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Research Article

A New Strong Convergence Theorem for Equilibrium Problems and Fixed Point Problems in Banach Spaces

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We introduce a new iterative sequence for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a Banach space. Then, we study the strong convergence of the sequences. With an appropriate setting, we obtain the corresponding results due to Takahashi-Takahashi and Takahashi-Zembayashi. Some of our results are established with weaker assumptions.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let E be a Banach space, E^* the dual space of E and C a closed convex subsets of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. The equilibrium problems include fixed point problems, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases.

Let E be a smooth Banach space and J the normalized duality mapping from E to E^*. Alber [1] considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E). \tag{1.2}$$

Using this functional, Matsushita and Takahashi [2, 3] studied and investigated the following mappings in Banach spaces. A mapping $S : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(S) \neq \emptyset$,

(R2) $\varphi(p, Sx) \leq \varphi(p, x)$ for all $p \in F(S)$ and $x \in C$,

(R3) $F(S) = \widehat{F}(S)$,

where $F(S)$ and $\widehat{F}(S)$ denote the set of fixed points of S and the set of asymptotic fixed points of S, respectively. It is known that S satisfies condition (R3) if and only if $I - S$ is demiclosed at zero, where I is the identity mapping; that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Sx_n\}$ converges strongly to 0, it follows that $p \in F(S)$. In a Hilbert space H, the duality mapping J is an identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $S : C \to H$ is nonexpansive (i.e., $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$), then it is relatively nonexpansive.

Recently, many authors studied the problems of finding a common element of the set of fixed points for a mapping and the set of solutions of equilibrium problem in the setting of Hilbert space and uniformly smooth and uniformly convex Banach space, respectively (see, e.g., [4–21] and the references therein). In a Hilbert space H, S. Takahashi and W. Takahashi [17] introduced the iteration as follows: sequence $\{x_n\}$ generated by $u, x_1 \in C$,

$$\begin{gathered} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) z_n), \end{gathered} \tag{1.3}$$

for every $n \in \mathbb{N}$, where S is nonexpansive, $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$, and $\{r_n\}$ is an appropriate positive real sequence. They proved that $\{x_n\}$ converges strongly to some element in $F(S) \cap \mathrm{EP}(F)$. In 2009, Takahashi and Zembayashi [19] proposed the iteration in a uniformly smooth and uniformly convex Banach space as follows: a sequence $\{x_n\}$ generated by $u_1 \in E$,

$$\begin{gathered} x_n \in C \text{ such that } F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \end{gathered} \tag{1.4}$$

for every $n \in \mathbb{N}$, S is relatively nonexpansive, $\{\alpha_n\}$ is an appropriate sequence in $[0, 1]$, and $\{r_n\}$ is an appropriate positive real sequence. They proved that if J is weakly sequentially continuous, then $\{x_n\}$ converges *weakly* to some element in $F(S) \cap \mathrm{EP}(F)$.

Motivated by S. Takahashi and W. Takahashi [17] and Takahashi and Zembayashi [19], we prove a strong convergence theorem for finding a common element of the fixed points set of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a uniformly smooth and uniformly convex Banach space.

2. Preliminaries

We collect together some definitions and preliminaries which are needed in this paper. We say that a Banach space E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \qquad x \neq y \text{ imply } \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1-\delta. \tag{2.2}$$

It is known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. We say that E is *uniformly smooth* if the dual space E^* of E is uniformly convex. A Banach space E is *smooth* if the limit $\lim_{t\to 0}((\|x+ty\|-\|x\|)/t)$ exists for all norm one elements x and y in E. It is not hard to show that if E is reflexive, then E is smooth if and only if E^* is strictly convex.

Let E be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ (see [1]) is defined by

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E), \tag{2.3}$$

where the *duality mapping* $J : E \to E^*$ is given by

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2 \quad (x \in E). \tag{2.4}$$

It is obvious from the definition of the function φ that

$$(\|x\| - \|y\|)^2 \leq \varphi(x,y) \leq (\|x\| + \|y\|)^2, \tag{2.5}$$

$$\varphi\Big(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\Big) \leq \lambda\varphi(x,y) + (1-\lambda)\varphi(x,z), \tag{2.6}$$

for all $\lambda \in [0,1]$ and $x, y, z \in E$. The following lemma is an analogue of Xu's inequality [22, Theorem 2] with respect to φ.

Lemma 2.1. *Let E be a uniformly smooth Banach space and $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\varphi\Big(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\Big) \leq \lambda\varphi(x,y) + (1-\lambda)\varphi(x,z) - \lambda(1-\lambda)g(\|Jy - Jz\|), \tag{2.7}$$

for all $\lambda \in [0,1]$, $x \in E$, and $y, z \in B_r$.

It is also easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space E, then $x_n - y_n \to 0$ implies that $\varphi(x_n, y_n) \to 0$.

Lemma 2.2 (see [23, Proposition 2]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

Remark 2.3. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space E, we have

$$\varphi(x_n, y_n) \longrightarrow 0 \iff x_n - y_n \longrightarrow 0 \iff Jx_n - Jy_n \longrightarrow 0. \tag{2.8}$$

Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. It is known that [1, 23] for any $x \in E$, there exists a unique point $\hat{x} \in C$ such that

$$\varphi(\hat{x}, x) = \min_{y \in C} \varphi(y, x). \tag{2.9}$$

Following Alber [1], we denote such an element $\hat{x}$ by $\Pi_C x$. The mapping Π_C is called the *generalized projection* from E onto C. It is easy to see that in a Hilbert space, the mapping Π_C coincides with the metric projection P_C. Concerning the generalized projection, the following are well known.

Lemma 2.4 (see [23, Propositions 4 and 5]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E, $x \in E$, and $\hat{x} \in C$. Then,*

(a) $\hat{x} = \Pi_C x$ *if and only if* $\langle y - \hat{x}, Jx - J\hat{x} \rangle \leq 0$ *for all* $y \in C$,

(b) $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$ *for all* $y \in C$.

Remark 2.5. The generalized projection mapping Π_C above is relatively nonexpansive and $F(\Pi_C) = C$.

Let E be a reflexive, strictly convex and smooth Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^* = J^{-1}$. We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in Alber [1]

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \tag{2.10}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma (see [1] and [24, Lemma 3.2]).

Lemma 2.6. *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in* (2.10). *Then,*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*), \tag{2.11}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.7 (see [25, Lemma 2.1]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \tag{2.12}$$

for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0,1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy conditions: $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Then, $\lim_{n\to\infty}a_n = 0$.

Lemma 2.8 (see [26, Lemma 3.1]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,*

$$a_{m_k} \leq a_{m_k+1}, \qquad a_k \leq a_{m_k+1}, \tag{2.13}$$

for all $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

For solving the equilibrium problem, we usually assume that a bifunction $F : C\times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$,

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$, for all $x, y \in C$,

(A3) for all $x, y, z \in C$, $\limsup_{t\to 0}F(tz + (1-t)x, y) \leq F(x,y)$,

(A4) for all $x \in C$, $F(x,\cdot)$ is convex and lower semicontinuous.

The following lemma gives a characterization of a solution of an equilibrium problem.

Lemma 2.9 (see [19, Lemma 2.8]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and uniformly smooth Banach space E. Let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)–(A4). *For $r > 0$, define a mapping $T_r : E \to C$ so-called the resolvent of F as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \ \forall y \in C\right\}, \tag{2.14}$$

for all $x \in E$. Then, the following hold:

(i) *T_r is single-valued,*

(ii) *T_r is a firmly nonexpansive-type mapping* [27], *that is, for all $x, y \in E$*

$$\langle T_rx - T_ry, JT_rx - JT_ry\rangle \leq \langle T_rx - T_ry, Jx - Jy\rangle, \tag{2.15}$$

(iii) $F(T_r) = \mathrm{EP}(F)$,

(iv) $\mathrm{EP}(F)$ *is closed and convex,*

Lemma 2.10 (see [4, Lemma 2.3]). *Let C be a nonempty closed convex subset of a Banach space E, F a bifunction from $C\times C \to \mathbb{R}$ satisfying conditions* (A1)–(A4) *and $z \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $F(y,z) \leq 0$ for all $y \in C$.*

Remark 2.11 (see [27]). Let C be a nonempty subset of a smooth Banach space E. If $S : C \to E$ is a firmly nonexpansive-type mapping, then

$$\varphi(z, Sx) \leq \varphi(z, Sx) + \varphi(Sx, x) \leq \varphi(z, x), \tag{2.16}$$

for all $x \in C$ and $z \in F(S)$. In particular, S satisfies condition (R2).

Lemma 2.12 (see [3, Proposition 2.4]). *Let C be a nonempty closed convex subset of a strictly convex and smooth Banach space E and $S : C \to E$ a relatively nonexpansive mapping. Then, $F(S)$ is closed and convex.*

3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the fixed points set of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions* (A1)–(A4) *and $S : C \to E$ a relatively nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \neq \emptyset$. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by $u \in C$, $u_1 \in E$ and*

$$\begin{aligned} F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n), \\ u_{n+1} &= J^{-1}(\beta_n Jx_n + (1-\beta_n)JSy_n), \end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{F(S)\cap \mathrm{EP}(F)}u$.

Proof. Note that x_n can be rewritten as $x_n = T_{r_n}u_n$. Since $F(S) \cap \mathrm{EP}(F)$ is nonempty, closed, and convex, we put $\widehat{u} = \Pi_{F(S)\cap \mathrm{EP}(F)}u$. Since Π_C, T_{r_n}, and S satisfy condition (R2), by (2.6), we get

$$\begin{aligned} \varphi(\widehat{u}, y_n) &\leq \varphi\Big(\widehat{u}, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n)\Big) \\ &\leq \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, x_n) \\ &\leq \alpha_n\varphi(\widehat{u}, u) + (1-\alpha_n)\varphi(\widehat{u}, u_n), \end{aligned} \tag{3.2}$$

and so

$$\begin{aligned}\varphi(\widehat{u},u_{n+1}) &\le \beta_n\varphi(\widehat{u},x_n)+(1-\beta_n)\varphi(\widehat{u},Sy_n)\\ &\le \beta_n\varphi(\widehat{u},u_n)+(1-\beta_n)\varphi(\widehat{u},y_n)\\ &\le \alpha_n(1-\beta_n)\varphi(\widehat{u},u)+(1-\alpha_n(1-\beta_n))\varphi(\widehat{u},u_n)\\ &\le \max\{\varphi(\widehat{u},u),\varphi(\widehat{u},u_n)\}.\end{aligned} \tag{3.3}$$

By induction, we have

$$\varphi(z,u_{n+1})\le \max\{\varphi(\widehat{u},u),\varphi(\widehat{u},u_1)\}, \tag{3.4}$$

for all $n\in\mathbb{N}$. This implies that $\{u_n\}$ is bounded and so are $\{x_n\}$, $\{y_n\}$, and $\{Sy_n\}$. Put

$$z_n \equiv J^{-1}(\alpha_n Ju+(1-\alpha_n)Jx_n). \tag{3.5}$$

Then, $y_n\equiv \Pi_C z_n$. Using Lemma 2.6 gives

$$\begin{aligned}\varphi(\widehat{u},y_n) &\le \varphi(\widehat{u},z_n)=V(\widehat{u},Jz_n)\\ &\le V(\widehat{u},Jz_n-\alpha_n(Ju-J\widehat{u}))-2\langle z_n-\widehat{u},-\alpha_n(Ju-J\widehat{u})\rangle\\ &=\varphi\Big(\widehat{u},J^{-1}(\alpha_n J\widehat{u}+(1-\alpha_n)Jx_n)\Big)+2\alpha_n\langle z_n-\widehat{u},Ju-J\widehat{u}\rangle\\ &\le \alpha_n\varphi(\widehat{u},\widehat{u})+(1-\alpha_n)\varphi(\widehat{u},x_n)+2\alpha_n\langle z_n-\widehat{u},Ju-J\widehat{u}\rangle\\ &\le (1-\alpha_n)\varphi(\widehat{u},u_n)+2\alpha_n\langle z_n-\widehat{u},Ju-J\widehat{u}\rangle.\end{aligned} \tag{3.6}$$

Let $g:[0,2r]\to[0,\infty)$ be a function satisfying the properties of Lemma 2.1, where $r=\sup\{\|x_n\|,\|Sy_n\|:n\in\mathbb{N}\}$. Then, by Remark 2.11 and (3.6), we get

$$\begin{aligned}\varphi(\widehat{u},u_{n+1}) &\le \beta_n\varphi(\widehat{u},x_n)+(1-\beta_n)\varphi(\widehat{u},Sy_n)-\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\\ &\le \beta_n(\varphi(\widehat{u},u_n)-\varphi(x_n,u_n))+(1-\beta_n)\varphi(\widehat{u},y_n)\\ &\quad -\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\\ &\le \beta_n\varphi(\widehat{u},u_n)+(1-\beta_n)((1-\alpha_n)\varphi(\widehat{u},u_n)+2\alpha_n\langle z_n-\widehat{u},Ju-J\widehat{u}\rangle)\\ &\quad -\beta_n\varphi(x_n,u_n)-\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\end{aligned} \tag{3.7}$$

$$\begin{aligned} &=(1-\gamma_n)\varphi(\widehat{u},u_n)+2\gamma_n\langle z_n-\widehat{u},Ju-J\widehat{u}\rangle\\ &\quad -\beta_n\varphi(x_n,u_n)-\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\\ &\le (1-\gamma_n)\varphi(\widehat{u},u_n)+2\gamma_n\langle z_n-\widehat{u},Ju-J\widehat{u}\rangle,\end{aligned} \tag{3.8}$$

where $\gamma_n=\alpha_n(1-\beta_n)$ for all $n\in\mathbb{N}$. Notice that $\{\gamma_n\}\subset(0,1)$ satisfying $\lim_{n\to\infty}\gamma_n=0$ and $\sum_{n=1}^{\infty}\gamma_n=\infty$.

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\widehat{u}, u_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(\widehat{u}, u_n)\}$ is then convergent. Then,

$$\varphi(\widehat{u}, u_n) - \varphi(\widehat{u}, u_{n+1}) \longrightarrow 0. \tag{3.9}$$

It follows from (3.7) and $\gamma_n \to 0$ that

$$\beta_n \varphi(x_n, u_n) + \beta_n(1-\beta_n) g(\|Jx_n - JSy_n\|) \longrightarrow 0. \tag{3.10}$$

Since $\{\beta_n\} \subset [a,b] \subset (0,1)$,

$$\varphi(x_n, u_n) \longrightarrow 0, \qquad g(\|Jx_n - JSy_n\|) \longrightarrow 0. \tag{3.11}$$

Consequently, by Remark 2.3,

$$x_n - u_n \longrightarrow 0, \qquad Jx_n - JSy_n \longrightarrow 0, \qquad x_n - Sy_n \longrightarrow 0. \tag{3.12}$$

From (2.6) and $\alpha_n \to 0$, we obtain

$$\varphi(x_n, y_n) \le \varphi(x_n, z_n) \le \alpha_n \varphi(x_n, u) + (1-\alpha_n)\varphi(x_n, x_n) = \alpha_n \varphi(x_n, u) \longrightarrow 0. \tag{3.13}$$

This implies that

$$x_n - y_n \longrightarrow 0, \qquad z_n - y_n \longrightarrow 0. \tag{3.14}$$

Therefore,

$$y_n - Sy_n \longrightarrow 0. \tag{3.15}$$

Since $\{y_n\}$ is bounded and E is reflexive, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z$ and

$$\limsup_{n\to\infty} \langle y_n - \widehat{u}, Ju - J\widehat{u} \rangle = \lim_{i\to\infty} \langle y_{n_i} - \widehat{u}, Ju - J\widehat{u} \rangle. \tag{3.16}$$

Then, $x_{n_i} \rightharpoonup z$. Since $x_n - u_n \to 0$ and $r_n \ge c > 0$, by Remark 2.3,

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Ju_n\| = 0. \tag{3.17}$$

Notice that

$$F(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \ge 0, \quad \forall y \in C. \tag{3.18}$$

Replacing n by n_i, we have from (A2) that

$$\frac{1}{r_{n_i}}\langle y - x_{n_i}, Jx_{n_i} - Ju_{n_i}\rangle \geq -F(x_{n_i}, y) \geq F(y, x_{n_i}), \quad \forall y \in C. \tag{3.19}$$

Letting $i \to \infty$, we have from (3.17) and (A4) that

$$F(y, z) \leq 0, \quad \forall y \in C. \tag{3.20}$$

From Lemma 2.10, we have $z \in \mathrm{EP}(F)$. Since S satisfies condition (R3) and (3.15), $z \in F(S)$. It follows that $z \in F(S) \cap \mathrm{EP}(F)$. By Lemma 2.4(a), we immediately obtain that

$$\limsup_{n\to\infty} \langle y_n - \hat{u}, Ju - J\hat{u}\rangle = \langle z - \hat{u}, Ju - J\hat{u}\rangle \leq 0. \tag{3.21}$$

Since $z_n - y_n \to 0$,

$$\limsup_{n\to\infty} \langle z_n - \hat{u}, Ju - J\hat{u}\rangle \leq 0. \tag{3.22}$$

It follows from Lemma 2.7 and (3.8) that $\varphi(\hat{u}, u_n) \to 0$. Then, $u_n \to \hat{u}$ and so $x_n \to \hat{u}$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\hat{u}, u_{n_i}) < \varphi(\hat{u}, u_{n_i+1}), \tag{3.23}$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\varphi(\hat{u}, u_{m_k}) \leq \varphi(\hat{u}, u_{m_k+1}), \qquad \varphi(\hat{u}, u_k) \leq \varphi(\hat{u}, u_{m_k+1}) \tag{3.24}$$

for all $k \in \mathbb{N}$. From (3.7) and $\gamma_n \to 0$, we have

$$\begin{aligned}
&\beta_{m_k}\varphi(x_{m_k}, u_{m_k}) + \beta_{m_k}(1-\beta_{m_k})g(\|Jx_{m_k} - JSy_{m_k}\|) \\
&\quad\leq (\varphi(\hat{u}, u_{m_k}) - \varphi(\hat{u}, u_{m_k+1})) - \gamma_{m_k}\varphi(\hat{u}, u_{m_k}) + 2\gamma_{m_k}\langle z_{m_k} - \hat{u}, Ju - J\hat{u}\rangle \\
&\quad\leq -\gamma_{m_k}\varphi(\hat{u}, u_{m_k}) + 2\gamma_{m_k}\langle z_{m_k} - \hat{u}, Ju - J\hat{u}\rangle \longrightarrow 0.
\end{aligned} \tag{3.25}$$

Using the same proof of Case 1, we also obtain

$$\limsup_{k\to\infty} \langle z_{m_k} - \hat{u}, Ju - J\hat{u}\rangle \leq 0. \tag{3.26}$$

From (3.8), we have

$$\varphi(\hat{u}, u_{m_k+1}) \leq (1-\gamma_{m_k})\varphi(\hat{u}, u_{m_k}) + 2\gamma_{m_k}\langle z_{m_k} - \hat{u}, Ju - J\hat{u}\rangle. \tag{3.27}$$

Since $\varphi(\widehat{u}, u_{m_k}) \leq \varphi(\widehat{u}, u_{m_k+1})$, we have

$$\begin{aligned} \gamma_{m_k}\varphi(\widehat{u}, u_{m_k}) &\leq \varphi(\widehat{u}, u_{m_k}) - \varphi(\widehat{u}, u_{m_k+1}) + 2\gamma_{m_k}\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \\ &\leq 2\gamma_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle. \end{aligned} \tag{3.28}$$

In particular, since $\gamma_{m_k} > 0$, we get

$$\varphi(\widehat{u}, u_{m_k}) \leq 2\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle. \tag{3.29}$$

It follows from (3.26) that $\varphi(\widehat{u}, u_{m_k}) \to 0$. This together with (3.27) gives

$$\varphi(\widehat{u}, u_{m_k+1}) \longrightarrow 0. \tag{3.30}$$

But $\varphi(\widehat{u}, u_k) \leq \varphi(\widehat{u}, u_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $u_k \to \widehat{u}$, and $x_k \to \widehat{u}$.

From two cases, we can conclude that $\{u_n\}$ and $\{x_n\}$ converge strongly to $\widehat{u}$ and the proof is finished. □

Applying Theorem 3.1 and [28, Theorem 3.2], we have the following result.

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $\{T_i : C \to E\}_{i=1}^{\infty}$ a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{EP}(F) \neq \emptyset$. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by* (3.1)*, where $S : C \to E$ is defined by*

$$Sx = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i J T_i x\right) \quad \text{for each } x \in C. \tag{3.31}$$

Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{\bigcap_{i=1}^{\infty} F(T_i)\cap \mathrm{EP}(F)} u$.

Setting $F \equiv 0$ and $r_n \equiv 1$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $S : C \to E$ a relatively nonexpansive mapping. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by $u \in C$, $u_1 \in E$ and*

$$\begin{aligned} x_n &= \Pi_C u_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n J u + (1-\alpha_n) J x_n), \\ u_{n+1} &= J^{-1}(\beta_n J x_n + (1-\beta_n) J S y_n), \end{aligned} \tag{3.32}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$. Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{F(S)} u$.

Letting $S : C \to C$ in Corollary 3.3, we have the following result.

Corollary 3.4. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $S : C \to C$ a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in C defined by $u \in C, x_1 \in C$ and*

$$
\begin{aligned}
y_n &= \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n),\\
x_{n+1} &= J^{-1}(\beta_n Jx_n + (1-\beta_n)JSy_n),
\end{aligned}
\tag{3.33}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(S)}u$.

Let S be the identity mapping in Theorem 3.1, we also have the following result.

Corollary 3.5. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and $F : C\times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4) such that $\mathrm{EP}(F) \neq \varnothing$. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by $u \in C, u_1 \in E$ and*

$$
\begin{aligned}
F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle &\geq 0, \quad \forall y \in C,\\
y_n &= \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n),\\
u_{n+1} &= J^{-1}(\beta_n Jx_n + (1-\beta_n)Jy_n),
\end{aligned}
\tag{3.34}
$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{\mathrm{EP}(F)}u$.

4. Deduced Theorems in Hilbert Spaces

In Hilbert spaces, every nonexpansive mappings are relatively nonexpansive, and J is the identity operator. We obtain the following result.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H, $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $S : C \to H$ a nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \neq \varnothing$. Let $\{x_n\}$ be a sequence in C defined by $u \in C, x_1 \in H$ and*

$$
x_{n+1} = \beta_n T_{r_n} x_n + (1-\beta_n)S(\alpha_n u + (1-\alpha_n)T_{r_n}x_n),
\tag{4.1}
$$

for all $n \in \mathbb{N}$, where T_{r_n} is the resolvent of F, $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{EP}(F)}u$.

Remark 4.2. In Theorem 4.1, we have the same conclusion if the mapping $S : C \to H$ is only quasinonexpansive (i.e., $F(S) \neq \varnothing$ and $\|p - Sx\| \leq \|p - x\|$ for all $x \in C$ and $p \in F(S)$) such that $I - T$ is demiclosed at zero.

Letting $F \equiv 0$ in Theorem 4.1, we have the following result.

Corollary 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H and $S : C \to H$ a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by $u \in C, x_1 \in H$ and*

$$x_{n+1} = \beta_n P_C x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) P_C x_n), \tag{4.2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\beta_n\} \subset [a,b] \subset (0,1)$. Then, $\{x_n\}$ converges strongly to $P_{F(S)}u$.

Let S be the identity mapping in Theorem 4.1, we have the following result.

Corollary 4.4. *Let C be a nonempty closed convex subset of a Hilbert space H and $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4). Let $\{x_n\}$ be a sequence in H defined by $u, x_1 \in H$ and*

$$x_{n+1} = \gamma_n u + (1 - \gamma_n) T_{r_n} x_n, \tag{4.3}$$

for all $n \in \mathbb{N}$, where T_{r_n} is the resolvent of F, $\{\gamma_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{EP}(F)}u$.

Proof. We may assume without loss of generality that $\gamma_n < 1/2$ for all $n \in \mathbb{N}$. Setting $\alpha_n = 2\gamma_n$ and $\beta_n = 1/2$ for all $n \in \mathbb{N}$, we get

$$x_{n+1} = \frac{1}{2} T_{r_n} x_n + \frac{1}{2} I(\alpha_n u + (1 - \alpha_n) T_{r_n} x_n), \tag{4.4}$$

$\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Applying Theorem 4.1, $\{x_n\}$ converges strongly to $P_{\mathrm{EP}(F)}u$. □

Remark 4.5. Corollary 4.4 improves and extends [29, Corollary 5.3]. More precisely, the conditions $\lim_{n\to\infty}(\gamma_{n+1}/\gamma_n) = 1$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ are removed.

Applying Corollary 4.4 and [30, Theorem 8], we have the following result.

Corollary 4.6. *Let C be a nonempty closed convex subset of a Hilbert space H, $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $f : C \to C$ a contraction of H into itself. Let $\{x_n\}$ be a sequence in H defined by u, $x_1 \in H$ and*

$$x_{n+1} = \gamma_n f(x_n) + (1 - \gamma_n) T_{r_n} x_n, \tag{4.5}$$

for all $n \in \mathbb{N}$, where T_{r_n} is the resolvent of F, $\{\gamma_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{x_n\}$ converges strongly to $z = P_{\mathrm{EP}(F)} f(z)$.

Remark 4.7. Corollary 4.6 improves and extends [16, Corollary 3.4]. More precisely, the conditions $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ are removed.

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