Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 716971, 14 pages
doi:10.1155/2010/716971

*Research Article*

# Convergence of the Sequence of Successive Approximations to a Fixed Point

**Tomonari Suzuki**

*Department of Basic Sciences, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan*

Correspondence should be addressed to Tomonari Suzuki, suzuki-t@mns.kyutech.ac.jp

Received 29 September 2009; Accepted 21 December 2009

Academic Editor: Mohamed A. Khamsi



If $(X, d)$ is a complete metric space and $T$ is a contraction on $X$, then the conclusion of the Banach-Caccioppoli contraction principle is that the sequence of successive approximations $\{T^n x\}$ of $T$ starting from any point $x \in X$ converges to a unique fixed point. In this paper, using the concept of $\tau$-distance, we obtain simple, sufficient, and necessary conditions of the above conclusion.

## 1. Introduction

The following famous theorem is referred to as the *Banach-Caccioppoli contraction principle*. This theorem is very forceful and simple, and it became a classical tool in nonlinear analysis.

**Theorem 1.1** (see Banach [1] and Caccioppoli [2]). *Let $(X, d)$ be a complete metric space and let $T$ be a self contraction on $X$, that is, there exists $r \in [0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then the following holds.*

(A) *$T$ has a unique fixed point $z$, and $\{T^n x\}$ converges to $z$ for any $x \in X$.*

We note that the conclusion of Kannan's fixed point theorem [3] is also (A). See Kirk's survey [4]. Recently, we obtained that (A) holds if and only if $T$ is a strong Leader mapping [5, 6].

**Theorem 1.2** (see [6]). *Let $T$ be a mapping on a complete metric space $(X, d)$. Then the following are equivalent.*

(i) *(A) holds.*

(ii) *$T$ is a strong Leader mapping, that is, the following hold.*

(a) *For $x, y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that*

$$d\left(T^i x, T^j y\right) < \varepsilon + \delta \Longrightarrow d\left(T^{i+\nu} x, T^{j+\nu} y\right) < \varepsilon, \tag{1.1}$$

*for all $i, j \in \mathbb{N} \cup \{0\}$, where $T^0$ is the identity mapping on $X$.*

(b) *For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that*

$$d\left(T^i x, T^j y\right) < \alpha_n \Longrightarrow d\left(T^{i+\nu} x, T^{j+\nu} y\right) < \frac{1}{n}, \tag{1.2}$$

*for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.*

The following theorem is proved in [7, 8].

**Theorem 1.3** (see Rus [7] and Subrahmanyam [8]). *Let $(X, d)$ be a complete metric space and let $T$ be a continuous mapping on $X$. Assume that there exists $r \in [0, 1)$ satisfying $d(Tx, T^2 x) \leq r d(x, Tx)$ for all $x \in X$. Then the following holds.*

(B) *$\{T^n x\}$ converges to a fixed point for every $x \in X$.*

We obtained a condition equivalent to (B) in [9].

**Theorem 1.4** (see [9]). *Let $T$ be a mapping on a complete metric space $(X, d)$. Then the following are equivalent.*

(i) *(B) holds.*

(ii) *The following hold.*

(a) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that*

$$d\left(T^i x, T^j x\right) < \varepsilon + \delta \Longrightarrow d\left(T^{i+\nu} x, T^{j+\nu} x\right) < \varepsilon, \tag{1.3}$$

*for all $i, j \in \mathbb{N} \cup \{0\}$.*

(b) *For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that*

$$d\left(T^i x, T^j y\right) < \alpha_n \Longrightarrow d\left(T^{i+\nu} x, T^{j+\nu} y\right) < \frac{1}{n}, \tag{1.4}$$

*for all $i, j \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.*

We sometimes call a mapping satisfying (A) a *Picard operator* [10]. We also call a mapping satisfying (B) a *weakly Picard operator* [11–13].

We cannot tell that the conditions (ii) of Theorems 1.2 and 1.4 are simple. Motivated by this, we obtain simpler conditions which are equivalent to Conditions (A) and (B).

## 2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the sets of positive integers, integers and real numbers, respectively.

In 2001, Suzuki introduced the concept of $\tau$-distance in order to improve results in Tataru [14], Zhong [15, 16], and others. See also [17].

*Definition 2.1* (see [18]). Let $(X, d)$ be a metric space. Then a function $p$ from $X \times X$ into $[0, \infty)$ is called a *$\tau$-distance* on $X$ if there exists a function $\eta$ from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

($\tau$1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$,

($\tau$2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and $\eta$ is concave and continuous in its second variable,

($\tau$3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$,

($\tau$4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply that $\lim_n \eta(y_n, t_n) = 0$,

($\tau$5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_n d(x_n, y_n) = 0$.

The metric $d$ is a $\tau$-distance on $X$. Many useful examples and propositions are stated in [9, 18–23] and references therein. The following fixed point theorems are proved in [18].

**Theorem 2.2** (see [18]). *Let $X$ be a complete metric space and let $T$ be a mapping on $X$. Assume that there exist a $\tau$-distance $p$ and $r \in [0, 1)$ such that $p(Tx, T^2x) \leq rp(x, Tx)$ for all $x \in X$. Assume the following.*

(i) *If $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$, then $Ty = y$.*

*Then (B) holds. Moreover, if $Tz = z$, then $p(z, z) = 0$.*

**Theorem 2.3** (see[18]). *Let $X$ be a complete metric space and let $T$ be a mapping on $X$. Assume that $T$ is a contraction with respect to some $\tau$-distance $p$, that is, there exist a $\tau$-distance $p$ and $r \in [0, 1)$ such that*

$$p(Tx, Ty) \leq rp(x, y), \tag{2.1}$$

*for all $x, y \in X$. Then (A) and $p(z, z) = 0$ hold.*

The following lemmas are useful in our proofs.

**Lemma 2.4** (see [18]). *Let $(X, d)$ be a metric space and let $p$ be a $\tau$-distance on $X$. If sequences $\{x_n\}$ and $\{y_n\}$ in $X$ satisfy $\lim_n p(z, x_n) = 0$ and $\lim_n p(z, y_n) = 0$ for some $z \in X$, then $\lim_n d(x_n, y_n) = 0$. In particular for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply that $x = y$.*

**Lemma 2.5** (see [18]). *Let $(X, d)$ be a metric space and let $p$ be a $\tau$-distance on $X$. If a sequence $\{x_n\}$ in $X$ satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence. Moreover if a sequence $\{y_n\}$ in $X$ satisfies $\lim_n p(x_n, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$.*

The following lemmas are easily deduced from Lemmas 2.4 and 2.5.

**Lemma 2.6.** *Let $(X, d)$ be a metric space and let $p$ be a $\tau$-distance on $X$. Then for every $z \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply that $d(x, y) \leq \varepsilon$.*

**Lemma 2.7.** *Let $X$ be a metric space and let $p$ be a $\tau$-distance on $X$. Assume that a sequence $\{x_n\}$ in $X$ satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, y) = 0$, and $\lim_n p(x_n, z) = 0$. Then $y = z$.*

The following is proved at Page 442 of [18]. However we give a proof because we use reductio ad absurdum in [18].

**Lemma 2.8** (see [18]). *Let $g$ be a nondecreasing function from $[0, \infty)$ into itself satisfying $\inf\{g(t) : t > 0\} = 0$. Define a function $f$ from $[0, \infty)$ into itself by*

$$f(t) = t + \sup\left\{ \sum_{i=1}^{n} \alpha_i \min\{g(s_i), 1\} : t = \sum_{i=1}^{n} \alpha_i s_i,\ s_i \geq 0,\ \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}. \tag{2.2}$$

*Then $f(0) = 0$, $f(t) \geq t + g(t)$ for all $t \in [0, \infty)$; and $f$ is concave and continuous.*

*Proof.* It is clear that $f(0) = 0$, $f(t) \geq t + g(t)$, and $f$ is concave. We shall prove that $f$ is continuous at 0. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $g(\delta) \leq \varepsilon$. Choose $\tau > 0$ with $\tau + \tau/\delta \leq \varepsilon$. Fix $t \in (0, \tau)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $s_1, s_2, \ldots, s_n \geq 0$ such that $t = \sum_{i=1}^{n} \alpha_i s_i$ and $\sum_{i=1}^{n} \alpha_i = 1$. Since $\delta \sum\{\alpha_i : s_i \geq \delta\} \leq t$, we have

$$\begin{aligned} t + \sum_{i=1}^{n} \alpha_i \min\{g(s_i), 1\} &\leq t + \sum_{s_i < \delta} \alpha_i g(s_i) + \sum_{s_i \geq \delta} \alpha_i \leq t + \sum_{s_i < \delta} \alpha_i \varepsilon + \sum_{s_i \geq \delta} \alpha_i \\ &\leq t + \varepsilon + \frac{t}{\delta} \leq \tau + \varepsilon + \frac{\tau}{\delta} \leq 2\varepsilon. \end{aligned} \tag{2.3}$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $s_1, s_2, \ldots, s_n \geq 0$ are arbitrary, we obtain $f(t) \leq 2\varepsilon$. Thus, $\lim_{t \to +0} f(t) = 0 = f(0)$. □

The following is obvious.

**Lemma 2.9.** *Let $T$ be a mapping on a set $X$. Let $A_0$ be a subset of $X$ such that $T(A_0) \subset A_0$. Define a sequence $\{A_n\}$ of subsets of $X$ by*

$$A_1 = T^{-1}(A_0) \setminus A_0, \qquad A_{n+1} = T^{-1}(A_n). \tag{2.4}$$

*Then the following hold.*

(i) *For every $n \in \mathbb{N}$ and $x \in X$, $x \in A_n$ if and only if $T^j x \notin A_0$ for $j = 0, 1, \ldots, n-1$ and $T^n x \in A_0$.*

(ii) *$A_m \cap A_n = \emptyset$ for $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$.*

(iii) *$T(A_{n+1}) = A_n$ for every $n \in \mathbb{N}$.*

## 3. Condition (B)

In this section, we discuss Condition (B).

**Theorem 3.1.** *Let $X$ be a complete metric space and let $T$ be a mapping on $X$. Assume that there exist a $\tau$-distance $p$, $r \in [0,1)$, and $M \in [0,\infty)$ such that*

$$p\left(Tx, T^2x\right) \le rp(x, Tx), \qquad p(Tx, Ty) \le Mp(x, y), \tag{3.1}$$

*for all $x, y \in X$. Then (B) holds. Moreover, if $Tz = z$, then $p(z,z) = 0$.*

*Proof.* Assume that $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$. Then we have

$$p(x_n, Ty) \le p(x_n, Tx_n) + p(Tx_n, Ty) \le p(x_n, Tx_n) + Mp(x_n, y), \tag{3.2}$$

and hence, $\lim_n p(x_n, Ty) = 0$. By Lemma 2.7, we obtain $Ty = y$. By Theorem 2.2, we obtain the desired result. □

As a direct consequence of Theorem 3.1, we obtain the following.

**Corollary 3.2.** *Let $X$ be a complete metric space and let $T$ be a mapping on $X$. Assume that there exist a $\tau$-distance $p$ and $r \in (0,1)$ such that*

$$p\left(Tx, T^2x\right) \le rp(x, Tx), \qquad p(Tx, Ty) \le p(x, y), \tag{3.3}$$

*for all $x, y \in X$. Then (B) holds.*

Corollary 3.2 characterizes Condition (B).

**Theorem 3.3.** *Let $T$ be a mapping on a metric space $(X, d)$ such that (B) holds. Then there exist a $\tau$-distance $p$ and $r \in (0,1)$ satisfying* (3.3).

*Proof.* Let $r \in (0,1)$ be fixed. We note that every periodic point is a fixed point. That is, if $x \in X$ satisfies $T^n x = x$ for some $n \in \mathbb{N}$, then $Tx = x$. Define a mapping $T^\infty$ from $X$ onto $F(T)$ by $T^\infty x = \lim_n T^n x$ for $x \in X$, where $F(T)$ is the set of all fixed points of $T$. Define a mapping $C$ from $X$ into the set of subsets of $X$ by

$$Cx = \left\{Tx, T^2x, T^3x, \dots, T^\infty x\right\}. \tag{3.4}$$

Since $T^\infty x$ is a fixed point of $T$, we have

$$y \in Cx \Longrightarrow Cy \subset Cx. \tag{3.5}$$

Next, we define a function $f$ from $X$ into $\mathbb{Z} \cup \{\infty\}$ satisfying

$$f(Tx) \geq f(x) + 1, \qquad f(x) = \infty \Longleftrightarrow Tx = x, \tag{3.6}$$

for all $x \in X$. We put $f(x) = \infty$ for $x \in F(T)$. It is obvious that $f(Tx) = f(x) = \infty = f(x) + 1$ for $x \in F(T)$. Define a sequence $\{A_n\}$ of subsets of $X$ by

$$A_1 = T^{-1}(F(T)) \setminus F(T), \qquad A_{n+1} = T^{-1}(A_n). \tag{3.7}$$

Then by Lemma 2.9,

$$F(T) \cap A_n = \emptyset, \qquad A_m \cap A_n = \emptyset, \tag{3.8}$$

for $m, n \in \mathbb{N}$ with $m \neq n$. We put $f(x) = -n$ for $x \in A_n$. We note that

$$f(Tx) = \begin{cases} \infty & \text{if } x \in A_1, \\ f(x) + 1 & \text{if } x \in \bigsqcup_{n=2}^{\infty} A_n. \end{cases} \tag{3.9}$$

Put

$$Y = X \setminus \left( F(T) \sqcup \left( \bigsqcup_{n \in \mathbb{N}} A_n \right) \right). \tag{3.10}$$

It is obvious that $T(Y) \subset Y$, $T^{-1}(Y) = Y$, and $Y \cap F(T) = \emptyset$. So,

$$T^m x = T^n x \Longleftrightarrow m = n, \tag{3.11}$$

for $x \in Y$ and $m, n \in \mathbb{N} \cup \{0\}$. Define an equivalence relation $\sim$ on $Y$ as follows: $x \sim y$ if and only if there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $T^m x = T^n y$. By Axiom of Choice, there exists a mapping $B$ on $Y$ such that

$$Bx \sim x, \quad x \sim y \Longleftrightarrow Bx = By. \tag{3.12}$$

Let $u \in Y$ with $Bu = u$. Then we put $f(T^n u) = n$ for $n \in \mathbb{N} \cup \{0\}$. Define a sequence $\{D_n\}$ of subsets of $Y$ by

$$D_0 = \left\{ u, Tu, T^2 u, T^3 u, \ldots \right\}, \quad D_1 = T^{-1}(D_0) \setminus D_0, \quad D_{n+1} = T^{-1}(D_n). \tag{3.13}$$

Then we have $D_m \cap D_n = \emptyset$ for $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$; and

$$\{x \in Y : x \sim u\} = \bigsqcup_{n \in \mathbb{N} \cup \{0\}} D_n. \tag{3.14}$$

We put $f(x) = -n$ for $x \in Y$ with $n \in \mathbb{N}$ and $x \in D_n$. We have defined $f$. We note that $f(x) \in \mathbb{N}$ implies that $x \in Y$.

Next, we define a $\tau$-distance $p$ by

$$p(x,y) = \begin{cases} r^{f(x)} + r^{f(y)} & \text{if } y \in Cx, \\ r^{f(x)} + r^{f(y)} + 1 & \text{if } y \notin Cx, \end{cases} \tag{3.15}$$

where $r^\infty = 0$. We note that $p(x,y) < 1$ implies either of the following.

(i) $Tx = x = y$.

(ii) There exist $u \in Y$, $k \in \mathbb{N}$, and $\ell \in \mathbb{N} \cup \{\infty\}$ such that $Bu = u$, $k < l$, $x = T^k u$, and $y = T^\ell u$. (In this case, $x \in Y$, $u = Bx$, $f(x) = k$, and $f(y) = \ell$ hold.)

We shall show that $p$ is a $\tau$-distance. Let $x, y, z \in X$. If $y \in Cx$ and $z \in Cy$, then $z \in Cx$. So we have

$$p(x,z) = r^{f(x)} + r^{f(z)} \le r^{f(x)} + r^{f(y)} + r^{f(y)} + r^{f(z)} = p(x,y) + p(y,z). \tag{3.16}$$

If $y \notin Cx$ or $z \notin Cy$, then

$$p(x,z) \le r^{f(x)} + r^{f(z)} + 1 \le r^{f(x)} + r^{f(y)} + r^{f(y)} + r^{f(z)} + 1 \le p(x,y) + p(y,z). \tag{3.17}$$

These imply ($\tau$1). We shall define a function $\eta$ from $X \times [0,\infty)$ into $[0,\infty)$. For $x \in X \setminus Y$, we put $\eta(x,t) = t$. For $x \in Y$, we put $u = Bx$. Since $\{T^n u\}$ converges to $T^\infty u$, there exists a strictly increasing sequence $\{h_u(n)\}$ in $\mathbb{N}$ such that $j \ge h_u(n)$ implies that $d(T^j u, T^\infty u) \le 1/n$ for $j \in \mathbb{N} \cup \{\infty\}$. Since $\lim_n h_u(n) = \infty$, we can define a nondecreasing function $g_u$ from $[0,\infty)$ into $[0,1]$ such that $g_u(r^{h_u(n)}) = 1/n$. It is obvious that $g_u(0) = \lim_{t \to +0} g_u(t) = 0$. Put

$$\eta(x,t) = t + \sup\left\{ \sum_{i=1}^{n} \alpha_i g_u(s_i) : t = \sum_{i=1}^{n} \alpha_i s_i,\ s_i \ge 0,\ \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}. \tag{3.18}$$

Then $\eta(x,t)$ satisfies ($\tau$2) and $\eta(x,t) \ge t + g_u(t)$ by Lemma 2.8. In order to show ($\tau$3), we assume that $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$. Then without loss of generality, we may assume that $\sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} < 1$. Thus $\sup\{p(z_n, x_m) : m \ge n\} < 1$ for $n \in \mathbb{N}$. It is obvious that $x_m \in Cz_n$ for $m, n \in \mathbb{N}$ with $m \ge n$. We consider the following two cases.

(i) There exists $\nu \in \mathbb{N}$ such that $x_n \in F(T)$ for $n \ge \nu$.

(ii) There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \notin F(T)$.

In the first case, since $F(T) \cap Cz_\nu$ exactly consists of one element and $x_n \in F(T) \cap Cz_\nu$ for $n \ge \nu$, $x_n = x_\nu$ holds for all $n \ge \nu$. So $x = x_\nu$. Thus, $p(w,x) = \lim_n p(w, x_n)$ holds for every $w \in X$. In the second case, we note that $z_n \notin F(T)$ for all $n \in \mathbb{N}$. Hence $z_n \in Y$. Put $u = Pz_1$. Since $x_n \in Cz_1$, there exists a sequence $\{\ell_n\}$ in $\mathbb{N} \cup \{\infty\}$ such that $x_n = T^{\ell_n} u$. Since $\ell_{n_j} \in \mathbb{N}$ for all $j \in \mathbb{N}$, there exists a sequence $\{k_n\}$ in $\mathbb{N}$ such that $z_n = T^{k_n} u$. Since $\lim_n p(z_n, x_n) = 0$, we have $\lim_n k_n = \infty$ and $\lim_n \ell_n = \infty$. So we obtain $x = T^{\infty}u$. We note that $x \in Cx_n$ for all $n \in \mathbb{N}$. Let $w \in X$. In the case where $x \in Cw$, we have

$$p(w,x) = r^{f(w)} = \lim_{n\to\infty}\left(r^{f(w)} + r^{f(x_n)}\right) \le \liminf_{n\to\infty} p(w,x_n). \tag{3.19}$$

In the other case, where $x \notin Cw$, we have $x_n \notin Cw$, and hence,

$$p(w,x) = r^{f(w)} + 1 = \lim_{n\to\infty}\left(r^{f(w)} + r^{f(x_n)} + 1\right) = \lim_{n\to\infty} p(w,x_n). \tag{3.20}$$

Therefore we have shown ($\tau$3). Let us prove ($\tau$4). We assume that $\lim_n \sup\{p(x_n,y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$. Without loss of generality, we may assume that $\sup\{p(x_n,y_m) : m \ge n\} < 1$. We consider the following two cases.

(i) There exists $\nu \in \mathbb{N}$ such that $y_n \in F(T)$ for $n \ge \nu$.

(ii) There exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \notin F(T)$.

In the first case, we have

$$\lim_{n\to\infty} \eta(y_n,t_n) = \lim_{n\to\infty} t_n \le \lim_{n\to\infty} \eta(x_n,t_n) = 0. \tag{3.21}$$

In the second case, as in the proof of ($\tau$3), there exist $u \in Y$, a sequence $\{k_n\}$ in $\mathbb{N}$, and a sequence $\{\ell_n\}$ in $\mathbb{N}\cup\{\infty\}$ such that $Bu = u$, $x_n = T^{k_n}u$, and $y_n = T^{\ell_n}u$. We note that $\eta(x_n,t) = \eta(u,t)$. If $y_n \in F(T)$, then $\eta(y_n,t) = t \le \eta(u,t)$. If $y_n \notin F(T)$, then $\eta(y_n,t) = \eta(u,t)$. Therefore

$$\lim_{n\to\infty} \eta(y_n,t_n) \le \lim_{n\to\infty} \eta(u,t_n) = \lim_{n\to\infty} \eta(x_n,t_n) = 0. \tag{3.22}$$

Let us prove ($\tau$5). We assume that $\eta(z,p(z,x)) < 1/n$. We note that $p(z,x) < 1$. In the case where $Tz = z = x$, we have $d(z,x) = 0 < 1/n$. In the other case, where there exist $u \in Y$, $k \in \mathbb{N}$, and $\ell \in \mathbb{N}\cup\{\infty\}$ such that $Bu = u$, $k < l$, $z = T^k u$, and $x = T^{\ell}u$, we have

$$\eta(z,p(z,x)) < \frac{1}{n} = g_u\left(r^{h_u(n)}\right) \le \eta\left(z, r^{h_u(n)}\right). \tag{3.23}$$

Hence

$$r^k + r^{\ell} = p(z,x) < r^{h_u(n)}. \tag{3.24}$$

Thus, we obtain $k > h_u(n)$ and $\ell > h_u(n)$. So we have

$$d(z,x) = d\left(T^k u, T^{\ell} u\right) \le d\left(T^k u, T^{\infty} u\right) + d\left(T^{\ell} u, T^{\infty} u\right) \le \frac{1}{n} + \frac{1}{n} = \frac{2}{n}. \tag{3.25}$$

Therefore

$$\eta(z, p(z,x)) < \frac{1}{n}, \quad \eta(z, p(z,y)) < \frac{1}{n} \Longrightarrow d(x,y) \le \frac{4}{n}, \tag{3.26}$$

which imply ($\tau$5). Therefore we have shown that $p$ is a $\tau$-distance on $X$.

We shall show (3.3). Let $x, y \in X$. Since $Tx \in Cx$, $T^2x \in C(Tx)$, $f(Tx) \ge f(x) + 1$, and $f(T^2x) \ge f(Tx) + 1$, we have

$$p\left(Tx, T^2x\right) = r^{f(Tx)} + r^{f(T^2x)} \le r^{f(x)+1} + r^{f(Tx)+1} = rp(x, Tx). \tag{3.27}$$

If $y \in Cx$, then $Ty \in C(Tx)$ holds. So we have

$$p(Tx, Ty) = r^{f(Tx)} + r^{f(Ty)} \le r^{f(x)+1} + r^{f(y)+1} = rp(x,y) \le p(x,y). \tag{3.28}$$

If $y \notin Cx$, then we have

$$p(Tx, Ty) \le r^{f(Tx)} + r^{f(Ty)} + 1 \le r^{f(x)} + r^{f(y)} + 1 = p(x,y). \tag{3.29}$$

Therefore (3.3) holds. □

*Remark 3.4.* We have proved that, for every $r \in (0,1)$, there exists a $\tau$-distance $p$ satisfying (3.3).

Combining Theorem 6 in [9], we obtain the following.

**Corollary 3.5.** *Let $T$ be a mapping on a complete metric space $(X,d)$. Then the following are equivalent.*

(i) *(B) holds.*

(ii) *There exists a $\tau$-distance $p$ on $X$ satisfying the following.*

(a) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that*

$$p\left(T^i x, T^j x\right) < \varepsilon + \delta \Longrightarrow p\left(T^{i+\nu} x, T^{j+\nu} x\right) < \varepsilon, \tag{3.30}$$

*for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.*

(b) *For $x, y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0, \infty)$ such that*

$$p\left(T^i x, T^j y\right) < \alpha_n \Longrightarrow p\left(T^{i+\nu} x, T^{j+\nu} y\right) < \frac{1}{n}, \tag{3.31}$$

*for all $n \in \mathbb{N}$ and $i, j \in \mathbb{N} \cup \{0\}$ with $i > j$.*

(iii) *There exist a $\tau$-distance $p$ and $r \in (0,1)$ such that $p(Tx, T^2x) \le rp(x, Tx)$ and $p(Tx, Ty) \le p(x,y)$ for all $x, y \in X$.*

## 4. Condition (A)

In this section, we discuss Condition (A).

Define a relation $\underset{0}{<}$ on $[0,\infty)$ as follows: $s \underset{0}{<} t$ if and only if either $s = t = 0$ or $s < t$ holds.

**Theorem 4.1.** *Let $X$ be a complete metric space and let $T$ be a mapping on $X$. Assume that there exist a $\tau$-distance $p$ and $r \in (0,1)$ such that*

$$p\left(Tx, T^2x\right) \le rp(x, Tx), \qquad p(Tx, Ty) \underset{0}{<} p(x, y), \tag{4.1}$$

*for all $x, y \in X$. Then (A) holds.*

*Proof.* By Theorem 3.1, (B) holds. Moreover, if $Tx = x$, then $p(x,x) = 0$. Let $z, w \in X$ be fixed points of $T$. Then

$$p(z, w) = p(Tz, Tw) \underset{0}{<} p(z, w), \tag{4.2}$$

which implies that $p(z,w) = 0$. Since $p(z,z) = 0$, we obtain $z = w$ by Lemma 2.4. Thus the fixed point is unique. □

**Theorem 4.2.** *Let $X$ be a complete metric space and let $T$ be a mapping on $X$. Assume that there exist a $\tau$-distance $p$ and $r \in (0,1)$ such that*

$$p\left(Tx, T^2x\right) \le rp(x, Tx), \qquad p(Tx, Ty) < p(x, y), \tag{4.3}$$

*for all $x, y \in X$ with $x \ne y$. Then (A) holds.*

*Proof.* In the case where $X$ consists of one element, the conclusion obviously holds. So we consider the other case. Assume that $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$. We consider the following two cases:

(i) $x_n \ne y$ for sufficiently large $n \in \mathbb{N}$,

(ii) there exists a sequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} = y$.

In the first case, we have

$$p(x_n, Ty) \le p(x_n, Tx_n) + p(Tx_n, Ty) < p(x_n, Tx_n) + p(x_n, y) \tag{4.4}$$

for sufficiently large $n$, and hence, $\lim_n p(x_n, Ty) = 0$. By Lemma 2.7, we obtain $Ty = y$. In the second case, we have

$$p(y, Ty) = \lim_{j\to\infty} p\left(x_{n_j}, Tx_{n_j}\right) = 0, \qquad p(y, y) = \lim_{j\to\infty} p\left(x_{n_j}, y\right) = 0. \tag{4.5}$$

By Lemma 2.4, we obtain $Ty = y$. By Theorem 2.2, (B) holds. Let $z, w \in X$ be distinct fixed points of $T$. Then

$$p(z,w) = p(Tz,Tw) < p(z,w), \tag{4.6}$$

which implies a contradiction. Thus the fixed point is unique. □

We shall show that Theorems 4.1 and 4.2 characterize Condition (A).

**Theorem 4.3.** *Let $T$ be a mapping on a metric space $(X,d)$ such that (A) holds. Then there exist a $\tau$-distance $p$ and $r \in (0,1)$ satisfying* (4.1).

*Proof.* Let $p$, $r$, $f$, and $C$ be as in the proof of Theorem 3.3. Then $p(Tx,T^2x) \leq rp(x,Tx)$ holds. Fix $x, y \in X$. We consider the following two cases:

(i) $Tx = x$ and $Ty = y$,

(ii) either $Tx \neq x$ or $Ty \neq y$.

In the first case, $x = y$ holds by (A). Since

$$p(x,x) = p\left(Tx,T^2x\right) \leq rp(x,Tx) = p(x,x), \tag{4.7}$$

we obtain $p(x,x) = 0$. Thus, $p(Tx,Ty) = p(x,y) = 0$. In the second case, we note that either $f(x) \in \mathbb{Z}$ or $f(y) \in \mathbb{Z}$ holds. Thus

$$r^{f(x)+1} + r^{f(y)+1} < r^{f(x)} + r^{f(y)}. \tag{4.8}$$

If $y \in Cx$, then $Ty \in C(Tx)$ holds. So we have

$$p(Tx,Ty) = r^{f(Tx)} + r^{f(Ty)} \leq r^{f(x)+1} + r^{f(y)+1} < r^{f(x)} + r^{f(y)} = p(x,y). \tag{4.9}$$

If $y \notin Cx$, then we have

$$p(Tx,Ty) \leq r^{f(Tx)} + r^{f(Ty)} + 1 < r^{f(x)} + r^{f(y)} + 1 = p(x,y). \tag{4.10}$$

Therefore (4.1) holds. □

**Theorem 4.4.** *Let $T$ be a mapping on a metric space $(X,d)$ such that (A) holds. Then there exist a $\tau$-distance $p$ and $r \in (0,1)$ satisfying* (4.3) *for all $x, y \in X$ with $x \neq y$.*

*Proof.* The proof of Theorem 4.3 works. □

Combining Theorem 7 in [9], we obtain the following.

**Corollary 4.5.** *Let $T$ be a mapping on a complete metric space $(X,d)$. Then the following are equivalent.*

(i) *(A) holds.*

(ii) *There exists a $\tau$-distance $p$ on $X$ satisfying the following.*

(a) *For $x,y \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that*

$$p\left(T^i x, T^j y\right) < \varepsilon + \delta \Longrightarrow p\left(T^{i+\nu} x, T^{j+\nu} y\right) < \varepsilon, \tag{4.11}$$

*for all $i,j \in \mathbb{N} \cup \{0\}$ with $i < j$.*

(b) *For $x,y \in X$, there exist $\nu \in \mathbb{N}$ and a sequence $\{\alpha_n\}$ in $(0,\infty)$ such that*

$$p\left(T^i x, T^j y\right) < \alpha_n \Longrightarrow p\left(T^{i+\nu} x, T^{j+\nu} y\right) < \frac{1}{n}, \tag{4.12}$$

*for all $n \in \mathbb{N}$ and $i,j \in \mathbb{N} \cup \{0\}$ with $i > j$.*

(iii) *There exist a $\tau$-distance $p$ and $r \in (0,1)$ such that $p(Tx,T^2x) \leq rp(x,Tx)$ and $p(Tx,Ty) \underset{0}{<} p(x,y)$ for all $x,y \in X$.*

(iv) *There exist a $\tau$-distance $p$ and $r \in (0,1)$ such that $p(Tx,T^2x) \leq rp(x,Tx)$ and $p(Tx,Ty) < p(x,y)$ for all $x,y \in X$ with $x \neq y$.*

## 5. Additional Result

Since Theorem 2.2 deduces Corollary 3.2, we can tell that Theorem 2.2 characterizes Condition (B). However, the following example tells that Theorem 2.3 does not characterize Condition (A).

*Example 5.1.* Let $A$ be the set of all real sequences $\{a_n\}$ such that $a_n \in (0,\infty)$ for $n \in \mathbb{N}$, $\{a_n\}$ is strictly decreasing, and $\{a_n\}$ converges to 0. Let $H$ be a Hilbert space consisting of all the functions $x$ from $A$ into $\mathbb{R}$ satisfying $\sum_{a\in A} |x(a)|^2 < \infty$ with inner product $\langle x,y\rangle = \sum_{a\in A} x(a)y(a)$ for all $x,y \in H$. Define a subset $X$ of $H$ by

$$X = \{0\} \cup \left(\bigcup_{a\in A} \{a_n e_a : \ n \in \mathbb{N}\}\right), \tag{5.1}$$

where $e_a \in H$ is defined by $e_a(a) = 1$ and $e_a(b) = 0$ for $b \in A \setminus \{a\}$. Define a mapping $T$ on $X$ by

$$T0 = 0, \qquad T(a_n e_a) = a_{n+1} e_a. \tag{5.2}$$

Then (A) holds. However, $T$ is not a contraction with respect to any $\tau$-distance $p$.

*Proof.* It is obvious that (A) holds. Arguing by contradiction, we assume that $T$ is a contraction with respect to some $\tau$-distance $p$. That is, there exist a $\tau$-distance $p$ and $r \in [0,1)$ such that $p(Tx,Ty) \leq rp(x,y)$ for all $x,y \in X$. Since

$$p(0,0) = p(T0,T0) \leq rp(0,0), \tag{5.3}$$

we have $p(0,0) = 0$. By Lemma 2.6, there exists a strictly increasing sequence $\{\kappa_n\}$ in $\mathbb{N}$ such that

$$p(0,x) \leq r^{\kappa_n} \Longrightarrow d(0,x) \leq \frac{1}{n}. \tag{5.4}$$

We choose $\alpha \in A$ such that $\alpha_{2\kappa_n+1} > 1/n$. Fix $\nu \in \mathbb{N}$ with $r^{\kappa_\nu}p(0,\alpha_1 e_\alpha) \leq 1$. Then we have

$$p(0,\alpha_{2\kappa_\nu+1}e_\alpha) = p\left(T^{2\kappa_\nu}0, T^{2\kappa_\nu}(\alpha_1 e_\alpha)\right) \leq r^{2\kappa_\nu}p(0,\alpha_1 e_\alpha) \leq r^{\kappa_\nu}, \tag{5.5}$$

and hence,

$$\frac{1}{\nu} < \alpha_{2\kappa_\nu+1} = d(0,\alpha_{2\kappa_\nu+1}e_\alpha) \leq \frac{1}{\nu}. \tag{5.6}$$

This is a contradiction. □

## Acknowledgments

The author is supported in part by Grant-in-Aid for Scientific Research from Japan Society for the Promotion of Science. The author wishes to express his gratitude to the referees for careful reading and giving a historical comment.

## References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] R. Caccioppoli, "Un teorema generale sull'esistenza di elementi uniti in una transformazione funzionale," *Rendiconti dell'Accademia Nazionale dei Lincei*, vol. 11, pp. 794–799, 1930.

[3] R. Kannan, "Some results on fixed points. II," *The American Mathematical Monthly*, vol. 76, pp. 405–408, 1969.

[4] W. A. Kirk, "Contraction mappings and extensions," in *Handbook of Metric Fixed Point Theory*, W. A. Kirk and B. Sims, Eds., pp. 1–34, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[5] S. Leader, "Equivalent Cauchy sequences and contractive fixed points in metric spaces," *Studia Mathematica*, vol. 76, no. 1, pp. 63–67, 1983.

[6] T. Suzuki, "A sufficient and necessary condition for the convergence of the sequence of successive approximations to a unique fixed point," *Proceedings of the American Mathematical Society*, vol. 136, no. 11, pp. 4089–4093, 2008.

[7] I. A. Rus, "The method of successive approximations," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 17, pp. 1433–1437, 1972.

[8] P. V. Subrahmanyam, "Remarks on some fixed-point theorems related to Banach's contraction principle," *Journal of Mathematical and Physical Sciences*, vol. 8, pp. 445–457, 1974.

[9] T. Suzuki, "Subrahmanyam's fixed point theorem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1678–1683, 2009.

[10] I. A. Rus, "Picard operators and applications," *Scientiae Mathematicae Japonicae*, vol. 58, no. 1, pp. 191–219, 2003.

[11] I. A. Rus, "The theory of a metrical fixed point theorem: theoretical and applicative relevances," *Fixed Point Theory*, vol. 9, no. 2, pp. 541–559, 2008.

[12] I. A. Rus, A. S. Mureşan, and V. Mureşan, "Weakly Picard operators on a set with two metrics," *Fixed Point Theory*, vol. 6, no. 2, pp. 323–331, 2005.

[13] I. A. Rus, A. Petruşel, and M. A. Şerban, "Weakly Picard operators: equivalent definitions, applications and open problems," *Fixed Point Theory*, vol. 7, no. 1, pp. 3–22, 2006.

[14] D. Tataru, "Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms," *Journal of Mathematical Analysis and Applications*, vol. 163, no. 2, pp. 345–392, 1992.

[15] C.-K. Zhong, "On Ekeland's variational principle and a minimax theorem," *Journal of Mathematical Analysis and Applications*, vol. 205, no. 1, pp. 239–250, 1997.

[16] C.-K. Zhong, "A generalization of Ekeland's variational principle and application to the study of the relation between the weak P.S. condition and coercivity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 29, no. 12, pp. 1421–1431, 1997.

[17] L.-J. Lin and W.-S. Du, "Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 360–370, 2006.

[18] T. Suzuki, "Generalized distance and existence theorems in complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 253, no. 2, pp. 440–458, 2001.

[19] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.

[20] T. Suzuki, "Several fixed point theorems concerning $\tau$-distance," *Fixed Point Theory and Applications*, no. 3, pp. 195–209, 2004.

[21] T. Suzuki, "Contractive mappings are Kannan mappings, and Kannan mappings are contractive mappings in some sense," *Commentationes Mathematicae. Prace Matematyczne*, vol. 45, no. 1, pp. 45–58, 2005.

[22] T. Suzuki, "The strong Ekeland variational principle," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 787–794, 2006.

[23] T. Suzuki, "On the relation between the weak Palais-Smale condition and coercivity given by Zhong," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 8, pp. 2471–2478, 2008.