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Research Article

Hybrid Proximal-Type Algorithms for Generalized Equilibrium Problems, Maximal Monotone Operators, and Relatively Nonexpansive Mappings

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The purpose of this paper is to introduce and consider new hybrid proximal-type algorithms for finding a common element of the set EP of solutions of a generalized equilibrium problem, the set $F(S)$ of fixed points of a relatively nonexpansive mapping S, and the set $T^{-1}0$ of zeros of a maximal monotone operator T in a uniformly smooth and uniformly convex Banach space. Strong convergence theorems for these hybrid proximal-type algorithms are established; that is, under appropriate conditions, the sequences generated by these various algorithms converge strongly to the same point in $\mathrm{EP} \cap F(S) \cap T^{-1}0$. These new results represent the improvement, generalization, and development of the previously known ones in the literature.

1. Introduction

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of positive integers and real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that if E is smooth, then J is single valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We will still denote by J the single valued duality mapping. Let

$f : C \times C \to \mathcal{R}$ be a bifunction and $A : C \to E^*$ be a nonlinear mapping. We consider the following generalized equilibrium problem:

$$\text{find } u \in C \text{ such that } f(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of such $u \in C$ is denoted by EP, that is,

$$\text{EP} = \{u \in C : f(u, y) + \langle Au, y - u \rangle \geq 0, \ \forall y \in C\}. \tag{1.3}$$

Whenever $E = H$ a Hilbert space, problem (1.2) was introduced and studied by S. Takahashi and W. Takahashi [1]. Similar problems have been studied extensively recently. See, for example, [2–11]. In the case of $A \equiv 0$, EP is denoted by EP(f). In the case of $f \equiv 0$, EP is also denoted by VI(C, A). The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for example, [12–14]. A mapping $S : C \to E$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(S)$ the set of fixed points of S, that is, $F(S) = \{x \in C : Sx = x\}$. A mapping $A : C \to E^*$ is called α-inverse-strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.4}$$

It is easy to see that if $A : C \to E^*$ is an α-inverse-strongly monotone mapping, then it is $1/\alpha$-Lipschitzian.

Let E be a real Banach space with the dual E^*. A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ is called monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator T is called maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. A method for solving the inclusion $0 \in Tx$ is the proximal point algorithm. Denote by I the identity operator on $E = H$ a Hilbert space. The proximal point algorithm generates, for any initial point $x_0 = x \in H$, a sequence $\{x_n\}$ in H, by the iterative scheme

$$x_{n+1} = (I + r_n T)^{-1} x_n, \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where $\{r_n\}$ is a sequence in the interval $(0, \infty)$. Note that this iteration is equivalent to

$$0 \in Tx_{n+1} + \frac{1}{r_n}(x_{n+1} - x_n), \quad n = 0, 1, 2, \ldots. \tag{1.6}$$

This algorithm was first introduced by Martinet [12] and generally studied by Rockafellar [15] in the framework of a Hilbert space. Later many authors studied its convergence in a Hilbert space or a Banach space. See, for instance, [16–21] and the references therein.

Let E be a reflexive, strictly convex, and smooth Banach space with the dual E^* and C be a nonempty closed convex subset of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator with domain $D(T) = C$ and $S : C \to C$ be a relatively nonexpansive mapping. Let $A : C \to X^*$ be an α-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ be a bifunction satisfying

(A1)–(A4): (A1) $f(x,x) = 0$, $\forall x \in C$; (A2) f is monotone, that is, $f(x,y) + f(y,x) \le 0$, $\forall x,y \in C$; (A3) $\limsup_{t\downarrow 0} f(x + t(z - x), y) \le f(x,y)$, $\forall x,y,z \in C$; (A4) the function $y \mapsto f(x,y)$ is convex and lower semicontinuous. The purpose of this paper is to introduce and investigate two new hybrid proximal-type Algorithms 1.1 and 1.2 for finding an element of $\mathrm{EP} \cap F(S) \cap T^{-1}0$.

Algorithm 1.1.

$$
\begin{gathered}
x_0 \in C \text{ arbitrarily chosen}, \\
0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\
z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\
y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\
u_n \in C \text{ such that} \\
f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\
H_n = \{v \in C : \phi(v,u_n) \le \alpha_n\phi(v,\widetilde{x}_n) + (1-\alpha_n)\phi(v,z_n),\ \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\
W_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\
x_{n+1} = \Pi_{H_n\cap W_n}x_0, \quad n = 0,1,2,\ldots,
\end{gathered}
\tag{1.7}
$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0,\infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

Algorithm 1.2.

$$
\begin{gathered}
x_0 \in C \text{ arbitrarily chosen}, \\
0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\
y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n), \\
u_n \in C \text{ such that} \\
f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\
H_n = \{v \in C : \phi(v,u_n) \le \alpha_n\phi(v,x_0) + (1-\alpha_n)\phi(v,\widetilde{x}_n),\ \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\
W_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\
x_{n+1} = \Pi_{H_n\cap W_n}x_0, \quad n = 0,1,2,\ldots,
\end{gathered}
\tag{1.8}
$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0,\infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.

In this paper, strong convergence results on these two hybrid proximal-type algorithms are established; that is, under appropriate conditions, the sequence $\{x_n\}$ generated by Algorithm 1.1 and the sequence $\{x_n\}$ generated by Algorithm 1.2, converge strongly to the same point $\Pi_{EP\cap F(S)\cap T^{-1}0}x_0$. These new results represent the improvement, generalization and development of the previously known ones in the literature including Solodov and Svaiter [22], Kamimura and Takahashi [23], Qin and Su [24], and Ceng et al. [25].

Throughout this paper the symbol $\rightharpoonup$ stands for weak convergence and $\to$ stands for strong convergence.

2. Preliminaries

Let E be a real Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A Banach space E is called strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $x_n - y_n \to 0$ for any two sequences $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E, then the Banach space E is called smooth if

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in U$. If E is smooth, then J is single valued. We still denote the single valued duality mapping by J.

It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. A Banach space E is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, whenever $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [26, 27] for more details.

Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. Nevertheless, Alber [28] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as in [28, 29] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.3}$$

It is clear that in a Hilbert space H, (2.3) reduces to $\phi(x, y) = \|x - y\|^2$, for all $x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.4}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [30]). In a Hilbert space H, $\Pi_C = P_C$. From [28], in uniformly smooth and uniformly convex Banach spaces, we have

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{2.5}$$

Let C be a nonempty closed convex subset of E, and let S be a mapping from C into itself. A point $p \in C$ is called an asymptotically fixed point of S [31] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Sx_n - x_n \to 0$. The set of asymptotical fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is called relatively nonexpansive [32–34] if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$.

We remark that if E is a reflexive, strictly convex and smooth Banach space, then for any $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.5), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [26, 27] for more details.

We need the following lemmas for the proof of our main results.

Lemma 2.1 (see [23]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2 (see [23, 28]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $x \in E$ and let $z \in C$, then*

$$z = \Pi_C x \Longleftrightarrow \langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C. \tag{2.6}$$

Lemma 2.3 (see [23, 28]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C, \ y \in E. \tag{2.7}$$

Lemma 2.4 (see [35]). *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space E, and let $S : C \to C$ be a relatively nonexpansive mapping, then $F(S)$ is closed and convex.*

The following result is according to Blum and Oettli [36].

Lemma 2.5 (see [36]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$, then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.8}$$

Motivated by Combettes and Hirstoaga [37] in a Hilbert space, Takahashi and Zembayashi [38] established the following lemma.

Lemma 2.6 (see [38]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\} \tag{2.9}$$

for all $x \in E$, then, the following hold:

(i) *T_r is single valued;*

(ii) *T_r is a firmly nonexpansive-type mapping, that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.10}$$

(iii) $F(T_r) = \widehat{F}(T_r) = \mathrm{EP}(f)$;

(iv) $\mathrm{EP}(f)$ *is closed and convex.*

Using Lemma 2.6, one has the following result.

Lemma 2.7 (see [38]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)–(A4), and let $r > 0$, then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.11}$$

Utilizing Lemmas 2.5, 2.6 and 2.7 as above, Chang [39] derived the following result.

Proposition 2.8 (see [39, Lemma 2.5]). *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping, let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)–(A4), and let $r > 0$, then there hold the following:*

(I) *for* $x \in E$*, there exists* $u \in C$ *such that*

$$f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C; \tag{2.12}$$

(II) *if* E *is additionally uniformly smooth and* $K_r : E \to C$ *is defined as*

$$K_r(x) = \left\{u \in C : f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\right\}, \quad \forall x \in E, \tag{2.13}$$

then the mapping K_r *has the following properties:*

(i) K_r *is single valued,*

(ii) K_r *is a firmly nonexpansive-type mapping, that is,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle, \quad \forall x, y \in E, \tag{2.14}$$

(iii) $F(K_r) = \widehat{F}(K_r) = \mathrm{EP}$,

(iv) EP *is a closed convex subset of* C,

(v) $\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x)$, *for all* $p \in F(K_r)$.

Proof. Define a bifunction $F : C \times C \to \mathcal{R}$ as follows:

$$F(x, y) = f(x, y) + \langle Ax, y - x\rangle, \quad \forall x, y \in C. \tag{2.15}$$

Then it is easy to verify that F satisfies the conditions (A1)–(A4). Therefore, The conclusions (I) and (II) of Proposition 2.8 follow immediately from Lemmas 2.5, 2.6 and 2.7. □

Lemma 2.9 (see [13, 14]). *Let* E *be a reflexive, strictly convex and smooth Banach space, and let* $T : E \to 2^{E^*}$ *be a maximal monotone operator with* $T^{-1}0 \neq \emptyset$*, then,*

$$\phi(z, J_r x) + \phi(J_r x, x) \leq \phi(z, x), \quad \forall r > 0,\ z \in T^{-1}0,\ x \in E. \tag{2.16}$$

3. Main Results

Throughout this section, unless otherwise stated, we assume that $T : E \to 2^{E^*}$ is a maximal monotone operator with domain $D(T) = C$, $S : C \to C$ is a relatively nonexpansive mapping, $A : C \to E^*$ is an α-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ is a bifunction satisfying (A1)–(A4), where C is a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. In this section, we study the following algorithm.

Algorithm 3.1.

$$
\begin{aligned}
&x_0 \in C \text{ arbitrarily chosen,}\\
&0 = v_n + \frac{1}{r_n}(J\tilde{x}_n - Jx_n), \quad v_n \in T\tilde{x}_n,\\
&z_n = J^{-1}(\beta_n J\tilde{x}_n + (1-\beta_n)JS\tilde{x}_n),\\
&y_n = J^{-1}(\alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n),\\
&u_n \in C \text{ such that}\\
&f(u_n, y) + \langle Au_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C,\\
&H_n = \{v \in C : \phi(v,u_n) \le \alpha_n\phi(v,\tilde{x}_n) + (1-\alpha_n)\phi(v,z_n),\ \langle v-\tilde{x}_n, v_n\rangle \le 0\},\\
&W_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\},\\
&x_{n+1} = \Pi_{H_n\cap W_n}x_0, \quad n = 0,1,2,\ldots,
\end{aligned}
\tag{3.1}
$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0,\infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

First we investigate the condition under which the Algorithm 3.1 is well defined. Rockafellar [40] proved the following result.

Lemma 3.2 (Rockafellar [40]). *Let E be a reflexive, strictly convex, and smooth Banach space and let $T : E \to 2^{E^*}$ be a multivalued operator, then there hold the following:*

(i) *$T^{-1}0$ is closed and convex if T is maximal monotone such that $T^{-1}0 \ne \emptyset$;*

(ii) *T is maximal monotone if and only if T is monotone with $R(J+rT) = E^*$ for all $r > 0$.*

Utilizing this result, we can show the following lemma.

Lemma 3.3. *Let E be a reflexive, strictly convex, and smooth Banach space. If $\mathrm{EP} \cap F(S) \cap T^{-1}0 \ne \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3.1 is well defined.*

Proof. For each $n \ge 0$, define two sets C_n and D_n as follows:

$$
\begin{aligned}
C_n &= \{v \in C : \phi(v,u_n) \le \alpha_n\phi(v,\tilde{x}_n) + (1-\alpha_n)\phi(v,z_n)\},\\
D_n &= \{v \in C : \langle v-\tilde{x}_n, v_n\rangle \le 0\}.
\end{aligned}
\tag{3.2}
$$

It is obvious that C_n is closed and D_n, W_n are closed convex sets for each $n \ge 0$. Let us show that C_n is convex. For $v_1, v_2 \in C_n$ and $t \in (0,1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. Indeed, observe that

$$
\phi(v,u_n) \le \alpha_n\phi(v,\tilde{x}_n) + (1-\alpha_n)\phi(v,z_n)
\tag{3.3}
$$

is equivalent to

$$2\alpha_n\langle v, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Ju_n\rangle \le \alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|u_n\|^2. \tag{3.4}$$

Note that there hold the following:

$$\begin{aligned}
\phi(v, u_n) &= \|v\|^2 - 2\langle v, Ju_n\rangle + \|u_n\|^2,\\
\phi(v, \widetilde{x}_n) &= \|v\|^2 - 2\langle v, J\widetilde{x}_n\rangle + \|\widetilde{x}_n\|^2,\\
\phi(v, z_n) &= \|v\|^2 - 2\langle v, Jz_n\rangle + \|z_n\|^2,
\end{aligned} \tag{3.5}$$

Thus we have

$$\begin{aligned}
&2\alpha_n\langle v, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Ju_n\rangle\\
&\quad= 2\alpha_n\langle tv_1 + (1-t)v_2, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle tv_1 + (1-t)v_2, Jz_n\rangle\\
&\qquad - 2\langle tv_1 + (1-t)v_2, Ju_n\rangle\\
&\quad= 2t\alpha_n\langle v_1, J\widetilde{x}_n\rangle + 2(1-t)\alpha_n\langle v_2, J\widetilde{x}_n\rangle + 2(1-\alpha_n)t\langle v_1, Jz_n\rangle\\
&\qquad + 2(1-\alpha_n)(1-t)\langle v_2, Jz_n\rangle - 2t\langle v_1, Ju_n\rangle - 2(1-t)\langle v_2, Ju_n\rangle\\
&\quad\le \alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|u_n\|^2.
\end{aligned} \tag{3.6}$$

This implies that $v \in C_n$. Therefore, C_n is convex and hence $H_n = C_n \cap D_n$ is closed and convex.

On the other hand, let $w \in \mathrm{EP} \cap F(S) \cap T^{-1}0$ be arbitrarily chosen, then $w \in \mathrm{EP}, w \in F(S)$ and $w \in T^{-1}0$. From Algorithm 3.1, it follows that

$$\begin{aligned}
\phi(w, u_n) = \phi(w, K_{r_n}y_n) &\le \phi(w, y_n)\\
&= \phi\Big(w, J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n)\Big)\\
&= \|w\|^2 - 2\langle w, \alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n\rangle + \|\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n\|^2\\
&\le \|w\|^2 - 2\alpha_n\langle w, J\widetilde{x}_n\rangle - 2(1-\alpha_n)\langle w, JSz_n\rangle + \alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|Sz_n\|^2\\
&\le \alpha_n\phi(w, \widetilde{x}_n) + (1-\alpha_n)\phi(w, Sz_n)\\
&\le \alpha_n\phi(w, \widetilde{x}_n) + (1-\alpha_n)\phi(w, z_n).
\end{aligned} \tag{3.7}$$

So $w \in C_n$ for all $n \ge 0$. Now, from Lemma 3.2 it follows that there exists $(\widetilde{x}_0, v_0) \in E \times E^*$ such that $0 = v_0 + (1/r_0)(J\widetilde{x}_0 - Jx_0)$ and $v_0 \in T\widetilde{x}_0$. Since T is monotone, it follows that $\langle \widetilde{x}_0 - w, v_0\rangle \ge 0$, which implies that $w \in D_0$ and hence $w \in H_0$. Furthermore, it is clear that $w \in W_0 = C$, then $w \in H_0 \cap W_0$, and therefore $x_1 = \Pi_{H_0 \cap W_0}x_0$ is well defined. Suppose that $w \in H_{n-1} \cap W_{n-1}$ and x_n is well defined for some $n \ge 1$. Again by Lemma 3.2, we deduce that $(\widetilde{x}_n, v_n) \in E \times E^*$ such that $0 = v_n + (1/r_n)(J\widetilde{x}_n - Jx_n)$ and $v_n \in T\widetilde{x}_n$, then from the monotonicity of T we conclude that $\langle \tilde{x}_n - w, v_n \rangle \geq 0$, which implies that $w \in D_n$ and hence $w \in H_n$. It follows from Lemma 2.4 that

$$\langle w - x_n, Jx_0 - Jx_n \rangle = \langle w - \Pi_{H_{n-1} \cap W_{n-1}} x_0, Jx_0 - J\Pi_{H_{n-1} \cap W_{n-1}} x_0 \rangle \leq 0, \tag{3.8}$$

which implies that $w \in W_n$. Consequently, $w \in H_n \cap W_n$ and so $\mathrm{EP} \cap F(S) \cap T^{-1}0 \subset H_n \cap W_n$. Therefore $x_{n+1} = \Pi_{H_n \cap W_n} x_0$ is well defined, then, by induction, the sequence $\{x_n\}$ generated by Algorithm 3.1, is well defined for each integer $n \geq 0$. □

Remark 3.4. From the above proof, we obtain that

$$\mathrm{EP} \cap F(S) \cap T^{-1}0 \subset H_n \cap W_n \tag{3.9}$$

for each integer $n \geq 0$.

We are now in a position to prove the main theorem.

Theorem 3.5. *Let E be a uniformly smooth and uniformly convex Banach space. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that*

$$\liminf_{n \to \infty} r_n > 0, \qquad \limsup_{n \to \infty} \alpha_n < 1, \qquad \lim_{n \to \infty} \beta_n = 1. \tag{3.10}$$

Let $\mathrm{EP} \cap F(S) \cap T^{-1}0 \neq \emptyset$. If S is uniformly continuous, then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $\Pi_{\mathrm{EP} \cap F(S) \cap T^{-1}0} x_0$.

Proof. First of all, if follows from the definition of W_n that $x_n = \Pi_{W_n} x_0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in W_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{3.11}$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. Also from $x_n = \Pi_{W_n} x_0$ and Lemma 2.3, we have that

$$\phi(x_n, x_0) = \phi(\Pi_{W_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0) \tag{3.12}$$

for each $w \in \mathrm{EP} \cap F(S) \cap T^{-1}0 \subset W_n$ and for each $n \geq 0$. Consequently, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, according to the inequality

$$(\|x_n\| - \|x_0\|)^2 \leq \phi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2, \tag{3.13}$$

we conclude that $\{x_n\}$ is bounded. Thus, we have that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. From Lemma 2.3, we derive the following:

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{W_n}x_0)\\ &\le \phi(x_{n+1}, x_0) - \phi(\Pi_{W_n}x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),\end{aligned} \tag{3.14}$$

for all $n \ge 0$. This implies that $\phi(x_{n+1}, x_n) \to 0$. So it follows from Lemma 2.1 that $x_{n+1} - x_n \to 0$. Since $x_{n+1} = \Pi_{H_n\cap W_n}x_0 \in H_n$, from the definition of H_n, we also have

$$\phi(x_{n+1}, u_n) \le \alpha_n\phi(x_{n+1}, \tilde{x}_n) + (1-\alpha_n)\phi(x_{n+1}, z_n), \quad \langle x_{n+1} - \tilde{x}_n, v_n\rangle \le 0. \tag{3.15}$$

Observe that

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \phi\Big(x_{n+1}, J^{-1}(\beta_n J\tilde{x}_n + (1-\beta_n)JS\tilde{x}_n)\Big)\\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J\tilde{x}_n + (1-\beta_n)JS\tilde{x}_n\rangle + \big\|\beta_n J\tilde{x}_n + (1-\beta_n)JS\tilde{x}_n\big\|^2\\ &\le \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, J\tilde{x}_n\rangle - 2(1-\beta_n)\langle x_{n+1}, JS\tilde{x}_n\rangle + \beta_n\|\tilde{x}_n\|^2 + (1-\beta_n)\|S\tilde{x}_n\|^2\\ &= \beta_n\phi(x_{n+1}, \tilde{x}_n) + (1-\beta_n)\phi(x_{n+1}, S\tilde{x}_n).\end{aligned} \tag{3.16}$$

At the same time,

$$\begin{aligned}\phi(\Pi_{H_n}x_n, x_n) - \phi(\tilde{x}_n, x_n) &= \|\Pi_{H_n}x_n\|^2 - \|\tilde{x}_n\|^2 + 2\langle\tilde{x}_n - \Pi_{H_n}x_n, Jx_n\rangle\\ &\ge 2\langle\Pi_{H_n}x_n - \tilde{x}_n, J\tilde{x}_n\rangle + 2\langle\tilde{x}_n - \Pi_{H_n}x_n, Jx_n\rangle\\ &= 2\langle\tilde{x}_n - \Pi_{H_n}x_n, Jx_n - J\tilde{x}_n\rangle.\end{aligned} \tag{3.17}$$

Since $\Pi_{H_n}x_n \in H_n$ and $v_n = (1/r_n)(Jx_n - J\tilde{x}_n)$, it follows that

$$\langle\tilde{x}_n - \Pi_{H_n}x_n, Jx_n - J\tilde{x}_n\rangle = r_n\langle\tilde{x}_n - \Pi_{H_n}x_n, v_n\rangle \ge 0 \tag{3.18}$$

and hence that $\phi(\Pi_{H_n}x_n, x_n) \ge \phi(\tilde{x}_n, x_n)$. Further, from $x_{n+1} \in H_n$, we have $\phi(x_{n+1}, x_n) \ge \phi(\Pi_{H_n}x_n, x_n)$, which yields

$$\phi(x_{n+1}, x_n) \ge \phi(\Pi_{H_n}x_n, x_n) \ge \phi(\tilde{x}_n, x_n). \tag{3.19}$$

Then it follows from $\phi(x_{n+1}, x_n) \to 0$ that $\phi(\tilde{x}_n, x_n) \to 0$. Hence it follows from Lemma 2.1 that $\tilde{x}_n - x_n \to 0$. Since from (3.15) we derive

$$\begin{aligned}
&\phi(x_{n+1}, \tilde{x}_n) - \phi(\tilde{x}_n, x_n) \\
&\quad = \|x_{n+1}\|^2 - 2\langle x_{n+1}, J\tilde{x}_n\rangle + \|\tilde{x}_n\|^2 - \left(\|\tilde{x}_n\|^2 - 2\langle \tilde{x}_n, Jx_n\rangle + \|x_n\|^2\right) \\
&\quad = \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1}, J\tilde{x}_n\rangle + 2\langle \tilde{x}_n, Jx_n\rangle \\
&\quad = \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1} - \tilde{x}_n, J\tilde{x}_n - Jx_n\rangle \\
&\qquad - 2\langle x_{n+1} - \tilde{x}_n, Jx_n\rangle + 2\langle \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle \\
&\quad = (\|x_{n+1}\| - \|x_n\|)(\|x_{n+1}\| + \|x_n\|) + 2r_n\langle x_{n+1} - \tilde{x}_n, v_n\rangle - 2\langle x_{n+1} - \tilde{x}_n, Jx_n\rangle \\
&\qquad + 2\langle \tilde{x}_n, Jx_n - J\tilde{x}_n\rangle \\
&\quad \le \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) + 2\|x_{n+1} - \tilde{x}_n\|\|x_n\| + 2\|\tilde{x}_n\|\|Jx_n - J\tilde{x}_n\| \\
&\quad \le \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) + 2(\|x_{n+1} - x_n\| + \|x_n - \tilde{x}_n\|)\|x_n\| + 2\|\tilde{x}_n\|\|Jx_n - J\tilde{x}_n\|,
\end{aligned} \tag{3.20}$$

we have

$$\begin{aligned}
\phi(x_{n+1}, \tilde{x}_n) \le \phi(\tilde{x}_n, x_n) &+ \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) \\
&+ 2(\|x_{n+1} - x_n\| + \|x_n - \tilde{x}_n\|)\|x_n\| + 2\|\tilde{x}_n\|\|Jx_n - J\tilde{x}_n\|.
\end{aligned} \tag{3.21}$$

Thus, from $\phi(\tilde{x}_n, x_n) \to 0$, $x_n - \tilde{x}_n \to 0$, and $x_{n+1} - x_n \to 0$, we know that $\phi(x_{n+1}, \tilde{x}_n) \to 0$. Consequently from (3.16), $\phi(x_{n+1}, \tilde{x}_n) \to 0$, and $\beta_n \to 1$ it follows that

$$\phi(x_{n+1}, z_n) \longrightarrow 0. \tag{3.22}$$

So it follows from (3.15), $\phi(x_{n+1}, \tilde{x}_n) \to 0$, and $\phi(x_{n+1}, z_n) \to 0$ that $\phi(x_{n+1}, u_n) \to 0$. Utilizing Lemma 2.1 we deduce that

$$\lim_{n\to\infty}\|x_{n+1} - u_n\| = \lim_{n\to\infty}\|x_{n+1} - \tilde{x}_n\| = \lim_{n\to\infty}\|x_{n+1} - z_n\| = 0. \tag{3.23}$$

Furthermore, for $u \in \mathrm{EP} \cap F(S) \cap T^{-1}0$ arbitrarily fixed, it follows from Proposition 2.8 that

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \le \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\
&= \phi\Big(u, J^{-1}(\alpha_n J\tilde{x}_n + (1-\alpha_n) JSz_n)\Big) - \phi(u, u_n) \\
&= \|u\|^2 - 2\langle u, \alpha_n J\tilde{x}_n + (1-\alpha_n) JSz_n\rangle + \|\alpha_n J\tilde{x}_n + (1-\alpha_n) JSz_n\|^2 - \phi(u, u_n) \\
&\le \|u\|^2 - 2\alpha_n \langle u, J\tilde{x}_n\rangle - 2(1-\alpha_n)\langle u, JSz_n\rangle + \alpha_n \|\tilde{x}_n\|^2 + (1-\alpha_n)\|Sz_n\|^2 - \phi(u, u_n) \\
&= \alpha_n \phi(u, \tilde{x}_n) + (1-\alpha_n)\phi(u, Sz_n) - \phi(u, u_n) \\
&\le (1-\alpha_n)\phi(u, z_n) + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= (1-\alpha_n)\phi\Big(u, J^{-1}(\beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n)\Big) + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= (1-\alpha_n)\Big[\|u\|^2 - 2\langle u, \beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n\rangle + \big\|\beta_n J\tilde{x}_n + (1-\beta_n) JS\tilde{x}_n\big\|^2\Big] \\
&\quad + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&\le (1-\alpha_n)\Big[\|u\|^2 - 2\beta_n \langle u, J\tilde{x}_n\rangle - 2(1-\beta_n)\langle u, JS\tilde{x}_n\rangle + \beta_n \|\tilde{x}_n\|^2 + (1-\beta_n)\|S\tilde{x}_n\|^2\Big] \\
&\quad + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= (1-\alpha_n)[\beta_n \phi(u, \tilde{x}_n) + (1-\beta_n)\phi(u, S\tilde{x}_n)] + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&\le (1-\alpha_n)[\beta_n \phi(u, \tilde{x}_n) + (1-\beta_n)\phi(u, \tilde{x}_n)] + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= (1-\alpha_n)\phi(u, \tilde{x}_n) + \alpha_n \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= \phi(u, \tilde{x}_n) - \phi(u, x_{n+1}) + \phi(u, x_{n+1}) - \phi(u, u_n) \\
&= \|\tilde{x}_n\|^2 - \|x_{n+1}\|^2 + 2\langle u, Jx_{n+1} - J\tilde{x}_n\rangle + \|x_{n+1}\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_{n+1}\rangle \\
&\le \|\tilde{x}_n - x_{n+1}\|(\|\tilde{x}_n\| + \|x_{n+1}\|) + 2\|u\|\|Jx_{n+1} - J\tilde{x}_n\| \\
&\quad + \|x_{n+1} - u_n\|(\|x_{n+1}\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_{n+1}\|.
\end{aligned} \tag{3.24}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, it follows from (3.23) that $\|Jx_{n+1} - J\tilde{x}_n\| \to 0$ and $\|Ju_n - Jx_{n+1}\| \to 0$, which hence yield $\phi(u_n, y_n) \to 0$. Utilizing Lemma 2.1, we get $\|u_n - y_n\| \to 0$. Observe that

$$\big\|x_{n+1} - y_n\big\| \le \|x_{n+1} - u_n\| + \big\|u_n - y_n\big\| \longrightarrow 0, \tag{3.25}$$

due to (3.23). Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have that

$$\lim_{n\to\infty} \big\|Jx_{n+1} - Jy_n\big\| = \lim_{n\to\infty} \|Jx_{n+1} - J\tilde{x}_n\| = 0. \tag{3.26}$$

On the other hand, we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \longrightarrow 0. \tag{3.27}$$

Noticing that

$$\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n J\tilde{x}_n + (1-\alpha_n)JSz_n)\| \\
&= \|\alpha_n (Jx_{n+1} - J\tilde{x}_n) + (1-\alpha_n)(Jx_{n+1} - JSz_n)\| \\
&= \|(1-\alpha_n)(Jx_{n+1} - JSz_n) - \alpha_n (J\tilde{x}_n - Jx_{n+1})\| \\
&\geq (1-\alpha_n)\|Jx_{n+1} - JSz_n\| - \alpha_n \|J\tilde{x}_n - Jx_{n+1}\|,
\end{aligned} \tag{3.28}$$

we have

$$\|Jx_{n+1} - JSz_n\| \leq \frac{1}{1-\alpha_n}\left(\|Jx_{n+1} - Jy_n\| + \alpha_n \|J\tilde{x}_n - Jx_{n+1}\|\right). \tag{3.29}$$

From (3.26) and $\limsup_{n\to\infty}\alpha_n < 1$, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1} - JSz_n\| = 0. \tag{3.30}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - Sz_n\| = 0. \tag{3.31}$$

Observe that

$$\|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| + \|Sz_n - Sx_n\|. \tag{3.32}$$

Since S is uniformly continuous, it follows from (3.27), (3.31) and $x_{n+1} - x_n \to 0$ that $x_n - Sx_n \to 0$.

Now let us show that $\omega_w(\{x_n\}) \subset \mathrm{EP} \cap F(S) \cap T^{-1}0$, where

$$\omega_w(\{x_n\}) := \left\{\hat{x} \in C : x_{n_k} \rightharpoonup \hat{x} \text{ for some subsequence } \{n_k\} \subset \{n\} \text{ with } n_k \uparrow \infty\right\}. \tag{3.33}$$

Indeed, since $\{x_n\}$ is bounded and X is reflexive, we know that $\omega_w(\{x_n\}) \neq \emptyset$. Take $\hat{x} \in \omega_w(\{x_n\})$ arbitrarily, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Hence $\hat{x} \in F(S)$. Let us show that $\hat{x} \in T^{-1}0$. Since $x_n - \tilde{x}_n \to 0$, we have that $\tilde{x}_{n_k} \rightharpoonup \hat{x}$. Moreover, since J is uniformly norm-to-norm continuous on bounded subsets of E and $\liminf_{n\to\infty} r_n > 0$, we obtain

$$v_n = \frac{1}{r_n}(Jx_n - J\tilde{x}_n) \longrightarrow 0. \tag{3.34}$$

It follows from $v_n \in T\tilde{x}_n$ and the monotonicity of T that

$$\langle z - \tilde{x}_n, z' - v_n \rangle \geq 0 \tag{3.35}$$

for all $z \in D(T)$ and $z' \in Tz$. This implies that

$$\langle z - \hat{x}, z' \rangle \geq 0 \tag{3.36}$$

for all $z \in D(T)$ and $z' \in Tz$. Thus from the maximality of T, we infer that $\hat{x} \in T^{-1}0$. Therefore, $\hat{x} \in F(S) \cap T^{-1}0$. Further, let us show that $\hat{x} \in \mathrm{EP}$. Since $u_n - y_n \to 0$ and $x_n - u_n \to 0$, from $x_{n_k} \rightharpoonup \hat{x}$ we obtain that $y_{n_k} \rightharpoonup \hat{x}$ and $u_{n_k} \rightharpoonup \hat{x}$.

Since J is uniformly norm-to-norm continuous on bounded subsets of E, from $u_n - y_n \to 0$ we derive

$$\lim_{n \to \infty} \| Ju_n - Jy_n \| = 0. \tag{3.37}$$

From $\liminf_{n \to \infty} r_n > 0$, it follows that

$$\lim_{n \to \infty} \frac{\| Ju_n - Jy_n \|}{r_n} = 0. \tag{3.38}$$

By the definition of $u_n := K_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \tag{3.39}$$

where

$$F(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n \rangle. \tag{3.40}$$

Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}} \langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k} \rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{3.41}$$

Since $y \mapsto f(x, y) + \langle Ax, y - x \rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (3.38) and (A4) we have

$$F(y, \hat{x}) \leq 0, \quad \forall y \in C. \tag{3.42}$$

For t, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \hat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \hat{x}) \leq tF(y_t, y). \tag{3.43}$$

Dividing by t, we have

$$F(y_t, y) \geq 0, \quad \forall y \in C. \tag{3.44}$$

Letting $t \downarrow 0$, from (A3) it follows that

$$F(\widehat{x}, y) \geq 0, \quad \forall y \in C. \tag{3.45}$$

So, $\widehat{x} \in \mathrm{EP}$. Therefore, we obtain that $\omega_w(\{x_n\}) \subset \mathrm{EP} \cap F(S) \cap T^{-1}0$ by the arbitrariness of $\widehat{x}$.

Next, let us show that $\omega_w(\{x_n\}) = \{\Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0\}$ and $x_n \to \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$.

Indeed, put $\overline{x} = \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. From $x_{n+1} = \Pi_{H_n\cap W_n}x_0$ and $\overline{x} \in \mathrm{EP} \cap F(S) \cap T^{-1}0 \subset H_n \cap W_n$, we have $\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive for each $\widehat{x} \in \omega_w(\{x_n\})$

$$\begin{aligned}
\phi(\widehat{x}, x_0) &= \|\widehat{x}\|^2 - 2\langle \widehat{x}, x_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty}\Big(\|x_{n_k}\|^2 - 2\langle x_{n_k}, x_0\rangle + \|x_0\|^2\Big) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \phi(\overline{x}, x_0).
\end{aligned} \tag{3.46}$$

It follows from the definition of $\Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$ that $\widehat{x} = \overline{x}$ and hence

$$\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(\overline{x}, x_0). \tag{3.47}$$

So we have $\lim_{k\to\infty}\|x_{n_k}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we conclude that $\{x_{n_k}\}$ converges strongly to $\Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. This completes the proof. □

Theorem 3.5 covers [25, Theorem 3.1] by taking $C = E, f \equiv 0$ and $A \equiv 0$. Also Theorem 3.5 covers [24, Theorem 2.1] by taking $f \equiv 0$, $A \equiv 0$ and $T \equiv 0$.

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator with domain $D(T) = C$, $S : C \to C$ be a relatively nonexpansive mapping, $A : C \to E^*$ be an α-inverse-strongly monotone mapping and $f : C\times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4). Assume that $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0,\infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$ and that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequences in $(0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$.*

Define a sequence $\{x_n\}$ by the following algorithm.

Algorithm 3.7.

$$\begin{gathered} x_0 \in C \text{ arbitrarily chosen}, \\ 0 = v_n + \frac{1}{r_n}(J\tilde{x}_n - Jx_n), \quad v_n \in T\tilde{x}_n, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\tilde{x}_n), \\ u_n \in C \text{ such that} \\ f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ H_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \tilde{x}_n),\ \langle v - \tilde{x}_n, v_n \rangle \leq 0\}, \\ W_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{gathered} \tag{3.48}$$

where J is the single valued duality mapping on E. Let $\mathrm{EP} \cap F(S) \cap T^{-1}0 \neq \emptyset$. If S is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{EP} \cap F(S) \cap T^{-1}0} x_0$.

Proof. For each $n \geq 0$, define two sets C_n and D_n as follows:

$$\begin{gathered} C_n = \{v \in C : \phi(v, u_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \tilde{x}_n)\}, \\ D_n = \{v \in C : \langle v - \tilde{x}_n, v_n \rangle \leq 0\}. \end{gathered} \tag{3.49}$$

It is obvious that C_n is closed and D_n, W_n are closed convex sets for each $n \geq 0$. Let us show that C_n is convex and so $H_n = C_n \cap D_n$ is closed and convex. Similarly to the proof of Lemma 3.3, since

$$\phi(v, u_n) \leq \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \tilde{x}_n) \tag{3.50}$$

is equivalent to

$$2\alpha_n \langle v, Jx_0 \rangle + 2(1-\alpha_n)\langle v, J\tilde{x}_n \rangle - 2\langle v, Ju_n \rangle \leq \alpha_n \|x_0\|^2 + (1-\alpha_n)\|\tilde{x}_n\|^2 - \|u_n\|^2, \tag{3.51}$$

we know that C_n is convex and so is $H_n = C_n \cap D_n$. Next, let us show that $\mathrm{EP} \cap F(S) \cap T^{-1}0 \subset C_n$ for each $n \geq 0$. Indeed, utilizing Proposition 2.8, we have, for each $w \in \mathrm{EP} \cap F(S) \cap T^{-1}0$,

$$\begin{aligned}
\phi(w, u_n) &= \phi(w, K_{r_n} y_n) \leq \phi(w, y_n) \\
&= \phi\Big(w, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J S \widetilde{x}_n)\Big) \\
&= \|w\|^2 - 2\langle w, \alpha_n J x_0 + (1-\alpha_n) J S \widetilde{x}_n \rangle + \|\alpha_n J x_0 + (1-\alpha_n) J S \widetilde{x}_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_n \langle w, J x_0 \rangle - 2(1-\alpha_n)\langle w, J S \widetilde{x}_n \rangle + \alpha_n \|x_0\|^2 + (1-\alpha_n)\|S \widetilde{x}_n\|^2 \\
&= \alpha_n \phi(w, x_0) + (1-\alpha_n)\phi(w, S\widetilde{x}_n) \\
&\leq \alpha_n \phi(w, x_0) + (1-\alpha_n)\phi(w, \widetilde{x}_n).
\end{aligned} \tag{3.52}$$

So $w \in C_n$ for all $n \geq 0$ and $\mathrm{EP} \cap F(S) \cap T^{-1}0 \subset C_n$. As in the proof of Lemma 3.3, we can obtain $w \in D_n$ and hence $w \in H_n$. It follows from Lemma 2.4 that

$$\langle w - x_n, J x_0 - J x_n \rangle = \langle w - \Pi_{H_{n-1} \cap W_{n-1}} x_0, J x_0 - J \Pi_{H_{n-1} \cap W_{n-1}} x_0 \rangle \leq 0, \tag{3.53}$$

which implies that $w \in W_n$. Consequently, $w \in H_n \cap W_n$ and so $\mathrm{EP} \cap F(S) \cap T^{-1}0 \subset H_n \cap W_n$ for all $n \geq 0$. Therefore, the sequence $\{x_n\}$ generated by Algorithm 3.7 is well defined. As in the proof of Theorem 3.5, we can obtain $\phi(x_{n+1}, x_n) \to 0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in H_n$, from the definition of H_n we also have

$$\phi(x_{n+1}, u_n) \leq \alpha_n \phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, \widetilde{x}_n), \qquad \langle x_{n+1} - \widetilde{x}_n, v_n \rangle \leq 0. \tag{3.54}$$

As in the proof of Theorem 3.5, we can deduce not only from $\phi(x_{n+1}, x_n) \to 0$ that $\phi(\widetilde{x}_n, x_n) \to 0$ but also from $\phi(\widetilde{x}_n, x_n) \to 0$, $x_n - \widetilde{x}_n \to 0$ and $x_{n+1} - x_n \to 0$ that

$$\lim_{n \to \infty} \phi(x_{n+1}, \widetilde{x}_n) = 0. \tag{3.55}$$

Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in H_n$, from the definition of H_n, we also have

$$\phi(x_{n+1}, u_n) \leq \alpha_n \phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, \widetilde{x}_n). \tag{3.56}$$

It follows from (3.55) and $\alpha_n \to 0$ that

$$\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0. \tag{3.57}$$

Utilizing Lemma 2.1 we have

$$\lim_{n \to \infty} \|x_{n+1} - u_n\| = \lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|x_{n+1} - \widetilde{x}_n\| = 0. \tag{3.58}$$

Furthermore, for $u \in \mathrm{EP} \cap F(S) \cap T^{-1}0$ arbitrarily fixed, it follows from Proposition 2.8 that

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \le \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\
&= \phi\Big(u, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J S \tilde{x}_n)\Big) - \phi(u, u_n) \\
&= \|u\|^2 - 2\langle u, \alpha_n J x_0 + (1-\alpha_n) J S \tilde{x}_n \rangle + \|\alpha_n J x_0 + (1-\alpha_n) J S \tilde{x}_n\|^2 - \phi(u, u_n) \\
&\le \|u\|^2 - 2\alpha_n \langle u, J x_0 \rangle - 2(1-\alpha_n)\langle u, J S \tilde{x}_n \rangle + \alpha_n \|x_0\|^2 + (1-\alpha_n)\|S\tilde{x}_n\|^2 - \phi(u, u_n) \\
&= \alpha_n \phi(u, x_0) + (1-\alpha_n)\phi(u, S\tilde{x}_n) - \phi(u, u_n) \\
&\le \alpha_n \phi(u, x_0) + \phi(u, \tilde{x}_n) - \phi(u, u_n) \\
&= \alpha_n \phi(u, x_0) + \phi(u, \tilde{x}_n) - \phi(u, x_{n+1}) + \phi(u, x_{n+1}) - \phi(u, u_n) \\
&= \alpha_n \phi(u, x_0) + \|\tilde{x}_n\|^2 - \|x_{n+1}\|^2 + 2\langle u, J x_{n+1} - J\tilde{x}_n \rangle + \|x_{n+1}\|^2 \\
&\quad - \|u_n\|^2 + 2\langle u, J u_n - J x_{n+1} \rangle \\
&\le \alpha_n \phi(u, x_0) + \|\tilde{x}_n - x_{n+1}\|(\|\tilde{x}_n\| + \|x_{n+1}\|) + 2\|u\|\|J x_{n+1} - J\tilde{x}_n\| \\
&\quad + \|x_{n+1} - u_n\|(\|x_{n+1}\| + \|u_n\|) + 2\|u\|\|J u_n - J x_{n+1}\|.
\end{aligned} \tag{3.59}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, it follows from (3.58) that $\|J x_{n+1} - J\tilde{x}_n\| \to 0$ and $\|J u_n - J x_{n+1}\| \to 0$, which together with $\alpha_n \to 0$, yield $\phi(u_n, y_n) \to 0$. Utilizing Lemma 2.1, we get $\|u_n - y_n\| \to 0$. Observe that

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - u_n\| + \|u_n - y_n\| \longrightarrow 0, \tag{3.60}$$

due to (3.58). Since J is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \|J x_{n+1} - J y_n\| = \lim_{n\to\infty} \|J x_{n+1} - J x_n\| = \lim_{n\to\infty} \|J x_{n+1} - J\tilde{x}_n\| = 0. \tag{3.61}$$

Note that

$$\|J S\tilde{x}_n - J y_n\| = \|J S\tilde{x}_n - (\alpha_n J x_0 + (1-\alpha_n) J S\tilde{x}_n)\| = \alpha_n \|J x_0 - J S\tilde{x}_n\|. \tag{3.62}$$

Therefore, from $\alpha_n \to 0$ we get

$$\lim_{n\to\infty} \|J S\tilde{x}_n - J y_n\| = 0. \tag{3.63}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|S\tilde{x}_n - y_n\| = 0. \tag{3.64}$$

It follows that

$$\|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S\widetilde{x}_n\| + \|S\widetilde{x}_n - Sx_n\|. \tag{3.65}$$

Since S is uniformly continuous, it follows from (3.58) and (3.64) that $x_n - Sx_n \to 0$.

Finally, we prove that $x_n \to \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. Indeed, for $\widehat{x} \in \mathrm{EP} \cap F(S) \cap T^{-1}0$ arbitrarily fixed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \widehat{x} \in C$, then $\widehat{x} \in F(S)$. Now let us show that $\widehat{x} \in T^{-1}0$. Since $x_n - \widetilde{x}_n \to 0$, we have that $\widetilde{x}_{n_k} \rightharpoonup \widehat{x}$. Moreover, since J is uniformly norm-to-norm continuous on bounded subsets of E, and $\liminf_{n\to\infty} r_n > 0$, we obtain that $v_n = (1/r_n)(Jx_n - J\widetilde{x}_n) \to 0$. It follows from $v_n \in T\widetilde{x}_n$ and the monotonicity of T that $\langle z - \widetilde{x}_n, z' - v_n\rangle \geq 0$ for all $z \in D(T)$ and $z' \in Tz$. This implies that $\langle z - \widehat{x}, z'\rangle \geq 0$ for all $z \in D(T)$ and $z' \in Tz$. Thus from the maximality of T, we infer that $\widehat{x} \in T^{-1}0$. Further, let us show that $\widehat{x} \in \mathrm{EP}$. Since $u_n - y_n \to 0$ and $x_n - u_n \to 0$, from $x_{n_k} \rightharpoonup \widehat{x}$ we obtain that $y_{n_k} \rightharpoonup \widehat{x}$ and $u_{n_k} \rightharpoonup \widehat{x}$.

Since J is uniformly norm-to-norm continuous on bounded subsets of E, from $u_n - y_n \to 0$ we derive $\lim_{n\to\infty}\|Ju_n - Jy_n\| = 0$. From $\liminf_{n\to\infty} r_n > 0$ it follows that

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.66}$$

By the definition of $u_n := K_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \tag{3.67}$$

where $F(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle$. Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C. \tag{3.68}$$

Since $y \mapsto f(x, y) + \langle Ax, y - x\rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (3.66) and (A4) we have $F(y, \widehat{x}) \leq 0$, for all $y \in C$. For t, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\widehat{x}$. Since $y \in C$ and $\widehat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \widehat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \widehat{x}) \leq tF(y_t, y). \tag{3.69}$$

Dividing by t, we have $F(y_t, y) \geq 0$, for all $y \in C$. Letting $t \downarrow 0$, from (A3) it follows that $F(\widehat{x}, y) \geq 0$, for all $y \in C$. So, $\widehat{x} \in \mathrm{EP}$. Therefore, we obtain that $\omega_w(\{x_n\}) \subset \mathrm{EP} \cap F(S) \cap T^{-1}0$ by the arbitrariness of $\widehat{x}$.

Next, let us show that $\omega_w(\{x_n\}) = \{\Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0\}$ and $x_n \to \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$.

Indeed, put $\overline{x} = \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. From $x_{n+1} = \Pi_{H_n\cap W_n}x_0$ and $\overline{x} \in \mathrm{EP} \cap F(S) \cap T^{-1}0 \subset H_n \cap W_n$, we have $\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive for each $\widehat{x} \in \omega_w(\{x_n\})$

$$\begin{aligned}
\phi(\widehat{x}, x_0) &= \|\widehat{x}\|^2 - 2\langle \widehat{x}, x_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty}\Big(\|x_{n_k}\|^2 - 2\langle x_{n_k}, x_0\rangle + \|x_0\|^2\Big) \\
&= \liminf_{k\to\infty}\phi(x_{n_k}, x_0) \\
&\leq \limsup_{k\to\infty}\phi(x_{n_k}, x_0) \\
&\leq \phi(\overline{x}, x_0).
\end{aligned} \tag{3.70}$$

It follows from the definition of $\Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$ that $\widehat{x} = \overline{x}$ and hence $\lim_{k\to\infty}\phi(x_{n_k}, x_0) = \phi(\overline{x}, x_0)$. So we have $\lim_{k\to\infty}\|x_{n_k}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we know that $x_{n_k} \to \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we know that $x_n \to \Pi_{\mathrm{EP}\cap F(S)\cap T^{-1}0}x_0$. This completes the proof. □

Theorem 3.6 covers [25, Theorem 3.2] by taking $C = E, f \equiv 0$ and $A \equiv 0$. Also Theorem 3.6 covers [24, Theorem 2.2] by taking $f \equiv 0$, $A \equiv 0$ and $T \equiv 0$.

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