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Research Article

Some Characterizations for a Family of Nonexpansive Mappings and Convergence of a Generated Sequence to Their Common Fixed Point

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Motivated by the method of Xu (2006) and Matsushita and Takahashi (2008) , we characterize the set of all common fixed points of a family of nonexpansive mappings by the notion of Mosco convergence and prove strong convergence theorems for nonexpansive mappings and semigroups in a uniformly convex Banach space.

1. Introduction

Let C be a nonempty bounded closed convex subset of a Banach space and $T : C \to C$ a nonexpansive mapping; that is, T satisfies $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$, and consider approximating a fixed point of T. This problem has been investigated by many researchers and various types of strong convergent algorithm have been established. For implicit algorithms, see Browder [1], Reich [2], Takahashi and Ueda [3], and others. For explicit iterative schemes, see Halpern [4], Wittmann [5], Shioji and Takahashi [6], and others. Nakajo and Takahashi [7] introduced a hybrid type iterative scheme by using the metric projection, and recently Takahashi et al. [8] established a modified type of this projection method, also known as the shrinking projection method.

Let us focus on the following methods generating an approximating sequence to a fixed point of a nonexpansive mapping. Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E and let T be a nonexpansive mapping of

C into itself. Xu [9] considered a sequence $\{x_n\}$ generated by

$$\begin{gathered}
x_1 = x \in C, \\
C_n = \operatorname{clco}\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}, \\
D_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{C_n \cap D_n} x
\end{gathered} \tag{1.1}$$

for each $n \in \mathbb{N}$, where clco D is the closure of the convex hull of D, $\Pi_{C_n \cap D_n}$ is the generalized projection onto $C_n \cap D_n$, and $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$ as $n \to \infty$. Then, he proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)} x$. Matsushita and Takahashi [10] considered a sequence $\{y_n\}$ generated by

$$\begin{gathered}
y_1 = x \in C, \\
C_n = \operatorname{clco}\{z \in C : \|z - Tz\| \leq t_n \|y_n - Ty_n\|\}, \\
D_n = \{z \in C : \langle y_n - z, J(x - y_n) \rangle \geq 0\}, \\
y_{n+1} = P_{C_n \cap D_n} x
\end{gathered} \tag{1.2}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap D_n}$ is the metric projection onto $C_n \cap D_n$ and $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$ as $n \to \infty$. They proved that $\{y_n\}$ converges strongly to $P_{F(T)} x$.

In this paper, motivated by these results, we characterize the set of all common fixed points of a family of nonexpansive mappings by the notion of Mosco convergence and prove strong convergence theorems for nonexpansive mappings and semigroups in a uniformly convex Banach space.

2. Preliminaries

Throughout this paper, we denote by E a real Banach space with norm $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. Let G be the family of all strictly increasing continuous convex functions $g : [0,\infty) \to [0,\infty)$ satisfying that $g(0) = 0$. We have the following theorem [11, Theorem 2] for a uniformly convex Banach space.

Theorem 2.1 (Xu [11]). *E is a uniformly convex Banach space if and only if, for every bounded subset B of E, there exists $g_B \in G$ such that*

$$\|\lambda x + (1-\lambda) y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda) g_B(\|x - y\|) \tag{2.1}$$

for all $x, y \in B$ and $0 \leq \lambda \leq 1$.

Bruck [12] proved the following result for nonexpansive mappings.

Theorem 2.2 (Bruck [12]). *Let C be a bounded closed convex subset of a uniformly convex Banach space E. Then, there exists $\gamma \in G$ such that*

$$\gamma\left(\left\| T\left(\sum_{i=1}^{n}\lambda_i x_i\right) - \sum_{i=1}^{n}\lambda_i T x_i \right\|\right) \leq \max_{1\leq j<k\leq n}\left(\|x_j - x_k\| - \|Tx_j - Tx_k\|\right) \tag{2.2}$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_n\} \subset C$, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and nonexpansive mapping T of C into E.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We denote the set of all strong limit points of $\{C_n\}$ by s-$\mathrm{Li}_n C_n$, that is, $x \in$ s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly the set of all weak subsequential limit points by w-$\mathrm{Ls}_n C_n$; $y \in$ w-$\mathrm{Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that $C_0 =$ s-$\mathrm{Li}_n C_n =$ w-$\mathrm{Ls}_n C_n$, then we say that $\{C_n\}$ converges to C_0 in the sense of Mosco and we write $C_0 =$ M-$\lim_n C_n$. By definition, it always holds that s-$\mathrm{Li}_n C_n \subset$ w-$\mathrm{Ls}_n C_n$. Therefore, to prove $C_0 =$ M-$\lim_n C_n$, it suffices to show that

$$\text{w-}\operatorname*{Ls}_n C_n \subset C_0 \subset \text{s-}\operatorname*{Li}_n C_n. \tag{2.3}$$

One of the simplest examples of Mosco convergence is a decreasing sequence $\{C_n\}$ with respect to inclusion. The Mosco limit of such a sequence is $\bigcap_{n=1}^{\infty} C_n$. For more details, see [13].

Suppose that E is smooth, strictly convex, and reflexive. The normalized duality mapping of E is denoted by J, that is,

$$Jx = \left\{x^* \in E^* : \|x\|^2 = \langle x, x^*\rangle = \|x^*\|^2\right\} \tag{2.4}$$

for $x \in E$. In this setting, we may show that J is a single-valued one-to-one mapping onto E^*. For more details, see [14–16].

Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. Then, for an arbitrarily fixed $x \in E$, a function $C \ni y \mapsto \|y - x\|^2 \in \mathbb{R}$ has a unique minimizer $y_x \in C$. Using such a point, we define the metric projection $P_C : E \to C$ by $P_C x = y_x$ for every $x \in E$. The metric projection has the following important property: $x_0 = P_C x$ if and only if $x_0 \in C$ and $\langle x_0 - z, J(x - x_0)\rangle \geq 0$ for all $z \in C$.

In the same manner, we define the generalized projection [17] $\Pi_C : E \to C$ for a nonempty closed convex subset C of a strictly convex, smooth, and reflexive Banach space E as follows. For a fixed $x \in E$, a function $C \ni y \mapsto \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2 \in \mathbb{R}$ has a unique minimizer and we define $\Pi_C x$ by this point. We know that the following characterization holds for the generalized projection [17, 18]: $x_0 = \Pi_C x$ if and only if $x_0 \in C$ and $\langle x_0 - z, Jx - Jx_0\rangle \geq 0$ for all $z \in C$.

Tsukada [19] proved the following theorem for a sequence of metric projections in a Banach space.

Theorem 2.3 (Tsukada [19]). *Let E be a reflexive and strictly convex Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 =$ M-$\lim_n C_n$ exists and nonempty, then,* *for each* $x \in E$, $\{P_{C_n}x\}$ *converges weakly to* $P_{C_0}x$, *where* P_K *is the metric projection onto a nonempty closed convex subset* K *of* E*. Moreover, if* E *has the Kadec-Klee property, the convergence is in the strong topology.*

On the other hand, Ibaraki et al. [20] proved the following theorem for a sequence of generalized projections in a Banach space.

Theorem 2.4 (Ibaraki et al. [20]). *Let* E *be a strictly convex, smooth, and reflexive Banach space and let* $\{C_n\}$ *be a sequence of nonempty closed convex subsets of* E*. If* $C_0 = \text{M-lim}_n C_n$ *exists and nonempty, then, for each* $x \in E$, $\{\Pi_{C_n}x\}$ *converges weakly to* $\Pi_{C_0}x$, *where* Π_K *is the generalized projection onto a nonempty closed convex subset* K *of* E*. Moreover, if* E *has the Kadec-Klee property, the convergence is in the strong topology.*

Kimura [21] obtained the further generalization of this theorem by using the Bregman projection; see also [22].

Theorem 2.5 (Kimura [21]). *Let* C *be a nonempty closed convex subset of a reflexive Banach space* E *and let* $f : E \to (-\infty, \infty]$ *be a Bregman function on* C*; that is,* (i) f *is lower semicontinuous and strictly convex;* (ii) C *is contained by the interior of the domain of* f*;* (iii) f *is Gâteaux differentiable on* C*;* (iv) *the subsets* $\{u \in C : D_f(y,u) \leq \alpha\}$ *and* $\{v \in C : D_f(v,x) \leq \alpha\}$ *of* C *are both bounded for all* $x, y \in C$ *and* $\alpha \geq 0$, *where* $D_f(y,x) = f(y) - f(x) + \langle \nabla f(x), x - y\rangle$ *for all* $y \in D$ *and* $x \in C$*. Let* $\{C_n\}$ *be a sequence of nonempty closed convex subsets of* C *such that* $C_0 = \text{M-lim}_n C_n$ *exists and is nonempty. Suppose that* f *is sequentially consistent; that is, for any bounded sequence* $\{x_n\}$ *of* C *and* $\{y_n\}$ *of the domain of* f, $\lim_{n\to\infty} D_f(y_n, x_n) = 0$ *implies* $\lim_{n\to\infty}\|y_n - x_n\| = 0$*. Then, the sequence* $\{\Pi^f_{C_n}x\}$ *of Bregman projections converges strongly to* $\Pi^f_{C_0}x$ *for all* $x \in C$.

We note that the generalized duality mapping J coincides with ∇f if the function f is defined by $f(x) = \|x\|^2/2$ for $x \in E$. In this case, the Bregman projection Π^f_K with respect to f becomes the generalized projection Π_K for any nonempty closed convex subset K of E.

3. Main Results

Let C be a nonempty closed convex subset of E and let $\{T_n\}$ be a sequence of mappings of C into itself such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We consider the following conditions.

(I) For every bounded sequence $\{z_n\}$ in C, $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$ implies $\omega_w(z_n) \subset F$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$; see [23–25].

(II) for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply $z \in F$.

We know that condition (I) implies condition (II). Then, we have the following results.

Theorem 3.1. *Let* C *be a nonempty bounded closed convex subset of a uniformly convex Banach space* E *and let* $\{T_n\}$ *be a family of nonexpansive mappings of* C *into itself with* $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$*. Let* $C_n(t_n) = \operatorname{clco}\{z \in C : \|z - T_n z\| \leq t_n\}$ *for each* $n \in \mathbb{N}$, *where* $\{t_n\} \subset [0, \infty)$*. Then, the following are equivalent:*

(i) $\{T_n\}$ *satisfies condition (I);*

(ii) *for every* $\{t_n\} \subset [0, \infty)$ *with* $t_n \to 0$ *as* $n \to \infty$, $\text{M-lim}_n C_n(t_n) = F$.

Proof. First, let us prove that (i) implies (ii). Let $\{t_n\} \subset [0,\infty)$ with $t_n \to 0$ as $n \to \infty$. It is obvious that $F \subset C_n(t_n)$ and $C_n(t_n)$ is closed and convex for all $n \in \mathbb{N}$. Thus we have

$$F \subset \operatorname*{s\text{-}Li}_n C_n(t_n). \tag{3.1}$$

Let $z \in \text{w-Ls}_n C_n(t_n)$. Then, there exists a sequence $\{z_i\}$ such that $z_i \in C_{n_i}(t_{n_i})$ for all $i \in \mathbb{N}$ and $z_i \rightharpoonup z$ as $i \to \infty$. Let $\{u_n\}$ be a sequence in C such that $u_n \in C_n(t_n)$ for every $n \in \mathbb{N}$ and that $u_{n_i} = z_i$ for all $i \in \mathbb{N}$. Fix $n \in \mathbb{N}$. From the definition of $C_n(t_n)$, there exist $m \in \mathbb{N}$, $\{\lambda_1, \lambda_2, \dots, \lambda_m\} \subset [0,1]$, and $\{y_1, y_2, \dots, y_m\} \subset C$ such that

$$\sum_{i=1}^{m} \lambda_i = 1, \qquad \left\| u_n - \sum_{i=1}^{m} \lambda_i y_i \right\| < t_n, \qquad \|y_i - T_n y_i\| \le t_n \tag{3.2}$$

for each $i = 1, 2, \dots, m$. On the other hand, by Theorem 2.2, there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that

$$\gamma\left(\left\| T\left(\sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i T x_i \right\| \right) \le \max_{1 \le j < k \le n} \left(\|x_j - x_k\| - \|Tx_j - Tx_k\| \right) \tag{3.3}$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \dots, x_n\} \subset C$, $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ and nonexpansive mapping T of C into E. Thus we get

$$\begin{aligned}
\|u_n - T_n u_n\| &\le \left\| u_n - \sum_{i=1}^{m} \lambda_i y_i \right\| + \left\| \sum_{i=1}^{m} \lambda_i y_i - \sum_{i=1}^{m} \lambda_i T_n y_i \right\| \\
&\quad + \left\| \sum_{i=1}^{m} \lambda_i T_n y_i - T_n\left(\sum_{i=1}^{m} \lambda_i y_i \right) \right\| + \left\| T_n\left(\sum_{i=1}^{m} \lambda_i y_i \right) - T_n u_n \right\| \\
&\le 3t_n + \gamma^{-1}\left(\max_{1 \le j < k \le m} \left(\|y_j - y_k\| - \|T_n y_j - T_n y_k\| \right) \right) \\
&\le 3t_n + \gamma^{-1}\left(\max_{1 \le j < k \le m} \left(\|y_j - T_n y_j\| + \|y_k - T_n y_k\| \right) \right) \\
&\le 3t_n + \gamma^{-1}(2t_n)
\end{aligned} \tag{3.4}$$

for every $n \in \mathbb{N}$, which implies $\|u_n - T_n u_n\| \to 0$ as $n \to \infty$. From condition (I), we get $z \in \omega_w(z_i) \subset \omega_w(u_n) \subset F$, that is,

$$\operatorname*{w\text{-}Ls}_n C_n(t_n) \subset F. \tag{3.5}$$

By (3.1) and (3.5), we have

$$\operatorname*{M\text{-}lim}_n C_n(t_n) = F. \tag{3.6}$$

Next we show that (ii) implies (i). Let $\{z_n\}$ be a sequence in C such that

$$\lim_{n\to\infty}\|z_n - T_n z_n\| = 0 \tag{3.7}$$

and define $\{t_n\}$ by $t_n = \|z_n - T_n z_n\|$ for each $n \in \mathbb{N}$. Suppose that a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converges weakly to z. Then since $z_n \in C_n(t_n)$ for all $n \in \mathbb{N}$ and $\text{M-lim}_n C_n(t_n) = F$, we have $z \in F$; that is, condition (I) holds. □

For a sequence of mappings satisfying condition (II), we have the following characterization.

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $D_0(t_0) = C$ and $D_n(t_n) = \operatorname{clco}\{z \in D_{n-1}(t_{n-1}) : \|z - T_n z\| \leq t_n\}$ for each $n \in \mathbb{N}$, where $\{t_n\} \subset [0,\infty)$. Then, the following are equivalent:*

(i) *$\{T_n\}$ satisfies condition (II);*

(ii) *for every $\{t_n\} \subset [0,\infty)$ with $t_n \to 0$ as $n \to \infty$, $\text{M-lim}_n D_n(t_n) = F$.*

Proof. Let us show that (i) implies (ii). Let $\{t_n\} \subset [0,\infty)$ with $t_n \to 0$ as $n \to \infty$. We have $F \subset D_n(t_n) \subset D_{n-1}(t_{n-1})$ for all $n \in \mathbb{N}$. Thus we get

$$F \subset \bigcap_{n=0}^{\infty} D_n(t_n) = \text{M-}\lim_n D_n(t_n). \tag{3.8}$$

Let $z \in \bigcap_{n=0}^{\infty} D_n(t_n)$. We have $z \in D_n(t_n)$ for all $n \in \mathbb{N}$. As in the proof of Theorem 3.1, we get $\lim_{n\to\infty}\|z - T_n z\| = 0$. By condition (II), we obtain $z \in F$, which implies $\bigcap_{n=0}^{\infty} D_n(t_n) \subset F$. Hence we have $\text{M-lim}_n D_n(t_n) = F$.

Suppose that condition (ii) holds. Let $\{z_n\}$ be a sequence in C and $z \in C$ such that $z_n \to z$ and that $T_n z_n \to z$. Since

$$\begin{aligned}\|z - T_n z\| &\leq \|z - z_n\| + \|z_n - T_n z_n\| + \|T_n z_n - T_n z\| \\ &\leq 2\|z_n - z\| + \|z_n - T_n z_n\|\end{aligned} \tag{3.9}$$

for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty}\|z - T_n z\| = 0$. Letting $t_n = \|z - T_n z\|$ for each $n \in \mathbb{N}$, we have $z \in D_n(t_n)$ for every $n \in \mathbb{N}$ and $t_n \to 0$ as $n \to \infty$, which implies $z \in \text{M-lim}_n D_n(t_n) = F$. Hence (i) holds, which is the desired result. □

Remark 3.3. *In Theorem 3.2, it is obvious by definition that $\{D_n(t_n)\}$ is a decreasing sequence with respect to inclusion. Therefore, conditions* (i) *and* (ii) *are also equivalent to*

(ii′) *for every $\{t_n\} \subset [0,\infty)$ with $t_n \to 0$ as $n \to \infty$, $\text{PK-lim}_n D_n(t_n) = F$,*

where $\text{PK-lim}_n D_n(t_n)$ is the Painlevé-Kuratowski limit of $\{D_n(t_n)\}$; see, for example, [13] for more details.

In the next section, we will see various types of sequences of nonexpansive mappings which satisfy conditions (I) and (II).

4. The Sequences of Mappings Satisfying Conditions (I) and (II)

First let us show some known results which play important roles for our results.

Theorem 4.1 (Browder [1]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and T a nonexpansive mapping on C with $F(T) \neq \emptyset$. If $\{x_n\}$ converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then z is a fixed point of T.*

Theorem 4.2 (Bruck [26]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E and $T_k : C \to C$ a nonexpansive mapping for each $k \in \mathbb{N}$. Let $\{\beta_k\}$ be a sequence of positive real numbers such that $\sum_{k=1}^{\infty} \beta_k = 1$. If $\bigcap_{k=1}^{\infty} F(T_k)$ is nonempty, then the mapping $T = \sum_{k=1}^{\infty} \beta_k T_k$ is well defined and*

$$F(T) = \bigcap_{k=1}^{\infty} F(T_k). \tag{4.1}$$

Theorems 4.3, 4.5(i), 4.6–4.9 show the examples of a family of nonexpansive mappings satisfying condition (I). Theorems 4.5(ii), 4.11, and 4.12 show those satisfying condition (II).

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Let $T_n = T$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$ and satisfies condition (I).*

Proof. This is a direct consequence of Theorem 4.1. □

Remark 4.4. *In the previous theorem, if C is bounded, then $F(T)$ is guaranteed to be nonempty by Kirk's fixed point theorem [27].*

Let E be a Banach space and A a set-valued operator on E. A is called an accretive operator if $\|x_1 - x_2\| \leq \|(x_1 - x_2) + \lambda(y_1 - y_2)\|$ for every $\lambda > 0$ and $x_1, x_2, y_1, y_2 \in E$ with $y_1 \in Ax_1$ and $y_2 \in Ax_2$.

Let A be an accretive operator and $r > 0$. We know that the operator $I + rA$ has a single-valued inverse, where I is the identity operator on E. We call $(I + rA)^{-1}$ the resolvent of A and denote it by J_r. We also know that J_r is a nonexpansive mapping with $F(J_r) = A^{-1}0$ for any $r > 0$, where $A^{-1}0 = \{z \in E : 0 \in Az\}$. For more details, see, for example, [15].

We have the following result for the resolvents of an accretive operator by [25]; see also [15, Theorem 4.6.3], and [16, Theorem 3.4.3] .

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $A \subset E \times E$ be an accretive operator with* $\operatorname{cl} D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$ *and $A^{-1}0 \neq \emptyset$. Let $T_n = J_{r_n}$ for every $n \in \mathbb{N}$, where $r_n > 0$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C*

into itself with $\bigcap_{n=1}^{\infty} F(T_n) = A^{-1}0$ *and the following hold:*

(i) *if* $\inf_{n\in\mathbb{N}} r_n > 0$*, then* $\{T_n\}$ *satisfies condition (I),*

(ii) *if there exists a subsequence* $\{r_{n_i}\}$ *of* $\{r_n\}$ *such that* $\inf_{i\in\mathbb{N}} r_{n_i} > 0$*, then* $\{T_n\}$ *satisfies condition (II).*

Proof. It is obvious that T_n is a nonexpansive mapping of C into itself and $F(T_n) = A^{-1}0$ for all $n \in \mathbb{N}$.

For (i), suppose $\inf_{n\in\mathbb{N}} r_n > 0$ and let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$. By [25, Lemma 3.5], we have $\lim_{n\to\infty}\|z_n - J_1 z_n\| = 0$. Using Theorem 4.1 we obtain $\omega_w(z_n) \subset F(J_1) = A^{-1}0$.

Let us show (ii). Let $\{r_{n_i}\}$ be a subsequence of $\{r_n\}$ with $\inf_{i\in\mathbb{N}} r_{n_i} > 0$ and let $\{z_n\}$ be a sequence in C and $z \in C$ such that $z_n \rightarrow z$ and $T_n z_n \rightarrow z$. As in the proof of (i), we get $\lim_{i\to\infty}\|z_{n_i} - J_1 z_{n_i}\| = 0$ and $z \in A^{-1}0$. □

Let C be a nonempty closed convex subset of E. Let $\{S_n\}$ be a family of mappings of C into itself and let $\{\beta_{n,k} : n, k \in \mathbb{N},\ 1 \le k \le n\}$ be a sequence of real numbers such that $0 \le \beta_{i,j} \le 1$ for every $i, j \in \mathbb{N}$ with $i \ge j$. Takahashi [16, 28] introduced a mapping W_n of C into itself for each $n \in \mathbb{N}$ as follows:

$$
\begin{aligned}
U_{n,n} &= \beta_{n,n} S_n + (1-\beta_{n,n}) I,\\
U_{n,n-1} &= \beta_{n,n-1} S_{n-1} U_{n,n} + (1-\beta_{n,n-1}) I,\\
&\ \ \vdots\\
U_{n,k} &= \beta_{n,k} S_k U_{n,k+1} + (1-\beta_{n,k}) I,\\
&\ \ \vdots\\
U_{n,2} &= \beta_{n,2} S_2 U_{n,3} + (1-\beta_{n,2}) I,\\
W_n = U_{n,1} &= \beta_{n,1} S_1 U_{n,2} + (1-\beta_{n,1}) I.
\end{aligned}
\tag{4.2}
$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. We have the following result for the W-mapping by [29, 30]; see also [25, Lemma 3.6].

Theorem 4.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\{S_n\}$ be a family of nonexpansive mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $\{\beta_{n,k} : n, k \in \mathbb{N},\ 1 \le k \le n\}$ be a sequence of real numbers such that $0 < a \le \beta_{i,j} \le b < 1$ for every $i, j \in \mathbb{N}$ with $i \ge j$ and let W_n be the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,1}$. Let $T_n = W_n$ for every $n \in \mathbb{N}$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) = F$ and satisfies condition (I).*

Proof. It is obvious that $\{T_n\}$ is a family of nonexpansive mappings of C into itself. By [29, Lemma 3.1], $F(T_n) = \bigcap_{i=1}^{n} F(S_i)$ for all $n \in \mathbb{N}$, which implies $\bigcap_{n=1}^{\infty} F(T_n) = F$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$. We have $\lim_{n\to\infty}\|z_n - S_1 U_{n,2} z_n\| = 0$.

Let $z \in F$. From Theorem 2.1, for a bounded subset B of C containing $\{z_n\}$ and z, there exists $g_{B_0} \in G$, where $B_0 = \{y \in E : \|y\| \le 2\sup_{x\in B}\|x\|\}$, such that

$$\begin{aligned}
\|z_n - z\|^2 &\le (\|z_n - S_1U_{n,2}z_n\| + \|S_1U_{n,2}z_n - z\|)^2 \\
&= \|z_n - S_1U_{n,2}z_n\|(\|z_n - S_1U_{n,2}z_n\| + 2\|S_1U_{n,2}z_n - z\|) \\
&\quad + \|S_1U_{n,2}z_n - z\|^2 \\
&\le M\|z_n - S_1U_{n,2}z_n\| + \|U_{n,2}z_n - z\|^2 \\
&\le M\|z_n - S_1U_{n,2}z_n\| + \beta_{n,2}\|S_2U_{n,3}z_n - z\|^2 + (1-\beta_{n,2})\|z_n - z\|^2 \\
&\quad - \beta_{n,2}(1-\beta_{n,2})g_{B_0}(\|S_2U_{n,3}z_n - z_n\|) \\
&\le M\|z_n - S_1U_{n,2}z_n\| + \|z_n - z\|^2 - \beta_{n,2}(1-\beta_{n,2})g_{B_0}(\|S_2U_{n,3}z_n - z_n\|)
\end{aligned} \tag{4.3}$$

for every $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}}(\|z_n - S_1U_{n,2}z_n\| + 2\|S_1U_{n,2}z_n - z\|)$. Thus we obtain $\lim_{n\to\infty}\|S_2U_{n,3}z_n - z_n\| = 0$. Let $m \in \mathbb{N}$. Similarly, we have

$$\lim_{n\to\infty}\|S_mU_{n,m+1}z_n - z_n\| = \lim_{n\to\infty}\|S_{m+1}U_{n,\mathrm{m}+2}z_n - z_n\| = 0. \tag{4.4}$$

As in the proof of [30, Theorem 3.1], we get $\lim_{n\to\infty}\|z_n - S_kz_n\| = 0$ for each $k \in \mathbb{N}$. Using Theorem 4.1 we obtain $\omega_w(z_n) \subset F$. □

We have the following result for a convex combination of nonexpansive mappings which Aoyama et al. [31] proposed.

Theorem 4.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that $F = \bigcap_{n=1}^{\infty}F(S_n) \neq \emptyset$. Let $\{\beta_n^k\}$ be a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \le n$ such that*

(i) $\sum_{k=1}^{n}\beta_n^k = 1$ *for every* $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty}\beta_n^k > 0$ *for each* $k \in \mathbb{N}$,

and let $T_n = \alpha_nI + (1-\alpha_n)\sum_{k=1}^{n}\beta_n^kS_k$ for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ with $a \le b$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty}F(T_n) = F$ and satisfies condition (I).

Proof. It is obvious that $\{T_n\}$ is a family of nonexpansive mappings of C into itself. By Theorem 4.2, we have $F(\sum_{k=1}^{n}\beta_n^kS_k) = \bigcap_{k=1}^{n}F(S_k)$ and thus $F(T_n) = \bigcap_{k=1}^{n}F(S_k)$. It follows that

$$F = \bigcap_{n=1}^{\infty}F(S_n) = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{n}F(S_k) = \bigcap_{n=1}^{\infty}F(T_n). \tag{4.5}$$

Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$. Let $z \in F$, $m \in \mathbb{N}$, and $\gamma_n^m = \alpha_n + (1-\alpha_n)\beta_n^m$ for $n \in \mathbb{N}$. By Theorem 2.1, for a bounded subset B of C containing $\{z_n\}$ and z, there exists $g_{B_0} \in G$ with $B_0 = \{y \in E : \|y\| \le 2\sup_{x\in B}\|x\|\}$ which satisfies that

$$\begin{aligned}
\|z_n - z\|^2 &\le (\|z_n - T_n z_n\| + \|T_n z_n - z\|)^2 \le M\|z_n - T_n z_n\| + \|T_n z_n - z\|^2 \\
&= M\|z_n - T_n z_n\| + \left\|\alpha_n(z_n - z) + (1-\alpha_n)\sum_{k=1}^{n}\beta_n^k(S_k z_n - z)\right\|^2 \\
&\le M\|z_n - T_n z_n\| + \gamma_n^m\left\|\frac{\alpha_n(z_n - z) + (1-\alpha_n)\beta_n^m(S_m z_n - z)}{\gamma_n^m}\right\|^2 \\
&\quad + (1-\gamma_n^m)\left\|\frac{(1-\alpha_n)\left(\sum_{k=1}^{m-1}\beta_n^k(S_k z_n - z) + \sum_{k=m+1}^{n}\beta_n^k(S_k z_n - z)\right)}{1-\gamma_n^m}\right\|^2 \\
&\le M\|z_n - T_n z_n\| + \alpha_n\|z_n - z\|^2 + (1-\alpha_n)\beta_n^m\|S_m z_n - z\|^2 \\
&\quad - \frac{\alpha_n(1-\alpha_n)\beta_n^m}{\gamma_n^m}g_{B_0}(\|z_n - S_m z_n\|) + (1-\gamma_n^m)\|z_n - z\|^2 \\
&= M\|z_n - T_n z_n\| + \|z_n - z\|^2 - \frac{\alpha_n(1-\alpha_n)\beta_n^m}{\alpha_n + (1-\alpha_n)\beta_n^m}g_{B_0}(\|z_n - S_m z_n\|)
\end{aligned} \tag{4.6}$$

for $n \in \mathbb{N}$, where $M = \sup_{n\in\mathbb{N}}\{\|z_n - T_n z_n\| + 2\|T_n z_n - z\|\}$. Since $a \le \alpha_n \le b$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\beta_n^m > 0$, we get $\lim_{n\to\infty}g_{B_0}(\|z_n - S_m z_n\|) = 0$ and hence $\lim_{n\to\infty}\|z_n - S_m z_n\| = 0$ for each $m \in \mathbb{N}$. Therefore, using Theorem 4.1 we obtain $\omega_w(z_n) \subset F$. □

Let C be a nonempty closed convex subset of a Banach space E and let S be a semigroup. A family $\mathcal{S} = \{T(t) : t \in S\}$ is said to be a nonexpansive semigroup on C if

(i) for each $t \in S$, $T(t)$ is a nonexpansive mapping of C into itself;

(ii) $T(st) = T(s)T(t)$ for every $s, t \in S$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \bigcap_{t\in S}F(T(t))$. We have the following result for nonexpansive semigroups by [25, Lemma 3.9]; see also [32, 33].

Theorem 4.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let S be a semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let X be a subspace of $B(S)$ such that X contains constants, X is l_s-invariant (i.e., $l_s(X) \subset X$) for each $s \in S$, and the function $t \mapsto \langle T(t)x, x^*\rangle$ belongs to X for every $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a sequence of means on X such that $\|\mu_n - l_s^*\mu_n\| \to 0$ as $n \to \infty$ for all $s \in S$ and let $T_n = T_{\mu_n}$ for each $n \in \mathbb{N}$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty}F(T_n) = F(\mathcal{S})$ and satisfies condition (I).*

Proof. It is obvious that $\{T_n\}$ is a family of nonexpansive mappings of C into itself. By [25, Lemma 3.9], we have $F(\mathcal{S}) = \bigcap_{n=1}^{\infty}F(T_n)$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$. Then we get $\lim_{n\to\infty}\|z_n - T(t)z_n\| = 0$ for every $t \in S$. Using Theorem 4.1 we have $\omega_w(z_n) \subset F(\mathcal{S})$. □

Let C be a nonempty closed convex subset of a Banach space E. A family $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$ for every $s, t \ge 0$;

(iii) $\|T(s)x - T(s)y\| \le \|x - y\|$ for each $s \ge 0$ and $x, y \in C$;

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We have the following result for one-parameter nonexpansive semigroups by [25, Lemma 3.12].

Theorem 4.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(s) : 0 \le s < \infty\}$ be a one-parameter nonexpansive semigroup on C with $F(\mathcal{S}) \ne \emptyset$. Let $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} r_n = \infty$ and let T_n be a mapping such that*

$$T_n x = \frac{1}{r_n}\int_0^{r_n} T(s)x\,ds \tag{4.7}$$

for all $x \in C$ and $n \in \mathbb{N}$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself satisfying that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{S})$ and condition (I).

Remark 4.10. *If C is bounded, then $F(\mathcal{S})$ is guaranteed to be nonempty; see [34].*

Proof. It is obvious that $\{T_n\}$ is a family of nonexpansive mappings of C into itself. By [25, Lemma 3.12], we have $F(\mathcal{S}) = \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$. We get $\lim_{n\to\infty}\|z_n - T(t)z_n\| = 0$ for every $t \in S$. Hence, using Theorem 4.1 we have $\omega_w(z_n) \subset F(\mathcal{S})$. □

Motivated by the idea of [23, page 256], we have the following result for nonexpansive mappings.

Theorem 4.11. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let I be a countable index set. Let $i : \mathbb{N} \to I$ be an index mapping such that, for all $j \in I$, there exist infinitely many $k \in \mathbb{N}$ satisfying $j = i(k)$. Let $\{S_i : i \in I\}$ be a family of nonexpansive mappings of C into itself satisfying $F = \bigcap_{i\in I} F(S_i) \ne \emptyset$ and let $T_n = S_{i(n)}$ for all $n \in \mathbb{N}$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) = F$ and satisfies condition (II).*

Proof. It is obvious that $\bigcap_{n=1}^{\infty} F(T_n) = F$. Let $\{z_n\}$ be a sequence in C and $z \in C$ such that $z_n \to z$ and $T_n z_n \to z$. Fix $j \in I$. There exists a subsequence $\{i(n_k)\}$ of $\{i(n)\}$ such that $i(n_k) = j$ for all $k \in \mathbb{N}$. Thus we have $\lim_{k\to\infty}\|z_{n_k} - T_{n_k} z_{n_k}\| = \lim_{n\to\infty}\|z_{n_k} - S_j z_{n_k}\| = 0$. Therefore, using Theorem 4.1 $z \in F(S_j)$ for every $j \in I$ and hence we get $z \in F$. □

From Theorem 4.11, we have the following result for one-parameter nonexpansive semigroups.

Theorem 4.12. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \ne \emptyset$. Let $S_n = T(r_n)$ for every $n \in \mathbb{N}$ with $\{r_n\} \subset (0,\infty)$ and $r_n \to 0$ as $n \to \infty$ and $T_n = S_{i(n)}$ for all $n \in \mathbb{N}$, where $i : \mathbb{N} \to \mathbb{N}$ is an index mapping satisfying, for all $j \in \mathbb{N}$, there exist infinitely many $k \in \mathbb{N}$ such that $j = i(k)$. Then, $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{S})$ and satisfies condition (II).*

Remark 4.13. *If C is bounded, it is guaranteed that $F(\mathcal{S}) \neq \emptyset$. See [34].*

Proof. We have $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{S})$ by [35, Lemma 2.7]; see also [36]. By Theorem 4.11, we obtain the desired result. □

5. Strong Convergence Theorems

Throughout this section, we assume that C is a nonempty bounded closed convex subset of a uniformly convex Banach space E and $\{T_n\}$ is a family of nonexpansive mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then, we know that F is closed and convex.

We get the following results for the metric projection by using Theorems 2.3, 3.1, and 3.2.

Theorem 5.1. *Let $x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} C_n = \operatorname{clco}\{z \in C : \|z - T_n z\| \leq t_n\}, \\ x_n = P_{C_n} x \end{gathered} \tag{5.1}$$

for each $n \in \mathbb{N}$, where $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$ as $n \to \infty$, and P_{C_n} is the metric projection onto C_n. If $\{T_n\}$ satisfies condition (I), then $\{x_n\}$ converges strongly to $P_F x$.

Theorem 5.2. *Let $x \in E$ and let $\{y_n\}$ be a sequence generated by*

$$\begin{gathered} C_0 = C, \\ C_n = \operatorname{clco}\{z \in C_{n-1} : \|z - T_n z\| \leq t_n\}, \\ y_n = P_{C_n} x \end{gathered} \tag{5.2}$$

for each $n \in \mathbb{N}$, where $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$ as $n \to \infty$. If $\{T_n\}$ satisfies condition (II), then $\{y_n\}$ converges strongly to $P_F x$.

On the other hand, we have the following results for the Bregman projection by using Theorems 2.5, 3.1, and 3.2.

Theorem 5.3. *Let $x \in C$ and let f be a Bregman function on C and let f be sequentially consistent. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} C_n = \operatorname{clco}\{z \in C : \|z - T_n z\| \leq t_n\}, \\ x_n = \Pi_{C_n}^f x \end{gathered} \tag{5.3}$$

for each $n \in \mathbb{N}$, where $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$ as $n \to \infty$ and $\Pi_{C_n}^f$ is the Bregman projection onto C_n. If $\{T_n\}$ satisfies condition (I), then $\{x_n\}$ converges strongly to $\Pi_F^f x$.

Theorem 5.4. *Let $x \in C$, let f be a Bregman function on C, and let f be sequentially consistent. Let $\{y_n\}$ be a sequence generated by*

$$\begin{aligned} C_0 &= C, \\ C_n &= \operatorname{clco}\{z \in C_{n-1} : \|z - T_n z\| \le t_n\}, \\ y_n &= \Pi^f_{C_n} x \end{aligned} \tag{5.4}$$

for each $n \in \mathbb{N}$, where $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$ as $n \to \infty$. If $\{T_n\}$ satisfies condition (II), then $\{y_n\}$ converges strongly to $\Pi^f_F x$.

In a similar fashion, we have the following results for the generalized projection by using Theorems 2.4, 3.1, and 3.2.

Theorem 5.5. *Suppose that E is smooth. Let $x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} C_n &= \operatorname{clco}\{z \in C : \|z - T_n z\| \le t_n\}, \\ x_n &= \Pi_{C_n} x \end{aligned} \tag{5.5}$$

for each $n \in \mathbb{N}$, where $\{t_n\} \subset (0, \infty)$ such that $t_n \to 0$ as $n \to \infty$ and Π_{C_n} is the generalized projection onto C_n. If $\{T_n\}$ satisfies condition (I), then $\{x_n\}$ converges strongly to $\Pi_F x$.

Theorem 5.6. *Suppose that E is smooth. Let $x \in E$ and let $\{y_n\}$ be a sequence generated by*

$$\begin{aligned} C_0 &= C, \\ C_n &= \operatorname{clco}\{z \in C_{n-1} : \|z - T_n z\| \le t_n\}, \\ y_n &= \Pi_{C_n} x \end{aligned} \tag{5.6}$$

for each $n \in \mathbb{N}$, where $\{t_n\} \subset (0, \infty)$ with $t_n \to 0$ as $n \to \infty$. If $\{T_n\}$ satisfies condition (II), then $\{y_n\}$ converges strongly to $\Pi_F x$.

Combining these theorems with the results shown in the previous section, we can obtain various types of convergence theorems for families of nonexpansive mappings.

6. Generalization of Xu's and Matsushita-Takahashi's Theorems

At the end of this paper, we remark the relationship between these results and the convergence theorems by Xu [9] and Matsushita and Takahashi [10] mentioned in the introduction.

Let us suppose the all assumptions in their results, respectively. Let $\{T_n\}$ be a countable family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and suppose that it satisfies condition (I). Let us define $C_n = \operatorname{clco}\{z \in C : \|z - T_n z\| \le t_n \|x_n - T_n x_n\|\}$ for $n \in \mathbb{N}$.

Then, by definition, we have that $\bigcap_{k=1}^{\infty} F(T_k) \subset C_n$ for every $n \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned} &\langle \Pi_{C_n \cap D_n} x - z, Jx - J\Pi_{C_n \cap D_n} x \rangle \geq 0, \\ &\langle P_{C_n \cap D_n} x - z, J(x - P_{C_n \cap D_n} x) \rangle \geq 0 \end{aligned} \tag{6.1}$$

for every $z \in C_n \cap D_n$ from basic properties of $P_{C_n \cap D_n}$ and $\Pi_{C_n \cap D_n}$. Therefore, for each theorem we have

$$\bigcap_{k=1}^{\infty} F(T_k) \subset C_n \cap D_n \tag{6.2}$$

for every $n \in \mathbb{N}$ by using mathematical induction. Since C is bounded, a sequence $\{t_n \|x_n - T_n x_n\|\}$ converges to 0 for any $\{x_n\}$ in C whenever $\{t_n\}$ converges to 0. Thus, using Theorem 3.1 we obtain

$$\bigcap_{k=1}^{\infty} F(T_k) \subset \operatorname{s-Li}_n (C_n \cap D_n) \subset \operatorname{w-Ls}_n (C_n \cap D_n) \subset \operatorname{M-lim}_n C_n = \bigcap_{k=1}^{\infty} F(T_k), \tag{6.3}$$

and therefore $\operatorname{M-lim}_n (C_n \cap D_n) = \bigcap_{k=1}^{\infty} F(T_k)$. Consequently, by using Theorems 2.3 and 2.4, we obtain the following results generalizing the theorems of Xu, and Matsushita and Takahashi, respectively.

Theorem 6.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E and $\{T_n\}$ a sequence of nonexpansive mappings of C into itself such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and suppose that it satisfies condition (I). Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ C_n &= \operatorname{clco} \{ z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\| \}, \\ D_n &= \{ z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0 \}, \\ x_{n+1} &= \Pi_{C_n \cap D_n} x \end{aligned} \tag{6.4}$$

for each $n \in \mathbb{N}$, where $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$ as $n \to \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_F x$.

Theorem 6.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex and smooth Banach space E and $\{T_n\}$ a sequence of nonexpansive mappings of C into itself such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and suppose that it satisfies condition (I). Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ C_n &= \operatorname{clco} \{ z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\| \}, \\ D_n &= \{ z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0 \}, \\ x_{n+1} &= P_{C_n \cap D_n} x \end{aligned} \tag{6.5}$$

for each $n \in \mathbb{N}$*, where* $\{t_n\}$ *is a sequence in* $(0,1)$ *with* $t_n \to 0$ *as* $n \to \infty$*. Then,* $\{x_n\}$ *converges strongly to* $P_F x$*.*

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