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Research Article

On Isosceles Sets in the 4-Dimensional Euclidean Space

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A subset X in the k-dimensional Euclidean space $\mathbb{R}^k$ that contains n points (elements) is called an n-point isosceles set if every triplet of points selected from them forms an isosceles triangle. In this paper, we show that there exist exactly two 11-point isosceles sets in $\mathbb{R}^4$ up to isomorphisms and that the maximum cardinality of isosceles sets in $\mathbb{R}^4$ is 11.

1. Introduction

Let $\mathbb{R}^k$ be the k-dimensional Euclidean space, let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ be in $\mathbb{R}^k$, and $d(x, y) = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}$.

For a finite set $X \subset \mathbb{R}^k$, let

$$A(X) = \{d(x, y) \mid x, y \in X,\ x \neq y\}. \tag{1.1}$$

If $|A(X)| = s$, we call X an *s-distance set*.

Two subsets in $\mathbb{R}^k$ are said to be *isomorphic* if there exists similar transformation from one to the other.

We have the following interesting problems on s-distance sets.

(1) What is the cardinality of points when the number of s-distance sets in $\mathbb{R}^k$ is finite up to isomorphisms?

(2) What is the maximum cardinality of s-distance sets in $\mathbb{R}^k$?

(3) Can we say something about the ratios of distances in an s-distance set?

Table 1: The maximum cardinality of 2-distance sets.

k	$\binom{k+2}{2}$	The maximum cardinality of 2-distance sets	The number of 2-distance sets giving the maximum cardinality
1	3	3	1
2	6	5	1
3	10	6	6
4	15	10	1
5	21	16	1
6	28	27	1
7	36	29	1
8	45	45	≥1

As regards question (1), Einhorn and Schoenberg [1] showed that the number of 2-distance sets in $\mathbb{R}^k$ is finite if cardinalities are more than or equal to $k+2$.

For question (2) with $s = 2$ and $k \le 8$, Erdös and Kelly [2], Croft [3], and Lisoněk [4] gave the maximum cardinalities. Their results are summarized in Table 1 (see [4, 5]).

As regards question (3), Larman et al. [6] showed that if $|X| > 2k+3$, the ratio of 2 distances in any 2-distance set X is given by $\sqrt{\alpha-1} : \sqrt{\alpha}$, where α is an integer α satisfying $\alpha \le 1/2 + \sqrt{k/2}$.

Bannai et al. [7] and Blokhuis [8] proved that the cardinality of an s-distance set in $\mathbb{R}^k$ is bounded above by $\binom{k+s}{s}$. For the case $s = 3$ and $k = 2$, Shinohara [9] gave the answers to questions (1) and (2) by classifying 3-distance sets in $\mathbb{R}^2$. He proved that there are finitely many 3-distance sets when cardinalities are more than or equal to 5. He also proved that the maximum cardinality of 3-distance sets is 7. The complete classification of 3-distance sets in $\mathbb{R}^2$ was also given. Recently Shinohara [10] showed uniqueness of maximum 3-distance sets in $\mathbb{R}^3$.

In this paper, we deal with isosceles sets which are defined in the following.

We call a set in $\mathbb{R}^k$ with n points an *n-point isosceles set* if every triplet of points selected from them forms an isosceles triangle.

Here three collinear points will be interpreted as forming an isosceles triangle if and only if one of them is the mid-point of the other pair.

We remark that all n-point 2-distance sets are n-point isosceles sets.

In this paper, we consider classification and the maximum cardinality of isosceles sets in $\mathbb{R}^4$. The following theorem and corollary are the main results.

Theorem 1.1. *There exist exactly two 11-point isosceles sets in $\mathbb{R}^4$ up to isomorphisms. They are X and Y, which will be explicitly defined in the following section.*

Corollary 1.2. *There is no 12-point isosceles set in $\mathbb{R}^4$. Therefore the maximum cardinality of isosceles sets in $\mathbb{R}^4$ is 11.*

We prove them by expanding the method by Croft [3] into $\mathbb{R}^4$.

2. Known Results and Example of Isosceles Sets

The following are the known facts about isosceles sets so far.

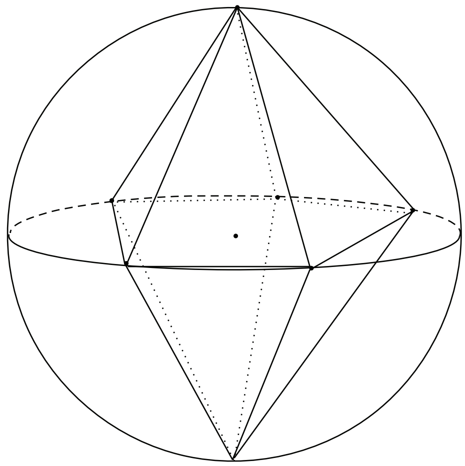


Figure 1: A unique 8-point isosceles set in $\mathbb{R}^3$ (from Kido [11]).

(i) Ten-point isosceles sets in $\mathbb{R}^4$ exist infinitely many up to isomorphisms. For example, $\{(\cos(2j/5)\pi,\ \sin(2j/5)\pi, 0, 0) \mid 0 \le j \le 4\} \cup \{c\ (0, 0, \cos(2k/5)\pi, \sin(2k/5)\pi) \mid 0 \le k \le 4\}$ is a 10-point isosceles set for any positive real number c. It is nonisomorphic to $\{(\cos(2j/5)\pi, \sin(2j/5)\pi, 0, 0) \mid 0 \le j \le 4\} \cup \{c'\ (0, 0, \cos(2k/5)\pi, \sin(2k/5)\pi) \mid 0 \le k \le 4\}$ for any positive real number c' satisfying $c' \neq c$.

(ii) No 9-point isosceles set in $\mathbb{R}^3$ exists (Croft [3]).

(iii) There exists a unique 8-point isosceles set in $\mathbb{R}^3$ up to isomorphisms. It is in Figure 1 (Kido [11]).

(iv) Seven-point isosceles sets in $\mathbb{R}^3$ exist infinitely many up to isomorphisms.

(v) No 7-point isosceles set in $\mathbb{R}^2$ exists (Erdös and Golomb [12], Erdös and Kelly [2]).

(vi) There exists a unique 6-point isosceles set in $\mathbb{R}^2$ up to isomorphisms. It consists of five points of a regular pentagon and its center (Erdös and Golomb [12], Erdös and Kelly [2]).

(vii) There exist exactly three 5-point isosceles sets in $\mathbb{R}^2$ up to isomorphisms. They are four points of a square and its center, five points of a regular pentagon, and four points of a regular pentagon and its center (Fishburn [13], Erdös and Golomb [12]).

(viii) Four-point isosceles sets in $\mathbb{R}^2$ exist infinitely many up to isomorphisms.

Now we define two examples X and Y which are mentioned in Theorem.

2.1. Example of 11-Point Isosceles Sets in $\mathbb{R}^4$

Let e_i, $1 \le i \le 4$ be the canonical basis of $\mathbb{R}^4$. Then 11-point sets X and Y in $\mathbb{R}^4$ defined as follows are isosceles sets:

$$X = X' \cup \{u_0\}, \tag{2.1}$$

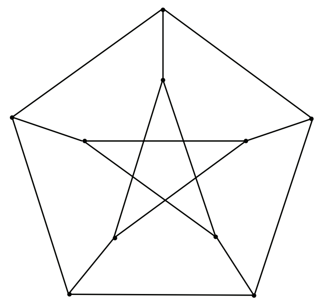


Figure 2: The Petersen graph.

where

$$X' = \{e_i + e_j \mid 1 \le i < j \le 4\} \cup \{-e_k + u \mid 1 \le k \le 4\}, \tag{2.2}$$

and $u_0 = ((5+\sqrt{5})/10, (5+\sqrt{5})/10, (5+\sqrt{5})/10, (5+\sqrt{5})/10)$ and $u = ((3+\sqrt{5})/4, (3+\sqrt{5})/4, (3+\sqrt{5})/4, (3+\sqrt{5})/4)$:

$$\begin{aligned} Y = &\left\{\left(\cos\frac{2j}{5}\pi, \sin\frac{2j}{5}\pi, 0, 0\right) \mid 0 \le j \le 4\right\} \\ &\cup \left\{\left(0, 0, \cos\frac{2k}{5}\pi, \sin\frac{2k}{5}\pi\right) \mid 0 \le k \le 4\right\} \cup \{(0,0,0,0)\}. \end{aligned} \tag{2.3}$$

Remark 2.1. In above X' is known as a unique 10-point 2-distance set (see Lisoněk [4]). It is constructed by the Petersen graph (Figure 2) and it is on a 3-dimensional sphere whose center is u_0. Also we can easily see that X' and X contain a square and that Y contains a regular pentagon.

3. Notation and Some Isosceles Set Configurations

We introduce the following notation (see [3]): apex: a point of a set of three or more points equidistant from all the others.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be an n-point isosceles set. We define the *vertex-number* $V(P_i)$ of a point $P_i \in \mathcal{P}$ by the number of distinct isosceles triangles of which P_i is an apex. It is easy to see that

$$V(P_1) + \cdots + V(P_n) \ge \binom{n}{3}. \tag{3.1}$$

Especially let α be the number of regular triangles in $\mathcal{P}$:

$$V(P_1) + \cdots + V(P_n) = 2\alpha + \binom{n}{3}. \tag{3.2}$$

We further say that a point $P_i \in \mathcal{P}$ is of *type* $(r, s, \ldots, u)$ if the lines joining it to the remaining points in $\mathcal{P}$ are constituted; thus: r of length a, s of length $b, \ldots, u$ of length l, where $a, b, \ldots, l$ are no two of them equal. Setting $r \geq s \geq \cdots \geq u$, $r + s + \cdots + u = n - 1$ clearly holds. Moreover if P_i is of type $(r, s, \ldots, u)$, then

$$V(P_i) = \binom{r}{2} + \binom{s}{2} + \cdots + \binom{u}{2}. \tag{3.3}$$

Lemma 3.1. *Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set in $\mathbb{R}^4$, and suppose that P_1 has the largest vertex-number. Then the type of P_1 satisfies one of the following cases (A)–(H):*

Case (A): (10),

Case (B): (9,1), (8,2), (8,1,1),

Case (C): (7,3), (7,2,1),

Case (D): (6,4), (6,3,1), (5,5), (5,4,1),

Case (E): (7,1,1,1),

Case (F): (6,2,2),

Case (G): (6,2,1,1),

Case (H): (6,1,1,1,1).

Proof. Since $V(P_1) + \cdots + V(P_{11}) \geq \binom{11}{3} = 165$ by (3.1), we have $V(P_1) \geq 15$. Let $(r, s, \ldots, u)$ be the type of P_1. Then we have

$$\binom{r}{2} + \binom{s}{2} + \cdots + \binom{u}{2} \geq 15, \tag{3.4}$$

and we have

$$r + s + \cdots + u = 10. \tag{3.5}$$

In order to satisfy (3.4) and (3.5), $(r, s, \ldots, u)$ must be one in the list of the lemma. □

Throughout this paper, we refer to the *condition* (X) as "four points in a set lie on a circle."

We first show the following lemma.

Lemma 3.2. *If an 11-point isosceles set in $\mathbb{R}^4$ exists, then the condition (X) is true for it.*

In Sections 4–11, we prove Lemma 3.2 case by case according to eight cases (A)–(H) of types of P_1 given in Lemma 3.1. In Sections 12 and 13, we deal with 11-point isosceles sets satisfying the condition (X). In Section 14, we complete the proofs of Theorem 1.1 and Corollary 1.2.

The following propositions are useful for us to prove Lemma 3.2 and Theorem 1.1. We can prove Propositions 3.3 and 3.4 using a similar method to Lemma 3.1.

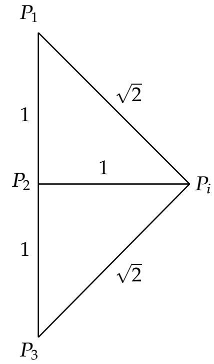


Figure 3

Proposition 3.3. *In a 10-point isosceles set in $\mathbb{R}^4$, let P be a point that has the largest vertex-number. Let $(r, s, \ldots, u)$ be the type of P. If $r < 6$, then it must be (5,4), (5,3,1), (5,2,2), or (4,4,1).*

Proposition 3.4. *In a 6-point isosceles set in $\mathbb{R}^3$, let P be a point that has the largest vertex-number. Then the type of P is one of (5), (4,1), and (3,2).*

Proposition 3.5. *Let an n-point isosceles set in $\mathbb{R}^4$ be constituted thus: P_1, which is the center of a 3-dimensional sphere S; upon S lie $P_2, P_3, P_4, \ldots$, being at least 3 and less than or equal to $n-2$ points; and at least one P, say P_n, does not lie on S. Then those points of the set that lie on S lie on one of two disjoint 2-dimensional spheres.*

Proof. We may assume that the equation of S is $x^2+y^2+z^2+w^2 = 1$ and $P_n = (k, 0, 0, 0)$, where $k > 0$ and $k \neq 1$. Then $P_1 = (0, 0, 0, 0)$. For a point $P_i = (x_i, y_i, z_i, w_i)$ on S, we consider $\Delta P_1 P_i P_n$.

When $P_1P_i = P_iP_n = 1$ holds, we have $x_i^2+y_i^2+z_i^2+w_i^2 = 1$ and $(x_i-k)^2+y_i^2+z_i^2+w_i^2 = 1$. Then $x_i = k/2$ and $y_i^2 + z_i^2 + w_i^2 = 1 - k^2/4$.

On the other hand, when $P_1P_n = P_iP_n = k$ holds, we have $x_i^2 + y_i^2 + z_i^2 + w_i^2 = 1$ and $(x_i - k)^2 + y_i^2 + z_i^2 + w_i^2 = k^2$. Then $x_i = 1/2k$ and $y_i^2 + z_i^2 + w_i^2 = 1 - 1/4k^2$.

Combining them, it holds that P_i is on one of two disjoint 2-dimensional spheres. □

Proposition 3.6. *If three points, P_1, P_2, P_3, say, in an n-point isosceles set in $\mathbb{R}^4$ are collinear in this order, then the other points of the set all lie on a 2-dimensional sphere.*

Proof. We may assume that $P_1 = (-1, 0, 0, 0), P_2 = (0, 0, 0, 0)$, and $P_3 = (1, 0, 0, 0)$. We consider the position of $P_i = (x_i, y_i, z_i, w_i)$ for $i = 4, \ldots, n$. By a similar method used in the proof of Kelly [2] or Lemma 6 in Croft [3], in a plane, P_1, P_2, P_3, and P_i must satisfy Figure 3.

Hence $x_i^2 + y_i^2 + z_i^2 + w_i^2 = 1$, $(x_i + 1)^2 + y_i^2 + z_i^2 + w_i^2 = 2$, and $(x_i - 1)^2 + y_i^2 + z_i^2 + w_i^2 = 2$ hold. Then we have $x_i = 0$ and $y_i^2 + z_i^2 + w_i^2 = 1$.

Therefore the other points lie on a 2-dimensional sphere. □

Corollary 3.7. *For $n \geq 11$, there is no n-point isosceles set in $\mathbb{R}^4$ which has three collinear points.*

Proof. We may show that this corollary holds for $n = 11$. By Proposition 3.6, the other eight points must lie on a 2-dimensional sphere. So they form an 8-point isosceles set in the 2-dimensional sphere ($\subset \mathbb{R}^3$). We know that there exists a unique 8-point isosceles set in $\mathbb{R}^3$

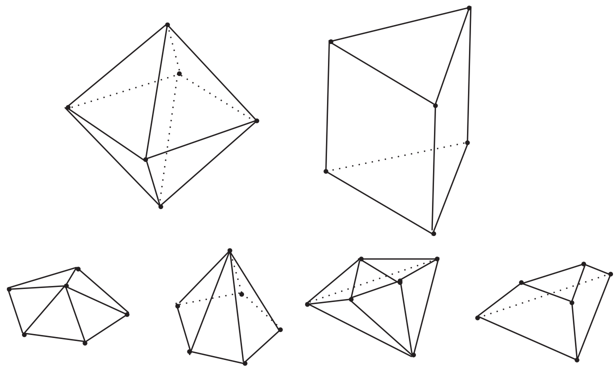


Figure 4: All 6-point 2-distance sets in $\mathbb{R}^3$ (from Einhorn and Schoenberg [14]).

and it is in Figure 1. But looking at the figure, we see that eight points in it do not lie on a 2-dimensional sphere. Hence there is no 8-point isosceles set in the 2-dimensional sphere.

Therefore there is no 11-point isosceles set in $\mathbb{R}^4$ which has three collinear points. □

Proposition 3.8. *Let $\mathcal{P} = \{P_1, \ldots, P_6\}$ be a 6-point isosceles set in a 2-dimensional sphere S. Then four points in $\mathcal{P}$ lie on a circle; the condition* (X) *holds.*

Proof. Let P_1 be a point that has the largest vertex-number in $\mathcal{P}$. By Proposition 3.4, the type of P_1 is one of (5), (4,1), and (3,2). If the type of P_1 is (5) or (4,1), then at least four points among $P_2, \ldots, P_6$ are on the intersection of S and the sphere whose center is P_1. So at least four points are on a circle, the condition (X) holds.

Thus we suppose that P_1 is of type (3,2) with corresponding distances r_1 and r_2. For $i = 1, 2$, let S_i be the sphere centered at P_1 with radius r_i. Let $U_1 = \mathcal{P} \cap S_1 = \{P_2, P_3, P_4\}$ and $U_2 = \mathcal{P} \cap S_2 = \{P_5, P_6\}$.

Now $\mathcal{P}$ is a 2- or an s-distance set ($s \geq 3$). We suppose that it is a 2-distance set. We know that there exist exactly six 6-point 2-distance sets in $\mathbb{R}^3$. These six figures are in Figure 4. Two figures contain all points of a square, and the others contain four points of a regular pentagon. All points of a square and four points of a regular pentagon are both on a circle. Therefore the condition (X) holds.

On the other hand, we suppose that $\mathcal{P}$ is an s-distance set ($s \geq 3$). So there exists a pair of points in $\mathcal{P}$ whose distance is c, that is, distinct from r_1 and r_2. Since $P_1P_i = r_1$ or $r_2 (i = 2, \ldots, 6)$, c is the distance apart of a pair of points in $\{P_2, \ldots, P_6\}$. If $P_iP_j = c$ holds for some $P_i \in U_1$ and $P_j \in U_2$, then $\Delta P_1P_iP_j$ is scalene with sides r_1, r_2, c. Thus the following condition holds:

$$P_iP_j = r_1 \text{ or } r_2 \quad \text{for any } P_i \in U_1,\ P_j \in U_2. \tag{3.6}$$

Because c is the distance apart of a pair of points in U_1 or U_2, at least one of P_2P_3, P_2P_4, P_3P_4, and P_5P_6 is c.

We suppose that $P_5P_6 = c$. Let $P_i \in U_1$ and consider $\Delta P_iP_5P_6$. Since P_iP_5 and P_iP_6 are of length r_1 or r_2 by (3.6), we have $P_iP_5 = P_iP_6$. Thus three points P_2, P_3, and P_4 are on the plane perpendicularly bisecting P_5P_6, the sphere S_1, and the sphere S. But the plane and the two spheres intersect at exactly two points. This is a contradiction. Therefore $P_5P_6 \neq c$, without loss of generality we may assume $P_2P_3 = c$.

Next we suppose that $P_2P_3 = c$ and $P_2P_4 = d$ ($d \neq r_1$, $d \neq r_2$, but we can admit $c = d$). Let $P_j \in U_2$ and consider $\Delta P_2P_3P_j$. Because P_2P_j and P_3P_j are of length r_1 or r_2 by (3.6), we have $P_2P_j = P_3P_j$. When we consider $\Delta P_2P_4P_j$ similarly, we have $P_2P_j = P_4P_j$. Thus P_6 and P_7 are on the plane perpendicularly bisecting P_2P_3, the plane perpendicularly bisecting P_2P_4, the sphere S_2, and the sphere S. Since the segment P_2P_3 and the segment P_2P_4 are not mutually parallel, the two planes and the two spheres have no intersection. Hence $P_2P_3 = c$ and $P_2P_4 = d$ do not hold. Similarly we can show that $P_2P_3 = c$ and $P_3P_4 = d$ do not hold.

So in $\mathcal{P}$, there is exactly one pair P_2P_3 whose distance is distinct from r_1 and r_2. When we consider $\Delta P_2P_3P_k$ for $k = 4, 5, 6$, $P_2P_k = P_3P_k$ holds by the configuration hypothesis. And we have $P_1P_2 = P_1P_3$. Then four points P_1, P_4, P_5, and P_6 on the plane perpendicularly bisecting P_2P_3 and the sphere S. The intersection of them is a circle, the condition (X) holds. □

4. Case (A) in Lemma 3.1

We consider the case (A) in Lemma 3.1. Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set in which P_1 is of type (6.1). Let S be the sphere centered at P_1 and $V = \mathcal{P} \cap S = \{P_2, \ldots, P_{11}\}$.

We notice that V is a 10-point isosceles set. Let P_2 be a point that has the largest vertex-number in V. Let $(r, s, \ldots, u)$ be the type of P_2 in V, the type of P_2 is $r \geq 6$, (5,4), (5,3,1), (5,2,2), or (4,4,1) by Proposition 3.3.

Proposition 4.1. *Let $(r, s, \ldots, u)$ be the type of P_2 in V. If the type of P_2 satisfies $r \geq 6$, then the condition* (X) *holds.*

Proof. If the type of P_2 in V satisfies $r \geq 6$, then at least six points among $P_3, \ldots, P_{11}$ are on the intersection of S and the sphere whose center is P_2. So they are on a 2-dimensional sphere. By Proposition 3.8, the condition (X) holds. □

Proposition 4.2. *If the type of P_2 is (5,4) in V, then the condition* (X) *holds.*

Proof. We suppose that P_2 is of type (5,4) in V. We see that five points in V are distributed on a 2-dimensional sphere which is the intersection of S and the sphere whose center is P_2 and another four points in V are distributed on another 2-dimensional sphere. These two spheres are disjoint.

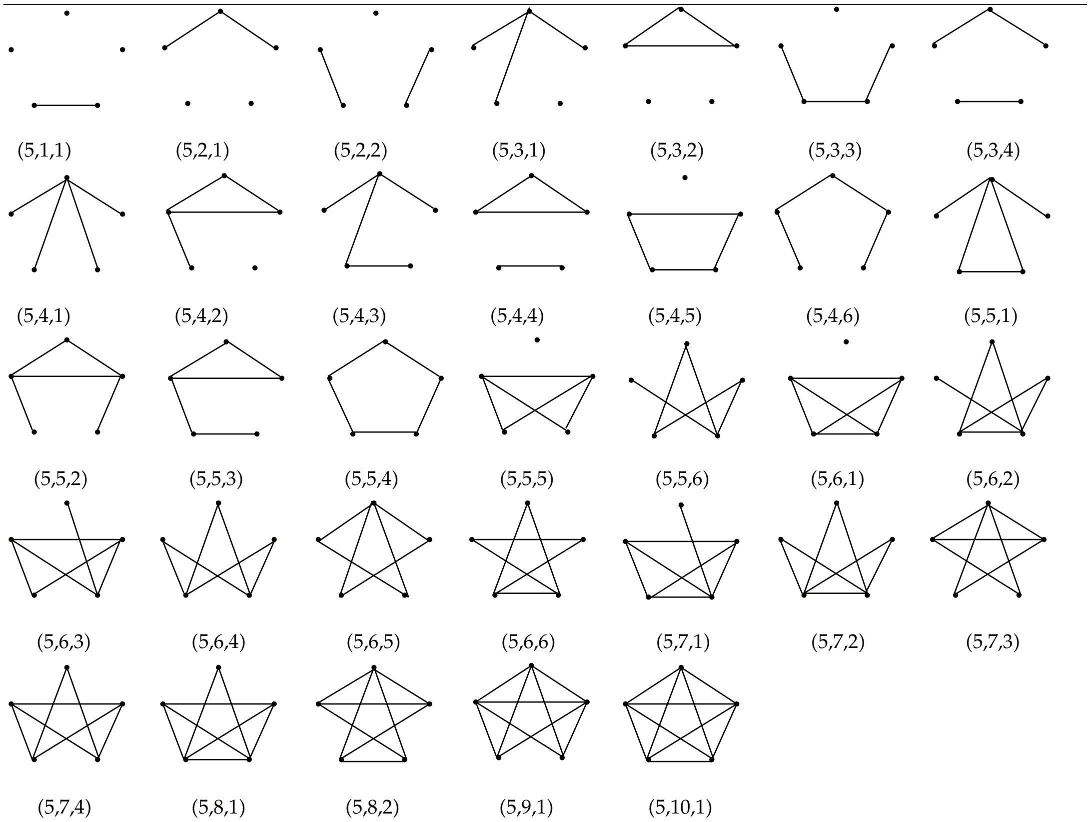
We will call them S_1 (on which $P_3, \ldots, P_7$ are) and S_2 (on which $P_8, \ldots, P_{11}$ are). Let $V_1 = V \cap S_1 = \{P_3, \ldots, P_7\}$ and $V_2 = V \cap S_2 = \{P_8, \ldots, P_{11}\}$. For $P_i \in V_1$, let $P_2P_i = a$ and for $P_j \in V_2$, let $P_2P_j = b$.

If V is a 2-distance set, then the types of ten points in V must be all (6,3) by looking at the Petersen graph (Figure 2). But P_2 is of type (5,4), V is not a 2-distance set. Hence V is an s-distance set ($s \geq 3$), there exists a pair of points in $P_2, \ldots, P_{11}$ whose distance is c, that is, distinct from a and b. Because $P_2P_k = a$ or b ($k = 3, \ldots, 11$), c is the distance between a pair of distinct points in $\{P_3, \ldots, P_{11}\}$. If $P_iP_j = c$ holds for some $P_i \in V_1$ and $P_j \in V_2$, then $\Delta P_2P_iP_j$ is scalene with sides a, b, c. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in V_1,\ P_j \in V_2. \tag{4.1}$$

So c is the distance between a pair of distinct points on the same 2-dimensional sphere.

Table 2: 5-point graphs.



We suppose that c is the distance between a pair of distinct points on S_1. Without loss of generality we may assume $P_3P_4 = c$. For $P_j \in V_2$ we consider $\Delta P_3P_4P_j$. Since P_3P_j and P_4P_j are of length a or b by (4.1), $P_3P_j = P_4P_j$ holds. Thus four points $P_8, \dots, P_{11}$ are on the hyperplane perpendicularly bisecting P_3P_4, the sphere S, and the sphere S_2. The intersection of them is a circle. Therefore the condition (X) holds.

We can repeat the similar discussion when we suppose that c is the distance between a pair of distinct points on S_2. □

Next we consider that the type of P_2 is (5,3,1) or (5,2,2) in V. We see that five points in V are distributed on a 2-dimensional sphere which is the intersection of S and the sphere whose center is P_2 and another two or three points in V are distributed on another 2-dimensional sphere. These two spheres are disjoint.

We will call them S_1 (on which $P_3, \dots, P_7$ are) and S_2 (on which P_8 and P_9 are). Let $V_1 = V \cap S_1 = \{P_3, \dots, P_7\}$ and $V_2 = V \cap S_2 = \{P_8, P_9\}$. For $P_i \in V_1$, let $P_2P_i = a$ and for $P_j \in V_2$, let $P_2P_j = b$. Moreover let $P_2P_{11} = c$.

Proposition 4.3. *Let $X_1 = \{P_2, \dots, P_9\}$. If X_1 is an s-distance set $(s \geq 3)$, then the condition* (X) *holds.*

Proof. Because we suppose that X_1 is an s-distance set ($s \geq 3$), there exists a pair of points $P_2, \ldots, P_9$ whose distance is d, that is, distinct from a and b (but we can admit $c = d$).

Since $P_2P_i = a$ or $b(i = 3, \ldots, 9)$, d is the distance apart of a pair of points in $\{P_3, \ldots, P_9\}$. If $P_iP_j = d$ holds for some $P_i \in V_1$ and $P_j \in V_2$, then $\Delta P_2P_iP_j$ is scalene with sides a, b, d. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in V_1,\ P_j \in V_2. \tag{4.2}$$

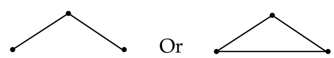
So d is the distance between a pair of distinct points on the same 2-dimensional sphere.

We suppose that d is the distance between a pair of distinct points on S_2, that is, $P_8P_9 = d$. In this case, if we repeat the similar discussion as Proposition 4.2, then the condition (X) holds. Hence we suppose that d is the distance between a pair of distinct points on S_1. For $P_3, \ldots, P_7$ on S_1, we consider 5-point graphs in Table 2. Edges in a graph represent the distance, that is, distinct from a and b. We regard the others, (i.e., transparent edges) as the distances a and b. Here we need not consider the graph which has no edge, because we suppose that there is at least one pair whose distance is distinct from a and b. We remark that 33 graphs in Table 2 and the graph which has no edge are all 5-point graphs.

We can classify 33 graphs into the following:

(i) a 4-point subgraph is "connected"; graphs satisfying it are (5,3,1), (5,3,3), (5,4,1), (5,4,2), (5,4,3), (5,4,5), (5,4,6), and (5,a,$*$) for $5 \leq a \leq 10$ ($*$ is arbitrary);

(ii) another four graphs whose a 3-point subgraph is



and no edge between them and the other two points; they are (5,2,1), (5,3,2), (5,3,4), and (5,4,4);

(iii) (5,2,2);

(iv) (5,1,1).

We observe each case. In the case (i), we may assume that the 4-point subgraph with $P_3, \ldots, P_6$ is connected. Without loss of generality we may assume $P_3P_4 = d$. For $i = 8, 9$, consider $\Delta P_3P_4P_i$. Then we have $P_3P_i = P_4P_i$ by (4.2). Since the 4-point subgraph with $P_3, \ldots, P_6$ is connected, we have $P_3P_i = P_4P_i = P_5P_i = P_6P_i$ by the similar discussion. Moreover we have $P_3P_j = P_4P_j = P_5P_j = P_6P_j$ for $j = 1, 2$ by the assumption. Then four points P_1, P_2, P_8, P_9 are equidistant from $P_3, \ldots, P_6$ on the 2-dimensional sphere S_1. If $P_3, \ldots, P_6$ are on a plane, then they are on a circle; the condition (X) holds. On the other hand, if $P_3, \ldots, P_6$ are not on a plane, then P_1, P_2, P_8, and P_9 are on a line. We cannot take four points on a line. This is a contradiction.

In the case (ii), we may assume that $P_3P_4 = d$ and $P_3P_5 = e$ (we can admit $d = e$). For $i = 8, 9$, consider $\Delta P_3P_4P_i$ and $\Delta P_3P_5P_i$. Then we have $P_3P_i = P_4P_i$ and $P_3P_i = P_5P_i$ by (4.2). In this case, P_3P_j, P_4P_j, and P_5P_j are a or b for $j = 6, 7$. When we consider $\Delta P_3P_4P_j$ and $\Delta P_3P_5P_j$, $P_3P_j = P_4P_j$ and $P_3P_j = P_5P_j$ hold. By the assumption we have $P_3P_k = P_4P_k$ and $P_3P_k = P_5P_k$ for $k = 1, 2$. Then six points $P_1, P_2, P_6, P_7, P_8, P_9$ are equidistant from P_3, P_4, and P_5. Hence they are in the 2-dimensional Euclidean space, that is, $\{P_1, P_2, P_6, P_7, P_8, P_9\}$ is a 6-point isosceles set in $\mathbb{R}^2$. We know that there exists a unique 6-point isosceles set in $\mathbb{R}^2$ up to isomorphisms. It consists of five points of a regular pentagon and its center. So four points in $\{P_1, P_2, P_6, P_7, P_8, P_9\}$ lie on a circle; the condition (X) holds.

In the case (iii), we may assume that $P_3P_4 = d$ and $P_5P_6 = e$ (we can admit $d = e$). For $i = 8, 9$, consider $\Delta P_3P_4P_i$ and $\Delta P_5P_6P_i$. Then we have $P_3P_i = P_4P_i$ and $P_5P_i = P_6P_i$ by (4.2). In this case, P_3P_7, P_4P_7, P_5P_7, and P_6P_7 are a or b. When we consider $\Delta P_3P_4P_7$ and $\Delta P_5P_6P_7$, $P_3P_7 = P_4P_7$ and $P_5P_7 = P_6P_7$ hold. By the assumption we have $P_3P_j = P_4P_j$ and $P_5P_j = P_6P_j$ for $j = 1, 2$. Then five points P_1, P_2, P_7, P_8, P_9 are on the hyperplane perpendicularly bisecting P_3P_4 and the hyperplane perpendicularly bisecting P_5P_6. For the intersection of them, there are two cases:

(α) since two hyperplanes are same, the intersection is a 3-dimensional Euclidean space.

(β) a 2-dimensional Euclidean space.

In the case (α), since $P_3, \ldots, P_6$ are on the 2-dimensional sphere S_1, the segment P_3P_4 and the segment P_5P_6 are mutually parallel. Then there is a plane that contains $P_3, \ldots, P_6$. So they are on a circle; the condition (X) holds.

In the case (β), $\{P_1, P_2, P_7, P_8, P_9\}$ is a 5-point isosceles set in $\mathbb{R}^2$. We know that there exist exactly three 5-point isosceles sets in $\mathbb{R}^2$ up to isomorphisms. They are four points of a square and its center, five points of a regular pentagon, and four points of a regular pentagon and its center. So four points in $\{P_1, P_2, P_7, P_8, P_9\}$ lie on a circle; the condition (X) holds.

In the case (iv), we may assume that $P_3P_4 = d$. Then we see that there is exactly one pair P_3P_4 whose distance is distinct from a and b in X_1. When we consider $\Delta P_3P_4P_i$ for $i = 2, 5, \ldots, 9$, $P_3P_i = P_4P_i$ holds by the configuration hypothesis. Thus six points P_2, P_5, P_6, P_7, P_8, and P_9 are on the hyperplane perpendicularly bisecting P_3P_4. This hyperplane is a 3-dimensional Euclidean space. Since $A(\{P_2, P_5, P_6, P_7, P_8, P_9\}) = \{a, b\}$, this is a 2-distance set in $\mathbb{R}^3$. We know that there exist exactly six 6-point 2-distance sets in $\mathbb{R}^3$. Any set contains four points lying on a circle. Hence the condition (X) holds. □

Proposition 4.4. *Similarly let* $\mathrm{X}_1 = \{P_2, \ldots, P_9\}$. *If* X_1 *is a 2-distance set, then the condition* (X) *holds.*

Proof. We consider the sum of all vertex-numbers in $\mathcal{D}$. Because P_2 has the largest vertex-number in V, $V(P_1) + \cdots + V(P_{11}) \leq \binom{10}{2} + 10 \times \{\binom{5}{2} + \binom{3}{2} + \binom{1}{2}\} = 175$. Let α be the number of regular triangles in $\mathcal{D}$. Then $2\alpha + \binom{11}{3} \leq 175$ holds by (3.2). Thus $\alpha \leq 5$.

Let $V_1 = \{P_3, \ldots, P_7\}$. We notice that V_1 on S_1 is a 2-distance set in $\mathbb{R}^3$. We consider 5-point graphs in Table 2 again. Edges in a graph represent the distance b. We regard the others, (i.e., transparent edges) as the distance a. Here we need not consider the graph which has no edge, because there is no 5-point 1-distance set in $\mathbb{R}^3$. Similarly we need not consider the complete graph (5,10,1).

If $P_iP_j = a$ for $i, j \in \{3, \ldots, 7\}$ $(i \neq j)$, then $\Delta P_2P_iP_j$ is a regular triangle. Since $\alpha \leq 5$, there are at most five pairs in V_1 whose distances are a. The number of pairs in V_1 is $\binom{5}{2} = 10$. Thus there are at least five pairs in V_1 whose distances are b.

Hence we have only to consider the 19 graphs between (5,5,1) and (5,9,1) in Table 2. In any graph, a 4-point subgraph is "connected". We may assume that their four points are $P_3, \ldots, P_6$ and that there is an edge between P_3 and P_4, that is, $P_3P_4 = b$. We consider $\Delta P_2P_3P_{11}$ and $\Delta P_2P_4P_{11}$. Since $P_2P_3 = P_2P_4 = a$ and $P_2P_{11} = c$, P_3P_{11} and P_4P_{11} are a or c. Then we consider $\Delta P_3P_4P_{11}$, we have $P_3P_{11} = P_4P_{11}$. Because the 4-point subgraph with $P_3, \ldots, P_6$ is connected, we have $P_3P_{11} = P_4P_{11} = P_5P_{11} = P_6P_{11}$ by the similar discussion. Moreover we have $P_3P_k = P_4P_k = P_5P_k = P_6P_k$ for $k = 1, 2$ by the assumption. Thus three points P_1, P_2, P_{11} are equidistant from $P_3, \ldots, P_6$ on the 2-dimensional sphere S_1. If $P_3, \ldots, P_6$ are on a plane, then they are on a circle; the condition (X) holds. On the other hand, if $P_3, \dots, P_6$ are not on a plane, then P_1, P_2, and P_{11} are on a line. By Corollary 3.7, this is a contradiction.

Therefore if X_1 is a 2-distance set, then the condition (X) holds. □

Combining Propositions 4.3 and 4.4, we have the following proposition.

Proposition 4.5. *If the type of P_2 is (5,3,1) or (5,2,2) in V, then the condition* (X) *holds.*

The last case is what the type of P_2 is (4,4,1) in V. We see that four points in V are distributed on a 2-dimensional sphere which is the intersection of S and the sphere whose center is P_2 and another four points in V are distributed on another 2-dimensional sphere. These two spheres are disjoint.

We will call these two spheres S_1 (on which $P_3, \dots, P_6$ are) and S_2 (on which $P_7, \dots, P_{10}$ are). Let $V_1 = V \cap S_1 = \{P_3, \dots, P_6\}$ and $V_2 = V \cap S_2 = \{P_7, \dots, P_{10}\}$. For $P_i \in V_1$, let $P_2P_i = a$ and for $P_j \in V_2$, let $P_2P_j = b$. Moreover let $P_2P_{11} = c$. Because $V(P_2) = 12$, $V(P_k) = 12$ for $k = 3, \dots, 11$. Thus the type of P_k is (4,4,1) in V for any k. (Since $V(P_k) = 12$, the type of P_k may be (5,2,2) in V. In this case, if we apply Proposition 4.5, then the condition (X) holds.)

Proposition 4.6. *If the type of P_2 is (4,4,1) in V, then P_{11} is equidistant from four points on one of the 2-dimensional spheres S_1 and S_2.*

Proof. Since the type of P_{11} is (4,4,1) in V and $P_2P_{11} = c$, the distance c corresponds to 1 or 4 of type (4,4,1). If c corresponds to 1 of type (4,4,1), then $P_iP_{11} \neq c$ for $i = 3, \dots, 10$. Considering $\Delta P_2P_iP_{11}$, we have $P_3P_{11} = P_4P_{11} = P_5P_{11} = P_6P_{11} = a$ and $P_7P_{11} = P_8P_{11} = P_9P_{11} = P_{10}P_{11} = b$. Thus this proposition holds.

On the other hand, if c corresponds to 4 of type (4,4,1), then for $j = 3, \dots, 10$, there are exactly three points such that $P_jP_{11} = c$. We may assume that $P_3P_{11} = c$. We have three means to select the other two points.

(i) $P_4P_{11} = P_5P_{11} = c$. (Both points are on S_1.)

(ii) $P_4P_{11} = P_7P_{11} = c$. (One is on S_1 and the other is on S_2.)

(iii) $P_7P_{11} = P_8P_{11} = c$. (Both points are on S_2.)

In the case (i), considering $\Delta P_2P_kP_{11}$ for $k = 6, \dots, 10$, we have $P_6P_{11} = a$ and $P_7P_{11} = P_8P_{11} = P_9P_{11} = P_{10}P_{11} = b$. Thus this proposition holds for S_2. In the case (ii), considering $\Delta P_2P_lP_{11}$ for $l = 5, 6, 8, 9, 10$, we have $P_5P_{11} = P_6P_{11} = a$ and $P_8P_{11} = P_9P_{11} = P_{10}P_{11} = b$. Then the type of P_{11} is (4,3,2), not (4,4,1). This is a contradiction. In the case (iii), considering $\Delta P_2P_mP_{11}$ for $m = 4, 5, 6, 9, 10$, we have $P_4P_{11} = P_5P_{11} = P_6P_{11} = a$ and $P_9P_{11} = P_{10}P_{11} = b$. Then the type of P_{11} is (4,3,2), not (4,4,1). This is a contradiction.

Therefore P_{11} is equidistant from four points on one of the 2-dimensional spheres S_1 and S_2. □

Proposition 4.7. *If the type of P_2 is (4,4,1) in V, then the condition* (X) *holds.*

Proof. By Proposition 4.6, P_{11} is equidistant from four points on one of the 2-dimensional spheres S_1 and S_2. We may assume that it is S_1. Moreover we have $P_iP_3 = P_iP_4 = P_iP_5 = P_iP_6$ for $i = 1, 2$ by the assumption. Thus three points P_1, P_2, P_{11} are equidistant from $P_3, \dots, P_6$ on the 2-dimensional sphere S_1. If $P_3, \dots, P_6$ are on a plane, then they are on a circle; the condition

(X) holds. On the other hand, if $P_3, \ldots, P_6$ are not on a plane, then P_1, P_2, and P_{11} are on a line. By Corollary 3.7, this is a contradiction.

Therefore if the type of P_2 is (4,4,1) in V, then the condition (X) holds. □

Summing up the results of Propositions 4.1, 4.2, 4.5, and 4.7, we have the following.

Lemma 4.8. *For any 11-point isosceles set in $\mathbb{R}^4$ in which P_1 is of type* (6.1)*, the condition* (X) *holds.*

5. Case (B) in Lemma 3.1

We consider the case (B) in Lemma 3.1. We see that at least eight points in an 11-point isosceles set are distributed on a 3-dimensioal sphere, and at least one point does not lie on the sphere.

Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set in which the type of P_1 satisfies the case (B). Let S be the sphere centered at P_1 with radius a and $V = \mathcal{P} \cap S = \{P_2, \ldots, P_9\}$. Let P_{11} be the point which is not on S and $P_1P_{11} = b$.

Lemma 5.1. *The condition* (X) *holds for any 11-point isosceles set in which the type of P_1 satisfies the case (B) in Lemma 3.1.*

Proof. By Proposition 3.5, eight points $P_2, \ldots, P_9$ are on one of two disjoint 2-dimensional spheres S_1 and S_2, where P_i on S_1 satisfies $P_iP_{11} = a$ and P_j on S_2 satisfies $P_jP_{11} = b$ (consider $\Delta P_1P_kP_{11}$ for $k = 2, \ldots, 9$).

If more than or equal to six points lie on one sphere, then the condition (X) holds by Proposition 3.8. So we consider the following cases.

(i) Five points lie on one sphere; the other three points lie on the other sphere.

(ii) Four points lie on one sphere; the other four points lie on the other sphere.

We consider the case (i). We may suppose that $P_2, \ldots, P_6$ are on S_1 and P_7, P_8, P_9 are on S_2. Let $V_1 = V \cap S_1 = \{P_2, \ldots, P_6\}$ and $V_2 = V \cap S_2 = \{P_7, P_8, P_9\}$.

Here the 10-point set $\{P_1, \ldots, P_9, P_{11}\}$ is not a 2-distance set, because P_1 is of type (8,1) in it, not of type (6,3) in the Petersen graph. Hence it is an s-distance set ($s \geq 3$), there exists a pair of points in $\{P_2, \ldots, P_9\}$ whose distance is c, that is, distinct from a and b. If $P_iP_j = c$ holds for some $P_i \in V_1$ and $P_j \in V_2$, then $\Delta P_iP_jP_{11}$ is scalene. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in V_1,\ P_j \in V_2. \tag{5.1}$$

So c is the distance between a pair of distinct points on the same 2-dimensional sphere.

We suppose that c is the distance between a pair of distinct points on S_2. Without loss of generality we may assume $P_7P_8 = c$. For $P_i \in V_1$ we consider $\Delta P_iP_7P_8$. Since P_iP_7 and P_iP_8 are of length a or b by (5.1), we have $P_iP_7 = P_iP_8$. Thus five points $P_2, \ldots, P_6$ are on the hyperplane perpendicularly bisecting P_7P_8 and the 2-dimensional sphere S_1. The intersection of them is a circle. Hence the condition (X) holds.

On the other hand, we suppose that c is the distance between a pair of distinct points on S_1. Without loss of generality we may assume $P_2P_3 = c$.

Next we suppose that there exist more than or equal to two pairs of points on S_1 whose distances are distinct from a and b. One is $P_2P_3 = c$. We have two cases as the second pair whose distance is distinct from a and b.

Case 1. $P_2P_3 = c$ and $P_4P_5 = d$ ($d \neq a$, $d \neq b$, but we can admit $c = d$).

Let $P_j \in V_2$ and consider $\Delta P_2P_3P_j$. Because P_2P_j and P_3P_j are of length a or b by (5.1), we have $P_2P_j = P_3P_j$. When we consider $\Delta P_4P_5P_j$ similarly, we have $P_4P_j = P_5P_j$. Thus three points P_7, P_8 and P_9 are on the hyperplane perpendicularly bisecting P_2P_3, the hyperplane perpendicularly bisecting P_4P_5, and the 2-dimensional sphere S_2. For the intersection of them, there are two cases:

(α) because two hyperplanes are same, the intersection is a circle.

(β) two points.

In the case (α), because $P_2, \ldots, P_5$ are on the 2-dimensional sphere S_1, the segment P_2P_3 and the segment P_4P_5 are mutually parallel. Then there is a plane that contains $P_2, \ldots, P_5$. So they are on a circle; the condition (X) holds.

In the case (β), we cannot put one of P_7, P_8, and P_9. This is a contradiction.

Case 2. $P_2P_3 = c$ and $P_2P_4 = d$ ($d \neq a$, $d \neq b$, but we can admit $c = d$).

We can repeat the same discussion. (But the case (α) does not exist, only the case (β) exists.)

Hence we suppose that there is exactly one pair P_2P_3 which is distinct from a and b in V_1. When we consider $\Delta P_2P_3P_k$ for $k = 1, 4, \ldots, 9, 11$, $P_2P_k = P_3P_k$ holds by the configuration hypothesis. Thus eight points $P_1, P_4, \ldots, P_9$ and P_{11} are on the hyperplane perpendicularly bisecting P_2P_3. This hyperplane is a 3-dimensional Euclidean space. Moreover $\{P_1, P_4, \ldots, P_9, P_{11}\}$ is a 2-distance set with the distances a and b. But we know that there exists no n-point 2-distance set in $\mathbb{R}^3$ for $n \geq 7$. This is a contradiction.

We consider the case (ii). If we repeat this discussion similarly, then we can see that the condition (X) holds. □

6. Case (C) in Lemma 3.1

We consider the case (C) in Lemma 3.1. We see that seven points in an 11-point isosceles set are distributed on a 3-dimensioal sphere, another at least two points are distributed on another 3-dimensioal sphere, where these are concentric spheres. The center of the spheres is in it.

Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set in which the type of P_1 satisfies the case (C). P_1 will denote the common center of the two spheres, which we will call S_1 (on which $P_2, \ldots, P_8$ are), S_2 (on which P_9 and P_{10} are), radii a, b, respectively.

Lemma 6.1. *The condition* (X) *holds for any 11-point isosceles set* $\mathcal{P}$ *in which the type of* P_1 *satisfies the case (C) in Lemma 3.1.*

Proof. The 10-point set $\{P_1, \ldots, P_{10}\}$ is not a 2-distance set, because P_1 is of type (7,2) in it, not of type (6,3) in the Petersen graph. Hence it is an s-distance set ($s \geq 3$), there exists a pair of points in $\{P_2, \ldots, P_{10}\}$ whose distance is c, that is, distinct from a and b. If $P_iP_j = c$ holds for some $P_i \in S_1$ and $P_j \in S_2$, then $\Delta P_1P_iP_j$ is scalene. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in S_1,\ P_j \in S_2. \tag{6.1}$$

So c is the distance between a pair of distinct points on the same 3-dimensional sphere.

We suppose that c is the distance between a pair of distinct points on S_2, that is, $P_9P_{10} = c$. For $P_i \in S_1$ we consider $\Delta P_iP_9P_{10}$. Since P_iP_9 and P_iP_{10} are of length a or b by (6.1), we have $P_iP_9 = P_iP_{10}$. Thus seven points $P_2, \ldots, P_8$ are on the hyperplane perpendicularly bisecting P_9P_{10} and the 3-dimensional sphere S_1. The intersection of them is a 2-dimensional sphere. By Proposition 3.8, the condition (X) holds.

Thus we suppose that c is the distance between a pair of distinct points on S_1. By Proposition 3.5, seven points $P_2, \ldots, P_8$ are on one of two disjoint 2-dimensional spheres S_{11} and S_{12}, where P_i on S_{11} satisfies $P_iP_9 = a$ and P_j on S_{12} satisfies $P_jP_9 = b$ (consider $\Delta P_1P_kP_9$ for $k = 2, \ldots, 8$).

If $P_iP_j = c$ holds for some $P_i \in S_{11}$ and $P_j \in S_{12}$, then $\Delta P_iP_jP_9$ is scalene. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in S_{11},\ P_j \in S_{12}. \tag{6.2}$$

So c is the distance between a pair of distinct points on the same 2-dimensional sphere.

If more than or equal to six points lie on one sphere, then the condition (X) holds by Proposition 3.8. So we consider the following cases:

(i) Five points lie on one sphere; the other two points lie on the other sphere.

(ii) Four points lie on one sphere; the other three points lie on the other sphere.

We consider the case (i). We may suppose that $P_2, \ldots, P_6$ are on S_{11} and P_7, P_8 are on S_{12}.

We suppose that c is the distance between a pair of distinct points on S_{12}, that is, $P_7P_8 = c$. We consider $\Delta P_iP_7P_8$ for $P_i \in S_{11}$. By (6.2), we have $P_iP_7 = P_iP_8$. Thus five points $P_2, \ldots, P_6$ are on the hyperplane perpendicularly bisecting P_7P_8 and the 2-dimensional sphere S_{11}. The intersection of them is a circle. Hence the condition (X) holds.

On the other hand, we suppose that c is the distance between a pair of distinct points on S_{11}. Without loss of generality we may assume $P_2P_3 = c$.

Next we suppose that there exist more than or equal to two pairs of points on S_{11} whose distances are distinct from a and b. One is $P_2P_3 = c$. We have two cases as the second pair whose distance is distinct from a and b.

Case 1. $P_2P_3 = c$ and $P_4P_5 = d$ ($d \neq a$, $d \neq b$, but we can admit $c = d$).

Let $P_j \in S_{12} \cup S_2$ and consider $\Delta P_2P_3P_j$. By (6.1) and (6.2), we have $P_2P_j = P_3P_j$. When we consider $\Delta P_4P_5P_j$ similarly, we have $P_4P_j = P_5P_j$. For P_1 we have $P_1P_2 = P_1P_3$ and $P_1P_4 = P_1P_5$. Thus five point P_1, P_7, P_8, P_9, and P_{10} are on the hyperplane perpendicularly bisecting P_2P_3 and the hyperplane perpendicularly bisecting P_4P_5. For the intersection of them, there are two cases:

(α) since two hyperplanes are same, the intersection is a 3-dimensional Euclidean space;

(β) a 2-dimensional Euclidean space.

In the case (α), since $P_2, \dots, P_5$ are on the 2-dimensional sphere S_{11}, the segment P_2P_3 and the segment P_4P_5 are mutually parallel. Then there is a plane that contains $P_2, \dots, P_5$. So they are on a circle; the condition (X) holds.

In the case (β), $\{P_1, P_7, P_8, P_9, P_{10}\}$ is a 5-point isosceles set in $\mathbb{R}^2$. We know that there exist three 5-point isosceles sets in $\mathbb{R}^2$ up to isomorphisms. They are four points of a square and its center, five points of a regular pentagon, and four points of a regular pentagon and its center. So four points in $\{P_1, P_7, P_8, P_9, P_{10}\}$ lie on a circle; the condition (X) holds.

Case 2. $P_2P_3 = c$ and $P_2P_4 = d$ ($d \neq a$, $d \neq b$, but we can admit $c = d$).

We can repeat the same discussion. (But the case (α) does not exist, only the case (β) exists.)

Hence we suppose that there is exactly one pair P_2P_3 whose distance is distinct from a and b in $\{P_1, \dots, P_{10}\}$. When we consider $\Delta P_2P_3P_k$ for $k = 1, 4, \dots, 10$, $P_2P_k = P_3P_k$ holds by the configuration hypothesis. Thus eight points $P_1, P_4, \dots, P_9$, and P_{10} are on the hyperplane perpendicularly bisecting P_2P_3. This hyperplane is a 3-dimensional Euclidean space. Moreover $\{P_1, P_4, \dots, P_{10}\}$ is a 2-distance set with the distances a and b. But there exists no n-point 2-distance set in $\mathbb{R}^3$ for $n \geq 7$. This is a contradiction.

We consider the case (ii). If we repeat this discussion similarly, then we see that the condition (X) holds. □

7. Case (D) in Lemma 3.1

Lemma 7.1. *The condition* (X) *holds for any 11-point isosceles set in which the type of P_1 satisfies the case (D) in Lemma 3.1.*

Proof. Let $\mathcal{P} = \{P_1, \dots, P_{11}\}$ be an 11-point isosceles set. When the type of P_1 is (6,4) or (6,3,1), P_1 will denote the common center of the two spheres, which we will call S_1 (on which $P_2, \dots, P_7$ are), S_2 (on which P_8, P_9, and P_{10} are).

Let $\mathcal{P}' = \{P_1, \dots, P_{10}\}$. $\mathcal{P}'$ can be the 2-distance set X' mentioned in Section 2. Since X' contains a square, the condition (X) holds.

Hence we may suppose that $\mathcal{P}'$ is an s-distance set ($s \geq 3$). In this case, we can show that the condition (X) holds by repeating the similar discussion as the proof of Lemma 6.1.

When the type of P_1 is (5,5) or (5,4,1), we can show that the condition (X) holds by repeating the similar discussion as the proof of Lemma 6.1. □

8. Case (E) in Lemma 3.1

We consider the case (E) in Lemma 3.1. Let $\mathcal{P} = \{P_1, \dots, P_{11}\}$ be an 11-point isosceles set in which the type of P_1 is (7,1,1,1). We may assume that $P_1P_2 = P_1P_3 = \cdots = P_1P_8 = a$, $P_1P_9 = b$, $P_1P_{10} = c$, and $P_1P_{11} = d$. Let $X_1 = \{P_1, \dots, P_8, P_9\}$, $X_2 = \{P_1, \dots, P_8, P_{10}\}$, and $X_3 = \{P_1, \dots, P_8, P_{11}\}$.

Proposition 8.1. *For $X_1, \dots, X_3$ above, if 2-distance sets exist, then the number of them is at most one.*

Proof. We suppose that X_1 and X_2 are 2-distance sets. We may prove that this leads a contradiction.

We have $A(X_1) = \{a,b\}$ and $A(X_2) = \{a,c\}$ by the hypothesis above. For $i,j = 2,\ldots,8$ $(i \neq j)$, P_iP_j must be a, b, or c.

If $P_iP_j = b$, then $A(X_2) \neq \{a,c\}$ for X_2. If $P_iP_j = c$, then $A(X_1) \neq \{a,b\}$ for X_1. So we have $P_iP_j = a$. Hence A $(\{P_1,\ldots,P_8\}) = \{a\}$, $\{P_1,\ldots,P_8\}$ is a 1-distance set. But there is no 8-point 1-distance set in $\mathbb{R}^4$. This is a contradiction.

Therefore the number of 2-distance sets is at most one. □

Lemma 8.2. *The condition* (X) *holds for any 11-point isosceles set* $\mathcal{P}$ *in which the type of* P_1 *is (7,1,1,1).*

Proof. By Proposition 8.1, at least two sets of $X_1,\ldots,X_3$ are s-distance sets ($s \geq 3$). We may suppose X_1 and X_2 are s-distance sets. Espesially we notice that X_1 is an s-distance set. Thus there is a distance apart of a pair of points in X_1 which is distinct from a and b. This is one of c, d, and e, where e is distinct from a, b, c, and d. We may assume that it is c.

Let S be the sphere centered at P_1 with radius a and $V = X_1 \cap S = \{P_2,\ldots,P_8\}$. By Proposition 3.5, seven points $P_2,\ldots,P_8$ are on one of two disjoint 2-dimensional spheres S_1 and S_2, where P_i on S_1 satisfies $P_iP_9 = a$ and P_j on S_2 satisfies $P_jP_9 = b$ (consider $\Delta P_1P_kP_9$ for $k = 2,\ldots,8$).

We remark that there is the distance c in V. If $P_iP_j = c$ holds for some $P_i \in S_1$ and $P_j \in S_2$, then $\Delta P_iP_jP_9$ is scalene. Thus $P_iP_j = a$ or b for any $P_i \in S_1$ and $P_j \in S_2$. So c is the distance between a pair of distinct points on the same 2-dimensional sphere.

If more than or equal to six points lie on one sphere, then the condition (X) holds by Proposition 3.8. So we consider the following cases:

(I) Five points lie on one sphere; the other two points lie on the other sphere.

(II) Four points lie on one sphere; the other three points lie on the other sphere.

We consider the case (I). We may suppose that $P_2,\ldots,P_6$ are on S_1 and P_7, P_8 are on S_2.

If c is the distance between a pair of distinct points on S_2, then we can show that the condition (X) holds by the similar discussion as the proof of Lemma 5.1.

Thus we suppose that c is the distance between a pair of distinct points on S_1. Without loss of generality we may assume $P_2P_3 = c$. For $P_2,\ldots,P_6$ on S_1, we consider 5-point graphs in Table 2 again. Edges in a graph represent the distance, that is, distinct from a and b. We regard the others, (i.e., transparent edges) as the distances a and b. Here we need not consider the graph which has no edge, because we suppose that there is at least one pair whose distance is distinct from a and b.

We observe the cases (i)–(iv) in the proof of Proposition 4.3 similarly. In any case, we can show that the condition (X) holds.

In the case (II), we can apply the similar discussion as the proof of Lemma 5.1. If we apply it, then we see that the condition (X) holds. □

9. Case (F) in Lemma 3.1

We consider the case (F) in Lemma 3.1. Let $\mathcal{P} = \{P_1,\ldots,P_{11}\}$ be an 11-point isosceles set in which the type of P_1 is (6,2,2). We may assume that $P_1P_2 = P_1P_3 = \cdots = P_1P_7 = a$, $P_1P_8 = P_1P_9 = b$, and $P_1P_{10} = P_1P_{11} = c$. Let $X_1 = \{P_1,\ldots,P_7,P_8,P_9\}$ and $X_2 = \{P_1,\ldots,P_7,P_{10},P_{11}\}$.

Proposition 9.1. *For X_1 and X_2 above, at least one of them is an s-distance set* $(s \geq 3)$.

Proof. We can show that the condition (X) holds by repeating the similar discussion as the proof of Proposition 8.1. □

Lemma 9.2. *The condition* (X) *holds for any 11-point isosceles set $\mathcal{P}$ in which the type of P_1 is (6,2,2).*

Proof. By Proposition 9.1, at least one of X_1 and X_2 is an s-distance set $(s \geq 3)$. We may suppose X_1 is an s-distance set.

P_1 will denote the common center of the two spheres, which we will call S_1 (on which $P_2, \ldots, P_7$ are), S_2 (on which P_8 and P_9 are), radii a, b, respectively.

There is a distance apart of a pair of points in $\{P_2, \ldots, P_9\}$ which is distinct from a and b. This is c or d, where d is distinct from a, b, and c. We may assume that it is c. If $P_iP_j = c$ holds for some $P_i \in S_1$ and $P_j \in S_2$, then $\Delta P_1P_iP_j$ is scalene. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in S_1,\ P_j \in S_2. \tag{9.1}$$

So c is the distance between a pair of distinct points on the same 3-dimensional sphere.

If c is the distance between a pair of distinct points on S_2, then we can show that the condition (X) holds by the similar discussion as the proof of Lemma 6.1.

Thus we suppose that c is the distance between a pair of distinct points on S_1. By Proposition 3.5, six points $P_2, \ldots, P_7$ are on one of two disjoint 2-dimensional spheres S_{11} and S_{12}, where P_i on S_{11} satisfies $P_iP_9 = a$ and P_j on S_{12} satisfies $P_jP_9 = b$ (consider $\Delta P_1P_kP_9$ for $k = 2, \ldots, 7$).

If $P_iP_j = c$ holds for some $P_i \in S_{11}$ and $P_j \in S_{12}$, then $\Delta P_iP_jP_9$ is scalene. Thus

$$P_iP_j = a \text{ or } b \quad \text{for any } P_i \in S_{11},\ P_j \in S_{12}. \tag{9.2}$$

So c is the distance between a pair of distinct points on the same 2-dimensional sphere.

If six points lie on one sphere, then the condition (X) holds by Proposition 3.8. So we consider the following cases:

(I) Five points lie on one sphere; the other one point lies on the other sphere.

(II) Four points lie on one sphere; the other two points lie on the other sphere.

(III) Three points lie on one sphere, the other three points lie on the other sphere.

As for the case (I), we can apply the similar discussion as the proof of the case (I) of Lemma 8.2. Thus the condition (X) holds in the case (I).

In the case (II), we can apply the similar discussion as the proof of Lemma 6.1 in Case (C). If we apply it, then we see that the condition (X) holds.

We consider the case (III). We suppose that $P_2, \ldots, P_4$ are on S_{11} and $P_5, \ldots, P_7$ are on S_{12}.

We may suppose that c is the distance between a pair of distinct points on S_{12}. Without loss of generality we may assume $P_6P_7 = c$. Next we suppose that there exist more than or equal to two pairs of points on S_{12} whose distances are distinct from a and b. One is $P_6P_7 = c$. Without loss of generality the second is $P_5P_7 = e$ ($e \neq a$, $e \neq b$, but we can admit $c = e$).

Let $P_i \in S_{11} \cup S_2$ and consider $\Delta P_6P_7P_i$. By (9.1) and (9.2), we have $P_6P_i = P_7P_i$. When we consider $\Delta P_5P_7P_i$ similarly, we have $P_5P_i = P_7P_i$. For P_1 we have $P_1P_6 = P_1P_7$ and $P_1P_5 = P_1P_7$. Thus six points P_1, P_2, P_3, P_4, P_8, and P_9 are on the hyperplane perpendicularly bisecting P_6P_7 and the hyperplane perpendicularly bisecting P_5P_7. The intersection of them is a 2-dimensional Euclidean space. Then $\{P_1, P_2, P_3, P_4, P_8, P_9\}$ is a 6-point isosceles set in $\mathbb{R}^2$. There exist a unique 6-point isosceles set in $\mathbb{R}^2$ up to isomorphisms and it contains four points on a circle. Thus the condition (X) holds.

Hence we suppose that there is exactly one pair P_6P_7 whose distance is distinct from a and b on S_{12}. If we repeat the similar discussion above, then there is also at most one pair whose distance is distinct from a and b on S_{11}. Without loss of generality this is P_2P_3.

When we consider $\Delta P_6P_7P_k$ for $k = 1, \ldots, 5, 8, 9$, $P_6P_k = P_7P_k$ holds by (9.1), (9.2), and the configuration hypothesis. Thus seven points $P_1, \ldots, P_5, P_8$, and P_9 are on the hyperplane perpendicularly bisecting P_6P_7. This hyperplane is a 3-dimensional Euclidean space. Particularly $\{P_1, P_3, P_4, P_5, P_8, P_9\}$ is a 6-point 2-distance set in $\mathbb{R}^3$ with distances a and b. There exist exactly six 6-point 2-distance sets in $\mathbb{R}^3$. Any set contains four points lying on a circle. Hence the condition (X) holds.

Therefore if the type of P_1 is (6,2,2), then the condition (X) holds. □

10. Case (G) in Lemma 3.1

We consider the case (G) in Lemma 3.1. Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set in which the type of P_1 is (6,2,1,1). We may assume that $P_1P_2 = P_1P_3 = \cdots = P_1P_7 = a$, $P_1P_8 = P_1P_9 = b$, $P_1P_{10} = c$, and $P_1P_{11} = d$. Let $X_1 = \{P_1, \ldots, P_7, P_8, P_9\}$, $X_2 = \{P_1, \ldots, P_7, P_{10}\}$, and $X_3 = \{P_1, \ldots, P_7, P_{11}\}$.

Proposition 10.1. *For $X_1, \ldots, X_3$ above, if 2-distance sets exist, then the number of them is at most one.*

Proof. We can show this proposition by repeating the similar discussion as Proposition 8.1. □

Lemma 10.2. *The condition (X) holds for any 11-point isosceles set $\mathcal{P}$ in which the type of P_1 is (6,2,1,1).*

Proof. By Proposition 10.1, at least two sets of $X_1, \ldots, X_3$ are s-distance sets ($s \geq 3$). If X_1 is an s-distance set, then we can show that the condition (X) holds by repeating the similar discussion as Lemma 9.2. Hence we may assume that X_1 is a 2-distance set and that X_2 and X_3 are s-distance sets. Since $|A(\{P_2, \ldots, P_7\})| \geq 2$ and X_1 is a 2-distance set with distances a and b, it holds that

$$A(\{P_2, \ldots, P_7\}) = \{a, b\}. \tag{10.1}$$

Thus b is the third distance in X_2 and X_3.

Let S be the sphere centered at P_1 with radius a. By Proposition 3.5, six points $P_2, \dots, P_7$ are on one of two disjoint 2-dimensional spheres S_1 and S_2, where P_i on S_1 satisfies $P_iP_{10} = a$ and P_j on S_2 satisfies $P_jP_{10} = c$ (consider $\Delta P_1P_kP_{10}$ for $k = 2, \dots, 7$).

We remark that there is the distance b in $\{P_2, \dots, P_7\}$. If $P_iP_j = b$ holds for some $P_i \in S_1$ and $P_j \in S_2$, then $\Delta P_iP_jP_{10}$ is scalene. Thus $P_iP_j = a$ or c for any $P_i \in S_1$ and $P_j \in S_2$. Combining this and (10.1), the following condition holds:

$$P_iP_j = a \quad \text{for any } P_i \in S_1,\ P_j \in S_2. \tag{10.2}$$

So b is the distance between a pair of distinct points on the same 2-dimensional sphere.

If six points lie on one sphere, then the condition (X) holds by Proposition 3.8. So we consider the following cases.

(I) Five points lie on one sphere; the other one point lies on the other sphere.

(II) Four points lie on one sphere; the other two points lie on the other sphere.

(III) Three points lie on one sphere, the other three points lie on the other sphere.

As for the case (I), we can apply the similar discussion as the proof of the case (I) of Lemma 8.2. Thus the condition (X) holds in the case (I).

In the case (II), we can apply the similar discussion as the proof of Lemma 6.1. If we apply it, then we see that the condition (X) holds.

We consider the case (III). We suppose that $P_2, \dots, P_4$ are on S_1 and $P_5, \dots, P_7$ are on S_2.

We may suppose that b is the distance between a pair of distinct points on S_2. Without loss of generality we may assume $P_6P_7 = b$. Next we suppose that there exist more than or equal to two pairs of points on S_2 whose distances are b. In this assumption, we can apply the similar discussion as the proof of the case (III) of Lemma 9.2. If we apply it, then we see that the condition (X) holds.

Hence we suppose that there is exactly one pair P_6P_7 whose distance is b on S_2. If we repeat the similar discussion above, then there is also at most one pair whose distance is b on S_1. Without loss of generality this is P_2P_3.

When we consider $\Delta P_6P_7P_k$ for $k = 2, \dots, 5$, $P_6P_k = P_7P_k$ holds by (10.2) and the configuration hypothesis. Thus $P_2, \dots, P_5$ are on the hyperplane perpendicularly bisecting P_6P_7 and on S. The intersection of them is a 2-dimensional sphere. By (10.1), (10.2), and the assumption, $P_2P_i = P_3P_i = P_4P_i = P_5P_i = a$ for $i = 1, 6, 7$. Thus P_1, P_6, and P_7 are equidistant from $P_2, \dots, P_5$ on a 2-dimensional sphere. If $P_2, \dots, P_5$ are on a plane, then they are on a circle; the condition (X) holds. On the other hand, if $P_2, \dots, P_5$ are not on a plane, then P_1, P_6, and P_7 are on a line. By Corollary 3.7, this is a contradiction.

Therefore if the type of P_1 is (6,2,1,1), then the condition (X) holds. □

11. Case (H) in Lemma 3.1

We consider the case (H) in Lemma 3.1. Let $\mathcal{P} = \{P_1, \dots, P_{11}\}$ be an 11-point isosceles set in which the type of P_1 is (6,1,1,1,1). We may assume that $P_1P_2 = P_1P_3 = \cdots = P_1P_7 = a$, $P_1P_8 = b$, $P_1P_9 = c$, $P_1P_{10} = d$, and $P_1P_{11} = e$.

We consider the sum of all vertex-numbers in $\mathcal{P}$. Since P_1 has the largest vertex-number in $\mathcal{P}$, $V(P_1) + \cdots + V(P_{11}) \le 11 \times \{\binom{6}{2} + \binom{1}{2} + \binom{1}{2} + \binom{1}{2} + \binom{1}{2}\} = 165$. On the other hand,

$V(P_1) + \cdots + V(P_{11}) \geq 165$ by (3.1). Thus $V(P_1) + \cdots + V(P_{11}) = 165$. Let α be the number of regular triangles in $\mathcal{P}$. Then $\alpha = 0$ holds by (3.2). Moreover $V(P_i) = 15$ holds for any $P_i \in \mathcal{P}$; the type of P_i is (6,1,1,1,1).

Lemma 11.1. *There is no 11-point isosceles set in which the type of P_1 is (6,1,1,1,1).*

Proof. We notice that the type of P_2 is (6,1,1,1,1). So the distance a corresponds to 6 or 1 of type (6,1,1,1,1). If a corresponds to 6, then at least one of $P_2P_3, \ldots, P_2P_7$ is a. We may suppose that $P_2P_3 = a$. Then $\Delta P_1P_2P_3$ is a regular triangle with the distance a. This contradicts $\alpha = 0$. Thus a corresponds to 1. Then $P_2P_8 = b$, $P_2P_9 = c$, $P_2P_{10} = d$, and $P_2P_{11} = e$ hold by considering $\Delta P_1P_2P_i$ for $i = 8, \ldots, 11$. This means that one of b, c, d, and e corresponds to 6 of type (6,1,1,1,1). We may assume that this is b. Then $P_2P_3 = \cdots = P_2P_8 = b$.

Next we notice that the type of P_3 is (6,1,1,1,1). We see that a corresponds to 1 of type (6,1,1,1,1) by repeating the discussion for P_2. Thus $P_3P_8 = b$, $P_3P_9 = c$, $P_3P_{10} = d$, and $P_3P_{11} = e$ hold by considering $\Delta P_1P_3P_i$ for $i = 8, \ldots, 11$, b corresponds to 6 of type (6,1,1,1,1). Then $P_2P_3 = P_3P_4 = \cdots = P_3P_8 = b$. But $\Delta P_2P_3P_4$ is a regular triangle with the distance b. This contradicts $\alpha = 0$.

Therefore there is no 11-point isosceles set in which the type of P_1 is (6,1,1,1,1). □

Therefore combining Lemmas 3.1, 4.8, 5.1, 6.1, 7.1, 8.2, 9.2, 10.2, and 11.1, we have Lemma 3.2.

By Lemma 3.2, at least four points, say $P_1, \ldots, P_4$, in $\mathcal{P}$ lie on a circle. We keep to this notation of suffixes in what follows. Lemma 11.2 can be proved by the same method given in the proof of Lemma 18 in Croft [3].

Lemma 11.2. *P_1, P_2, P_3, P_4 are either all the vertices of a square, or four of the vertices of a regular pentagon.*

From now on, we observe two cases in Lemma 11.2 respectively.

12. Observation of 11-Point Isosceles Sets in $\mathbb{R}^4$ Containing Four Points of a Regular Pentagon

Proposition 12.1. *Suppose an n-point isosceles set $\mathcal{P} = \{P_1, \ldots, P_n\}$ contains four vertices of a regular pentagon, P_1, P_2, P_3, P_4 (in order, with the "gap" between P_4 and P_1). We may suppose that $P_1 = ((-1-\sqrt{5})/4, \sqrt{(10+2\sqrt{5})}/4, 0, 0)$, $P_2 = (-1/2, 0, 0, 0)$, $P_3 = (1/2, 0, 0, 0)$, $P_4 = ((1+\sqrt{5})/4, \sqrt{(10+2\sqrt{5})}/4, 0, 0)$. (The mid-point of P_2P_3 is the origin. Each side of this regular pentagon is 1.)*

Then the only other possible coordinates for the remaining points are as follows:

(i) $(0, (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}, z, w)$, *where z and w are arbitrary,*

(ii) $T = (0, (5+3\sqrt{5})/2\sqrt{(10+2\sqrt{5})}, 0, 0)$, *the remaining vertex of the pentagon, or*

(iii) $(0, (1-\sqrt{5})/2\sqrt{(10+2\sqrt{5})}, z, w)$, *where z and w satisfy $z^2 + w^2 = (\sqrt{(10+2\sqrt{5})}/2\sqrt{5})^2$.*

Proof. We expand the proof of Lemma 22 in Croft [3] into $\mathbb{R}^4$, then we obtain this proposition. □

Table 3: The distribution of the remaining points.

	The number of points satisfying (i)	(ii)	The number of points satisfying (iii)
$\langle 1\rangle$	6	T	0
$\langle 2\rangle$	6		1
$\langle 3\rangle$	5		2
$\langle 4\rangle$	4		3
$\langle 5\rangle$	3		4
$\langle 6\rangle$	2		5

Proposition 12.2. *Let Q be a point satisfying (iii) in the previous proposition. Then no n-point isosceles set can contain P_1, P_2, P_3, P_4, T, and Q.*

Proof. It holds that $QT = \sqrt{(10+2\sqrt{5})}/2$. Then ΔP_1QT is scalene with 1, $(1+\sqrt{5})/2$, $\sqrt{(10+2\sqrt{5})}/2$. This is contrary to the configuration hypothesis.

Therefore no n-point isosceles set can contain P_1, P_2, P_3, P_4, T, and Q. □

We observe the detail for $n = 11$ in Proposition 12.1. The space which satisfies the case (i) in Proposition 12.1 is a plane and that satisfying the case (iii) in Proposition 12.1 is a circle. The maximum cardinality of isosceles sets in $\mathbb{R}^2$ is 6 and we see that that on a circle is 5. We consider them and Proposition 12.2. If an 11-point isosceles set exists, then it satisfies one row of Table 3.

Proposition 12.3. *Any 11-point isosceles set in $\mathbb{R}^4$ satisfying $\langle 1\rangle$ in Table 3 is isomorphic to Y in Theorem 1.1.*

Proof. Any 11-point isosceles set $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ in $\mathbb{R}^4$ satisfying $\langle 1\rangle$ in Table 3 contains all the vertices of a regular pentagon. And the other six points are in a 2-dimensional Euclidean space. Then they are all the vertices of a regular pentagon and its center. Hence we can fix $P_1, P_2, P_3, P_4, P_5 = T$, and $P_6 = (0, (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}, 0, 0)$, we consider the configuration of the other five points which form a regular pentagon in the 2-dimensional Euclidean space $x = 0$, $y = (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}$.

Let $P_i = (0, (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}, z, w)$ for $i \in \{7, \ldots, 11\}$. We consider $\Delta P_5P_6P_i$, we have $(P_5P_6)^2 = ((1+\sqrt{5})/\sqrt{(10+2\sqrt{5})})^2$, $(P_6P_i)^2 = z^2 + w^2 (> 0)$, and $(P_5P_i)^2 = (P_5P_6)^2 + (P_6P_i)^2$. Since $P_5P_i > P_5P_6$ and $P_5P_i > P_6P_i$, $P_5P_6 = P_6P_i$ holds by the configuration hypothesis. Thus $P_7, \ldots, P_{11}$ which form a regular pentagon are on the circle satisfying $z^2 + w^2 = ((1+\sqrt{5})/\sqrt{(10+2\sqrt{5})})^2$. This 11-point isosceles set $\mathcal{P}$ is isomorphic to Y in Theorem 1.1. □

Next we observe $\langle 5\rangle$ and $\langle 6\rangle$ in Table 3. For any 11-point isosceles set $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ in $\mathbb{R}^4$ satisfying $\langle 5\rangle$ or $\langle 6\rangle$ in Table 3, the other seven points $P_5, \ldots, P_{11}$ are in the 3-dimensional Euclidean space $x = 0$, and four points in $\{P_5, \ldots, P_{11}\}$ are on a circle. We may assume that they are $P_5, \ldots, P_8$. Then they are all the vertices of a square, or four points of a regular pentagon. Moreover we may assume that P_{10} and P_{11} are in the 2-dimensional Euclidean space $x = 0$, $y = (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}$.

The following two propositions in $\mathbb{R}^3$ are useful for us. We quote them from Kido [11] (or Croft [3]).

Proposition 12.4. *Let four points P_1, P_2, P_3, P_4 of an n-point isosceles set in $\mathbb{R}^3$ form a square. We may suppose that $P_1 = (-1/2, -1/2, 0)$, $P_2 = (1/2, -1/2, 0)$, $P_3 = (1/2, 1/2, 0)$, $P_4 = (-1/2, 1/2, 0)$. And let the center $(0,0,0)$ be O, and let the plane that contains the square be Π.*

Then the only other possible situations for the remaining points are:

(i) *on the vertical line L through O, or*

(ii) *at some of $Q_1, \ldots, Q_8$, where*

$$\begin{aligned} Q_1 &= \left(0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right), & Q_2 &= \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), \\ Q_3 &= \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right), & Q_4 &= \left(-\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), \\ Q_5 &= \left(0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), & Q_6 &= \left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right), \\ Q_7 &= \left(0, \frac{1}{2}, -\frac{\sqrt{3}}{2}\right), & Q_8 &= \left(-\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right). \end{aligned} \tag{12.1}$$

(The square Q_1, Q_2, Q_3, Q_4, and Q_5, Q_6, Q_7, Q_8 both have sides of length $\sqrt{2}/2$.)

Proposition 12.5. *Suppose an n-point isosceles set in $\mathbb{R}^3$ contains four vertices of a regular pentagon, P_1, P_2, P_3, P_4 (in order, with the "gap" between P_4 and P_1), lying in a horizontal plane. Then the only other possible situations for the remaining points are:*

(i) *at T the remaining vertex of the pentagon; or*

(ii) *at two points Q_1, Q_2, which are the only points Q such that ΔQP_4P_1 and ΔQP_2P_3 are both equilaternal; or*

(iii) *on the vertical line L through the center of the pentagon.*

Proposition 12.6. *Any 11-point isosceles set in $\mathbb{R}^4$ satisfies neither $\langle 5\rangle$ nor $\langle 6\rangle$ in Table 3.*

Proof. We consider when $P_5, \ldots, P_8$ form a square. If each side of the square $P_1P_2P_3P_4$ in Proposition 12.4 is $\sqrt{(10+2\sqrt{5})/2\sqrt{5}} \times \sqrt{2} = \sqrt{(5+\sqrt{5})/\sqrt{5}}$, then we can change them into $P_5, \ldots, P_8$. Now P_{10} and P_{11} are in $x = 0, y = (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}$. By Proposition 12.4, we see that there is exactly one point $(0, (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}, 0, 0)$ in $x = 0$, $y = (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}$. Thus we cannot put one of P_{10} and P_{11}. This is a contradiction.

On the other hand, we consider when $P_5, \ldots, P_8$ form four points of a regular pentagon. Here P_{10} and P_{11} are in $x = 0, y = (3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}$. By Proposition 12.5, we see that there is exactly one point $(0,(3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})},0,0)$ in $x=0, y=(3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})}$. Thus we cannot put one of P_{10} and P_{11}. This is a contradiction.

Therefore any 11-point isosceles set in $\mathbb{R}^4$ satisfies neither $\langle 5\rangle$ nor $\langle 6\rangle$ in Table 3. □

The last cases are $\langle 2\rangle$–$\langle 4\rangle$ in Table 3. For any 11-point isosceles set $\mathcal{P}=\{P_1,\ldots,P_{11}\}$ in $\mathbb{R}^4$ satisfying one of $\langle 2\rangle$–$\langle 4\rangle$ in Table 3, we may assume that P_5 satisfies (iii) in Proposition 12.1 and that $P_8,\ldots,P_{11}$ satisfy (i) in Proposition 12.1. We may suppose that $P_5=(0,(1-\sqrt{5})/2\sqrt{(10+2\sqrt{5})},0,\sqrt{(10+2\sqrt{5})/2\sqrt{5}})$ because of symmetry. We remark that $P_5,\ldots,P_{11}$ are in the 3-dimensional Euclidean space $x=0$.

Proposition 12.7. *Suppose that an 11-point isosceles set $\mathcal{P}=\{P_1,\ldots,P_{11}\}$ in $\mathbb{R}^4$ satisfying one of $\langle 2\rangle$–$\langle 4\rangle$ in Table 3, then the possible situations for $P_8,\ldots,P_{11}$ are the following:*

(I) *on the line L which satisfies $w=\sqrt{(10+2\sqrt{5})}/4\sqrt{5}$,*

(II) $R_1=(0,(3+\sqrt{5})/2\sqrt{(10+2\sqrt{5})},0,(1/20)(-5\sqrt{10+2\sqrt{5}}+\sqrt{50+10\sqrt{5}})$, *or*

(III) $R_2=(0,(3+\sqrt{5})/2\sqrt{10+2\sqrt{5}},0,(1/20)(5\sqrt{10+2\sqrt{5}}+\sqrt{50+10\sqrt{5}})$.

Proof. For $i=8,\ldots,11$, let $P_i=(0,(3+\sqrt{5})/2\sqrt{10+2\sqrt{5}},z,w)$. We consider $\Delta P_2P_5P_i$. Because $P_2P_5=1$, one of the following (a-1)–(a-3) must hold to satisfy the configuration hypothesis:

(a-1) $z^2+w^2=(5-\sqrt{5})/10$ when $P_2P_i=1$,

(a-2) $((1+\sqrt{5})/\sqrt{(10+2\sqrt{5})})^2+z^2+(w-\sqrt{(10+2\sqrt{5})/2\sqrt{5}})^2=1$ when $P_5P_i=1$,

(a-3) $w=\sqrt{(10+2\sqrt{5})}/4\sqrt{5}$ when $P_2P_i=P_5P_i$.

On the other hand, we consider $\Delta P_1P_5P_i$. Since $P_1P_5=(1+\sqrt{5})/2$, one of the following (b-1)–(b-3) must hold to satisfy the configuration hypothesis:

(b-1) $z^2+w^2=(5+2\sqrt{5})/5$ when $P_1P_i=(1+\sqrt{5})/2$,

(b-2) $((1+\sqrt{5})/\sqrt{(10+2\sqrt{5})})^2+z^2+(w-\sqrt{(10+2\sqrt{5})/2\sqrt{5}})^2=(3+\sqrt{5})/2$ when $P_5P_i=(1+\sqrt{5})/2$,

(b-3) $w=\sqrt{(10+2\sqrt{5})}/4\sqrt{5}$ when $P_1P_i=P_5P_i$.

Hence combining one of (a-1)–(a-3) and one of (b-1)–(b-3), we see that the possible situations for P_i must be in the list of the proposition. □

Proposition 12.8. *Any 11-point isosceles set in $\mathbb{R}^4$ cannot satisfy one of $\langle 2\rangle$–$\langle 4\rangle$ in Table 3.*

Proof. The previous proposition implies that $P_8,\ldots,P_{11}$ satisfy one of the following conditions:

(i) four points on L,

(ii) three points on L and the other is one of R_1 and R_2,

(iii) two points on L and the others are R_1 and R_2.

In the case (i), we cannot take four points on a line. This is a contradiction. In the case (ii), three collinear points are contained. By Corollary 3.7, this is a contradiction.

Table 4: The distribution of the remaining points.

	The number of points satisfying (i)	The number of points satisfying (ii)
$\langle 1\rangle$	0	7
$\langle 2\rangle$	1	6
$\langle 3\rangle$	2	5
$\langle 4\rangle$	3	4
$\langle 5\rangle$	4	3
$\langle 6\rangle$	5	2
$\langle 7\rangle$	6	1

In the case (iii), $\Delta P_5R_1R_2$ is scalene with 1, $(1+\sqrt{5})/2$, $\sqrt{1+((1+\sqrt{5})/2)^2}$. This is contrary to the configuration hypothesis.

Therefore any 11-point isosceles set in $\mathbb{R}^4$ cannot satisfy one of $\langle 2\rangle$–$\langle 4\rangle$ in Table 3. □

Thus we have the following lemma.

Lemma 12.9. *There exists a unique 11-point isosceles set in $\mathbb{R}^4$ containing four vertices of a regular pentagon. This is Y in Theorem 1.1.*

13. Observation of 11-Point Isosceles Sets in $\mathbb{R}^4$ Containing a Square

Proposition 13.1. *Let P_1, P_2, P_3, P_4 in an n-point isosceles set $\mathcal{P} = \{P_1, \ldots, P_n\}$ form a square. We may suppose that $P_1 = (-1/2, -1/2, 0, 0), P_2 = (1/2, -1/2, 0, 0), P_3 = (1/2, 1/2, 0, 0), P_4 = (-1/2, 1/2, 0, 0)$.*

Then the only other possible coordinates for the remaining points are

(i) $(0, 0, z, w)$, *where z and w are arbitrary, or*

(ii) *one of* $(0, -1/2, z, w)$, $(1/2, 0, z, w)$, $(0, 1/2, z, w)$, *and* $(-1/2, 0, z, w)$, *where z and w satisfy $z^2 + w^2 = 3/4$.*

Proof. We expand the proof of Lemma 19 in Croft [3] into $\mathbb{R}^4$, then we obtain this proposition. □

We observe the detail for $n = 11$ in Proposition 13.1. The space which satisfies the case (i) in Proposition 13.1 is a plane. The maximum cardinality of isosceles sets in $\mathbb{R}^2$ is 6. Hence if an 11-point isosceles set exists, then it satisfies one row of Table 4.

We observe $\langle 1\rangle$–$\langle 3\rangle$ in Table 4. We see that another point P_i of an 11-point isosceles set $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ which satisfies (ii) in Proposition 13.1 is on one of four circles.

Let S_1 be $x = 0$, $y = -1/2$, $z^2 + w^2 = 3/4$,S_2 be $x = 1/2$, $y = 0$, $z^2 + w^2 = 3/4$, S_3 be $x = 0$, $y = 1/2$, $z^2 + w^2 = 3/4$, and S_4 be $x = -1/2$, $y = 0$, $z^2 + w^2 = 3/4$. We remark that S_1 and S_3 are the subsets of the 3-dimensional Euclidean space $x = 0$, and S_2 and S_4 are the subsets of the 3-dimensional Euclidean space $y = 0$.

When $\mathcal{P}$ satisfies one of $\langle 1\rangle$–$\langle 3\rangle$ in Table 4, the remaining at least five points are distributed on some of $S_1, \ldots, S_4$. If they are distributed on one circle, then they form a regular pentagon. By Lemma 12.9, such any 11-point isosceles set is isomorphic to Y in Theorem 1.1.

Hence we may suppose that they are distributed on more than or equal to two circles. We may assume that we choose S_1 as the first circle because of symmetry. Now we separate the choice of the second circle into two cases whether S_i is the subset of the 3-dimensional Euclidean space $x = 0$ or not for $i = 2, 3, 4$. So one is S_3, the other is S_2 or S_4.

Proposition 13.2. *One considers the first case above. One fixes a point P_i on S_1. Then the possible situations for the points on S_3 are at most three. Moreover the distance between a pair of distinct points from these three points must be 1 or $2\sqrt{6}/3$.*

Proof. We may assume that $P_i = (0, -1/2, \sqrt{3}/2, 0)$ because S_1 and S_3 are on the 3-dimensional Euclidean space $x = 0$ and we have only to investigate the relation between the points on S_1 and those on S_3. Let $P_j = (0, 1/2, z, w)$ on S_3, where $z^2 + w^2 = 3/4$. We consider $\Delta P_1 P_i P_j$. Since $P_1 P_i = 1$ and $P_1 P_j = \sqrt{2}$, we have $P_i P_j = 1$ or $\sqrt{2}$. When $P_i P_j = 1$, P_j is $(0, 1/2, \sqrt{3}/2, 0)$. When $P_i P_j = \sqrt{2}$, P_j is $(0, 1/2, \sqrt{3}/6, \sqrt{6}/3)$ or $(0, 1/2, \sqrt{3}/6, -\sqrt{6}/3)$.

Therefore the possible situations for P_j are at most three. Moreover we see easily that the distance between a pair of distinct points from these three points must be 1 or $2\sqrt{6}/3$. □

We consider the other case. We may suppose that the choice of the second circle is S_2 because of symmetry.

Proposition 13.3. *One considers that the choice of the second circle is S_2. One fixes a point $P_i = (0, -1/2, z_i, w_i)$ on S_1, where $z_i^2 + w_i^2 = 3/4$. Then the possible situations for the points on S_2 are at most two. And the distance between the two points must be one of $\sqrt{3}$, $\sqrt{15}/3$, $(\sqrt{5}+1)/2$, and $(\sqrt{5}-1)/2$.*

Proof. Let $P_j = (1/2, 0, z, w)$ on S_2, where $z^2 + w^2 = 3/4$. We consider $\Delta P_1 P_i P_j$. Since $P_1 P_i = 1$ and $P_1 P_j = \sqrt{2}$, we have $P_i P_j = 1$ or $\sqrt{2}$. Then $(z - z_i)^2 + (w - w_i)^2 = 1/2$ or $(z - z_i)^2 + (w - w_i)^2 = 3/2$ holds. From them, we have $z_i z + w_i w - 1/2 = 0$ or $z_i z + w_i w = 0$. And z and w satisfy $z^2 + w^2 = 3/4$. Hence the possible situations for P_j are at most four.

Since (1/2,0,0,0) is on $z_i z + w_i w = 0$, the distance between (1/2,0,0,0) and $z_i z + w_i w - 1/2 = 0$ is $|-1/2|/\sqrt{z_i^2 + w_i^2} = 1/2/\sqrt{3}/2 = \sqrt{3}/3$ in spite of the way to fix P_i. So let Q_1, Q_2, Q_3, Q_4 be the four possible points for P_j, the distances between Q-points are in Figure 5 in spite of the way to fix P_i. Looking at Figure 5, all triangles that we choose from Q-points are scalene.

Therefore if we fix a point P_i on S_1, then the possible situations for the points on S_2 are at most two. Moreover we see easily that the distance between the two points must be one of $\sqrt{3}$, $\sqrt{15}/3$, $(\sqrt{5}+1)/2$, and $(\sqrt{5}-1)/2$ by Figure 5. □

For the supposition of Proposition 13.3, moreover we suppose that there is a point on S_4, too. Then the distance between two points on S_2 must be 1 or $2\sqrt{6}/3$ by an analogue of Proposition 13.2, we take at most one point on S_2. Thus we see that we cannot take five points satisfying (ii) in Proposition 13.1, we have the following proposition.

Proposition 13.4. *Any 11-point isosceles set in $\mathbb{R}^4$ cannot satisfy one of $\langle 1 \rangle$–$\langle 3 \rangle$ in Table 4.*

Next we observe $\langle 5 \rangle$–$\langle 7 \rangle$ in Table 4. Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set. We suppose that P_5 is on some of $S_1, \ldots, S_4$ and that $P_8, \ldots, P_{11}$ are on the plane $x = y = 0$. We may assume that P_5 is on S_1 because of symmetry. So P_5 is one of

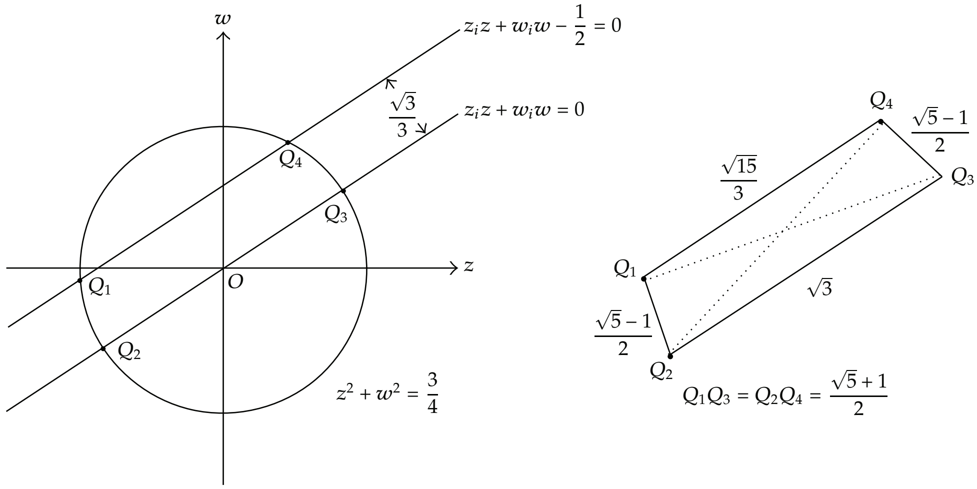


Figure 5

$(0, -1/2, a, \sqrt{3/4 - a^2})$ and $(0, -1/2, a, -\sqrt{3/4 - a^2})$, where $-\sqrt{3}/2 \leq a \leq \sqrt{3}/2$. We can choose that $P_5 = (0, -1/2, a, \sqrt{3/4 - a^2})$. (For the latter we can repeat the similar discussion.)

Proposition 13.5. *If an 11-point isosceles set $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ contains a square satisfying one of $\langle 5\rangle$–$\langle 7\rangle$ in Table 4, then the possible situations for $P_8, \ldots, P_{11}$ are as follows:*

(I) *on the line L which satisfies $az + w\sqrt{3/4 - a^2} = 1/4$, or*

(II) *at some of $R_1, \ldots, R_4$, where*

$$\begin{aligned}
R_1 &= \left(0, 0, \frac{-2a - \sqrt{5(3 - 4a^2)}}{6}, \frac{2\sqrt{5}a - \sqrt{3 - 4a^2}}{6}\right), \\
R_2 &= \left(0, 0, \frac{-2a + \sqrt{5(3 - 4a^2)}}{6}, \frac{-2\sqrt{5}a - \sqrt{3 - 4a^2}}{6}\right), \\
R_3 &= \left(0, 0, \frac{2a - \sqrt{3 - 4a^2}}{2}, \frac{2a + \sqrt{3 - 4a^2}}{2}\right), \\
R_4 &= \left(0, 0, \frac{2a + \sqrt{3 - 4a^2}}{2}, \frac{-2a + \sqrt{3 - 4a^2}}{2}\right).
\end{aligned} \tag{13.1}$$

Proof. For $i = 8, \ldots, 11$, let $P_i = (0, 0, z, w)$. We consider $\Delta P_1 P_5 P_i$. Because $P_1 P_5 = 1$, one of the following (a-1)–(a-3) must hold to satisfy the configuration hypothesis:

(a-1) $z^2 + w^2 = 1/2$ when $P_1 P_i = 1$,

(a-2) $(z - a)^2 + (w - \sqrt{3/4 - a^2})^2 = 3/4$ when $P_5 P_i = 1$,

(a-3) $az + w\sqrt{3/4 - a^2} = 1/4$ when $P_1 P_i = P_5 P_i$.

On the other hand, we consider $\Delta P_3P_5P_i$. Since $P_3P_5 = \sqrt{2}$, one of the following (b-1)–(b-3) must hold to satisfy the configuration hypothesis:

(b-1) $z^2 + w^2 = 3/2$ when $P_3P_i = \sqrt{2}$,

(b-2) $(z - a)^2 + (w - \sqrt{3/4 - a^2})^2 = 7/4$ when $P_5P_i = \sqrt{2}$,

(b-3) $az + w\sqrt{3/4 - a^2} = 1/4$ when $P_3P_i = P_5P_i$.

Hence conbining one of (a-1)–(a-3) and one of (b-1)–(b-3), we see that the possible situations for P_i must be in the list of the proposition. □

Proposition 13.6. *Any 11-point isoseles set in $\mathbb{R}^4$ cannot satisfy one of $\langle 5\rangle$–$\langle 7\rangle$ in Table 4.*

Proof. By Proposition 13.5, four points $P_8, \ldots, P_{11}$ are in the list of the proposition. We observe (II) in Proposition 13.5. Because $R_1R_2 = \sqrt{15}/3$, $R_1R_3 = R_2R_4 = \sqrt{(5 - \sqrt{5})/2}$, $R_1R_4 = R_2R_3 = \sqrt{(5 + \sqrt{5})/2}$, and $R_3R_4 = \sqrt{3}$, any triangle selected from R-points is scalene. So we choose at most two R-points. On the other hand, we observe (I) in Proposition 13.5. Since L is a line, we choose at most three points on L. By Corollary 3.7, we cannot choose three points. So we choose at most two points on L. Hence we must choose two R-points and two points on L for $P_8, \ldots, P_{11}$. Let P_8, P_9 be two R-points and P_{10}, P_{11} be two points on L. For each choice of two R-points, we see that the possible situations for P_{10} and P_{11} are five by considering $\Delta P_8P_9P_{10}$ and $\Delta P_8P_9P_{11}$ and the calculations.

The number of the choices of P_8 and P_9 is $\binom{4}{2} = 6$ and the number of the choices of P_{10} and P_{11} is $\binom{5}{2} = 10$. Thus the number of the choices of $P_8, \ldots, P_{11}$ is $6 \times 10 = 60$. We have only to check 60 cases whether $P_1, \ldots, P_5, P_8, \ldots, P_{11}$ form an isosceles set or not. But for all cases we see that they contain a scalene by the calculations.

Therefore any 11-point isoseles set cannot satisfy one of $\langle 5\rangle$–$\langle 7\rangle$ in Table 4. □

Finally we observe $\langle 4\rangle$ in Table 4. Let $\mathcal{P} = \{P_1, \ldots, P_{11}\}$ be an 11-point isosceles set. Four points $P_5, \ldots, P_8$ lie on some of $S_1, \ldots, S_4$. We may assume that one point $P_5 = (0, -1/2, \sqrt{3}/2, 0)$ because of symmetry. $P_9, \ldots, P_{11}$ are in the plane $x = y = 0$.

Proposition 13.7. *If there exists an 11-point isosceles set in $\mathbb{R}^4$ satisfying $\langle 4\rangle$ in Table 4, then it is isomorphic to X or Y in Theorem 1.1.*

Proof. If $P_5, \ldots, P_8$ are distributed on S_1, then they are all points of a square or four points of a regular pentagon.

We consider when $P_5, \ldots, P_8$ form a square. If each side of the square $P_1P_2P_3P_4$ in Proposition 12.4 is $\sqrt{3}/2 \times \sqrt{2} = \sqrt{6}/2$, then we can change them into $P_5, \ldots, P_8$ in the 3-dimensional Euclidean space $x = 0$. Now the other points P_9, P_{10}, P_{11} are in the 2-dimensional Euclidean space $x = y = 0$. By Proposition 12.4, we see that there is exactly one point $(0, 0, 0, 0)$ in $x = y = 0$. Thus we cannot take three points in $x = y = 0$. This is a contradiction. Hence $P_5, \ldots, P_8$ do not form a square.

If $P_5, \ldots, P_8$ form four points of a regular pentagon, then such any 11-point isosceles set is isomorphic to Y in Theorem 1.1 by Lemma 12.9.

Hence they are distributed on more than or equal to two circles. By the proof of Proposition 13.2, the number of the possible points on S_3 for P_6, P_7, P_8 is at most three. They are $U_1 = (0, 1/2, \sqrt{3}/2, 0)$, $U_2 = (0, 1/2, \sqrt{3}/6, \sqrt{6}/3)$, and $U_3 = (0, 1/2, \sqrt{3}/6, -\sqrt{6}/3)$. By the proof of Proposition 13.3, the number of the possible points on S_2 for P_6, P_7, P_8 is at most four.

They are $U_4 = (1/2, 0, \sqrt{3}/3, \sqrt{15}/6)$, $U_5 = (1/2, 0, \sqrt{3}/3, -\sqrt{15}/6)$, $U_6 = (1/2, 0, 0, \sqrt{3}/2)$, and $U_7 = (1/2, 0, 0, -\sqrt{3}/2)$. Similarly the number of the possible points on S_4 for P_6, P_7, P_8 is at most four by the proof of Proposition 13.3. They are $U_8 = (-1/2, 0, \sqrt{3}/3, \sqrt{15}/6)$, $U_9 = (-1/2, 0, \sqrt{3}/3, -\sqrt{15}/6)$, $U_{10} = (-1/2, 0, 0, \sqrt{3}/2)$, and $U_{11} = (-1/2, 0, 0, -\sqrt{3}/2)$.

If we apply Proposition 13.5 to P_5, then P_9, P_{10}, and P_{11} in the plane $x = y = 0$ must satisfy one of the following situations:

$$\begin{aligned} &\text{(I)} \quad \text{on the line which satisfies } z = \frac{\sqrt{3}}{6}, \\ &\text{(II)} \quad \text{at some of } \left(0, 0, -\frac{\sqrt{3}}{6}, \frac{\sqrt{15}}{6}\right), \left(0, 0, -\frac{\sqrt{3}}{6}, -\frac{\sqrt{15}}{6}\right), \\ &\qquad\qquad \left(0, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right), \left(0, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right). \end{aligned} \tag{13.2}$$

We apply Proposition 13.5 and its analogue to U-points, we have only to check whether there exist three points in $x = y = 0$ which satisfy (13.2) or not. When we take U_1, U_4, and U_8, there exist three points $(0, 0, \sqrt{3}/6, \sqrt{15}/30)$, $(0, 0, -\sqrt{3}/6, \sqrt{15}/6)$, and $(0, 0, \sqrt{3}/6, -\sqrt{15}/6)$ satisfying (13.2). Then $\{P_1, \ldots, P_5, U_1, U_4, U_8, (0, 0, \sqrt{3}/6, \sqrt{15}/30), (0, 0, -\sqrt{3}/6, \sqrt{15}/6), (0, 0, \sqrt{3}/6, -\sqrt{15}/6)\}$ is an 11-point isosceles set which is isomorphic to X in Theorem 1.1. Similarly when we take U_1, U_5, and U_9, there exist three points $(0, 0, \sqrt{3}/6, -\sqrt{15}/30)$, $(0, 0, -\sqrt{3}/6, -\sqrt{15}/6)$, and $(0, 0, \sqrt{3}/6, \sqrt{15}/6)$ satisfying (13.2). Then $\{P_1, \ldots, P_5, U_1, U_5, U_9, (0, 0, \sqrt{3}/6, -\sqrt{15}/30), (0, 0, -\sqrt{3}/6, -\sqrt{15}/6), (0, 0, \sqrt{3}/6, \sqrt{15}/6)\}$ is an 11-point isosceles set which is isomorphic to X in Theorem 1.1, too. In the other cases, three points satisfying (13.2) do not exist.

On the other hand, if we apply the proofs of Propositions 13.2 and 13.3 to U-points, then there are some possible points on S_1 except for P_5. We apply Proposition 13.5 and its analogue to them. But three points satisfying (13.2) do not exist. Hence we cannot take points on S_1 except for P_5.

Therefore if there exists an 11-point isosceles set in $\mathbb{R}^4$ satisfying $\langle 4 \rangle$ in Table 4, then it is isomorphic to X or Y in Theorem 1.1. □

We remark that Y in Theorem 1.1 does not contain a square. Thus we have the following lemma.

Lemma 13.8. *There exists a unique 11-point isosceles set in $\mathbb{R}^4$ containing a square. This is X in Theorem 1.1.*

14. Completion of the Proofs of Theorem 1.1 and Corollary 1.2

First, Lemma 3.1 holds if an 11-point isosceles set exists. In any case of Lemma 3.1, if there exists an 11-point isosceles set, then the condition (X) holds by Lemmas 4.8, 5.1, 6.1, 7.1, 8.2, 9.2, 10.2, and 11.1.

When the condition (X) holds, four points that lie on a circle are either all the vertices of a square, or four of the vertices of a regular pentagon by Lemma 11.2. If they are four of the vertices of a regular pentagon, then Lemma 12.9 implies that there exists a unique 11-point isosceles set Y. On the other hand, if they are all the vertices of a square, then there exists a unique 11-point isosceles set X by Lemma 13.8.

Therefore there are exactly two 11-point isosceles sets X and Y in $\mathbb{R}^4$ up to isomorphisms. Moreover we see that there is no 12-point isosceles set in $\mathbb{R}^4$ by the calculation, and he maximum cardinality of isosceles sets in $\mathbb{R}^4$ is 11.

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