pp. 329–350, 2003.

[18] J. Andres and L. Górniewicz, "On the Banach contraction principle for multivalued mappings," in *Approximation, Optimization and Mathematical Economics (Pointe-à-Pitre, 1999)*, pp. 1–23, Physica, Heidelberg, Germany, 2001.

[19] J. Andres and J. Fišer, "Metric and topological multivalued fractals," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 14, no. 4, pp. 1277–1289, 2004.

[20] R. Espínola and A. Petruşel, "Existence and data dependence of fixed points for multivalued operators on gauge spaces," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 420–432, 2005.

[21] M. Frigon, "Fixed point results for multivalued contractions on gauge spaces," in *Set Valued Mappings with Applications in Nonlinear Analysis*, vol. 4, pp. 175–181, Taylor & Francis, London, UK, 2002.

[22] M. Frigon, "Fixed point and continuation results for contractions in metric and gauge spaces," in *Fixed Point theory and Its Applications*, vol. 77, pp. 89–114, Polish Academy of Sciences, Warsaw, Poland, 2007.

[23] J. Jachymski, "Continuous dependence of attractors of iterated function systems," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 1, pp. 221–226, 1996.

[24] A. Petruşel and I. A. Rus, "Well-posedness of the fixed point problem for multivalued operators," in *Applied Analysis and Differential Equations*, O. Cârja and I. I. Vrabie, Eds., pp. 295–306, World Scientific Publishing, Hackensack, NJ, USA, 2007.

[25] A. Petruşel, I. A. Rus, and J.-C. Yao, "Well-posedness in the generalized sense of the fixed point problems for multivalued operators," *Taiwanese Journal of Mathematics*, vol. 11, no. 3, pp. 903–914, 2007.

[26] R. B. Fraser Jr. and S. B. Nadler Jr., "Sequences of contractive maps and fixed points," *Pacific Journal of Mathematics*, vol. 31, pp. 659–667, 1969.

[27] J. Jachymski and I. Jóźwik, "Nonlinear contractive conditions: a comparison and related problems," in *Fixed Point Theory and Its Applications*, vol. 77 of *Banach Center Publications*, pp. 123–146, Polish Academy of Sciences, Warsaw, Poland, 2007.

[28] T.-C. Lim, "On fixed point stability for set-valued contractive mappings with applications to generalized differential equations," *Journal of Mathematical Analysis and Applications*, vol. 110, no. 2, pp. 436–441, 1985.

[29] J. T. Markin, "Continuous dependence of fixed point sets," *Proceedings of the American Mathematical Society*, vol. 38, pp. 545–547, 1973.

[30] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.

[31] N. S. Papageorgiou, "Convergence theorems for fixed points of multifunctions and solutions of differential inclusions in Banach spaces," *Glasnik Matematički. Serija III*, vol. 23, no. 2, pp. 247–257, 1988.

[32] S. Reich, "Kannan's fixed point theorem," *Bollettino della Unione Matematica Italiana*, vol. 4, pp. 1–11, 1971.

[33] S. Reich, "Some remarks concerning contraction mappings," *Canadian Mathematical Bulletin*, vol. 14, pp. 121–124, 1971.

[34] I. A. Rus, "Fixed point theorems for multi-valued mappings in complete metric spaces," *Mathematica Japonica*, vol. 20, pp. 21–24, 1975.

[35] S. Reich and A. J. Zaslavski, "Well-posedness of fixed point problems," *Far East Journal of Mathematical Sciences*, pp. 393–401, 2001.

[36] L. B. Ćirić, "Fixed points for generalized multi-valued contractions," *Matematički Vesnik*, vol. 9(24), pp. 265–272, 1972.

[37] H.-K. Xu, "Metric fixed point theory for multivalued mappings," *Dissertationes Mathematicae*, vol. 389, p. 39, 2000.