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Research Article

A Noncommutative Enumeration Problem

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We tackle the combinatorics of coloured hard-dimer objects. This is achieved by identifying coloured hard-dimer configurations with a certain class of rooted trees that allow for an algebraic treatment in terms of noncommutative formal power series. A representation in terms of matrices then allows to find the asymptotic behaviour of these objects.

1. Introduction

The aim is to count coloured hard-dimer configurations which are objects as shown in Figure 1.

More precisely, we define a coloured hard-dimer configuration (CHDC) to be a finite sequence σ_n, of n blue and red vertices, together with coloured dimers on σ_n which must not intersect. Here a coloured dimer is an edge connecting two nearest vertices of the same colour see Figure 1. The dimer's colour is given by the colour of its boundary vertices. In graph-theoretic language CHDCs are a subclass of labeled graphs whose vertices and edges carry one of two possible labels. Without loss of generality, we may assume that a CHDC is a subset of $\mathbb{R}$ and its vertices belong to $\mathbb{Z}$. We include also the empty CHDC, that is, the configuration when no dimers are present.

The enumeration problem we have in mind goes as follows: given a sequence σ_n and a triple of nonnegative integers (i, j, k), where i, j count the numbers of blue, red dimers and k the number of inner vertices, that is, vertices that are connected to their left and right vertex. How many CHDCs of a given type (i, j, k) are there on σ_n, where σ_n as well as n may be arbitrary?

The strategy is to identify the set of CHDCs with a certain class of trees the elements of which can be encoded into monomials of a noncommutative formal power series $\sum_{\mathbf{x}} a_{\mathbf{x}}\mathbf{x}$, where the $\mathbf{x}$'s are words over the alphabet $\{b, r\}$. This means that the indeterminates b and r are supposed to be noncommutative and a particular word $\mathbf{x}$ just corresponds to a particular

Figure 1: A coloured hard-dimer configuration ($n = 12$) with 2 blue dimers, 1 red dimer, and 3 inner vertices.

sequence of blue and red vertices σ_n. The coefficients $a_{\mathbf{x}}$ collect the information of CHDCs of a given type on σ_n.

The motivation for studying this kind of objects is due to their appearance in causally triangulated $(2+1)$-dimensional gravity. It was shown in [1], by using special triangulations of spacetime, that the discrete Laplace transform of the one-step propagator can be expressed as follows:

$$Z(u,v,w) = \sum_{n\in\mathbb{N}} e^{-\gamma n} Z_n(u,v,w), \tag{1.1}$$

with

$$Z_n(u,v,w) = \sum_{\sigma_n} \frac{1}{Z^{\mathrm{hcd}}_{\sigma_n}(-u,-v,w)}, \qquad Z^{\mathrm{hcd}}_{\sigma_n}(u,v,w) = \sum_{\text{CHDCs } D|\sigma_n} u^{|D|_b} v^{|D|_r} w^{|\cap D|}. \tag{1.2}$$

The parameters γ, u, v, w are related to physical and geometrical constants. Now, the exponents $|D|_b, |D|_r, |\cap D|$ just count, for a given CHDC D, the number of blue dimers, red dimers, and inner vertices, respectively. A glance at $Z^{\mathrm{hcd}}_{\sigma_n}$ shows that it involves precisely the combinatorial numbers we are after in this paper.

In Section 2, we show how to make CHDCs into monomials, and thereby solve the enumeration problem from above. In Section 3, we give an alternative way to express the coefficients $a_{\mathbf{x}}$ by means of a linear representation of the indeterminates b, r. The formal power series that allow such a representation are called recognizable. Finally, we use this result to investigate the growth behaviour of the number of CHDCs.

2. From CHDCs to Noncommutative Series

In this section we will establish a bijection between CHDCs and a certain class of trees. This is achieved by first identifying CHDCs with a particular class of graphs denoted $\mathbf{G}_{\mathrm{hcd}}$ which in turn will be identified with the corresponding class of trees denoted $\mathbf{T}_{\mathrm{hcd}}$.

Bijection CHDCs $\Leftrightarrow$ $\mathbf{G}_{\mathrm{hcd}}$: for a given CHDC, we endow its vertices with the order inherited from $\mathbb{Z}$. An extra vertex $\times$ is added in the first place, which becomes the root of the graph. Then, all the vertices which are next neighbors are connected through an edge. Finally, the vertices which in the original CHDC are the starting and end points of a dimer are linked via an extra edge to the right, as seen from the root (see Figure 2(a)). We note that only the closest vertices of the same colour may get an extra link and that these become trivalent, unless we consider the last vertex, in which case it becomes bivalent. The set of rooted graphs we obtain in this manner is denoted as $\mathbf{G}_{\mathrm{hcd}}$, and it is obvious that we have established a bijection between the latter and the set of CHDCs.

Bijection $\mathbf{G}_{\mathrm{hcd}}$ $\Leftrightarrow$ $\mathbf{T}_{\mathrm{hcd}}$: when a graph in $\mathbf{G}_{\mathrm{hcd}}$ is given, we move beginning from the root in clockwise direction and cut an edge if the graph remains connected. The cut edge is replaced by two edges ending in univalent vertices called buds (black arrow) and leafs

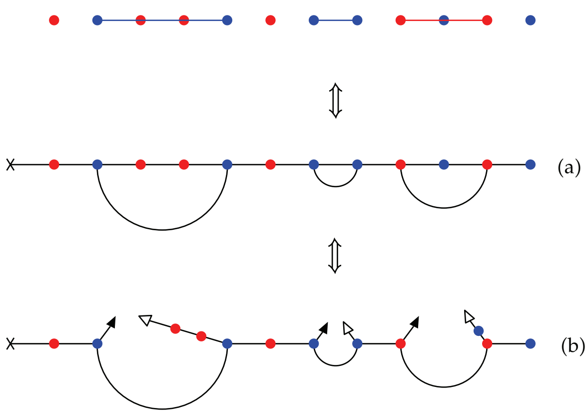

Figure 2: Illustration of the one-to-one correspondence between a coloured hard-dimer, its graph, and its tree.

(white arrow), respectively. The procedure is repeated until we reach the last vertex (see Figure 2(b)). We note that buds and three vertices are always connected by one edge only, whereas leafs and three vertices may be interlaced by a certain number of bivalent red or blue vertices. The set of rooted trees we obtain following this procedure is denoted as $\mathbf{T}_{\mathrm{hcd}}$. Vice versa, when a tree in $\mathbf{T}_{\mathrm{hcd}}$ is given, we may, starting from the root, move in clockwise direction and merge the bud leave pairs to edges getting a graph in $\mathbf{G}_{\mathrm{hcd}}$.

We have thus established a bijection between the set of CHDCs and the set $\mathbf{T}_{\mathrm{hcd}}$. The next goal is to find an algebraic formulation of the enumeration problem for elements in $\mathbf{T}_{\mathrm{hcd}}$. For this, we adapt ideas from [2] and assign the charges $+1$, -1 and 0 to leaves, buds and univalent coloured vertices, respectively. With this choice any tree has charge 0 and any subtree different from a bud has charge 0 or $+1$. For this reason, we distinguish between S (resp., R) trees, if and only if

(a) their total charge is 0 (resp., 1),

(b) any descendant subtree different from a bud has charge 0 or 1.

Furthermore, we subdivide these two groups into S_b, R_b and S_r, R_r trees depending on whether the first vertex after the root is blue or red.

Now, we are able to write down a system of equations for the four types of trees which will turn out to be linear. Since we are interested in counting the number of CHDCs on a fixed sequence of blue and red vertices we assign noncommutative variables b and r to blue and red vertices, respectively. Furthermore, for a given sequence σ_n we group together all CHDCs with the same numbers i, j, and k. This can be achieved by encoding a blue or red dimer through the assignment of commutative variables b_3 and r_3, respectively. Particularly, the variables b_3 and r_3 are assigned to those trivalent blue and red vertices, respectively, which possess a bud leg pair as subtree. Finally, to every bivalent vertex sitting in between a three vertex and a leaf, we assign a commutative variable y.

We are thus naturally led to consider the trees S_b, S_r, R_b, and R_r as elements of the noncommutative algebra $K\langle\langle b, r\rangle\rangle$ of formal power series, where K is the commutative polynomial ring $K = \mathbb{Z}[b_3, r_3, y]$. A formal series will be written in the form $P = \sum_{\mathbf{x}} a_{\mathbf{x}}\mathbf{x}$, where the sum runs over all words of the alphabet $\{b, r\}$ with coefficients $a_{\mathbf{x}} \equiv (P, \mathbf{x})$. For example, the tree and the corresponding graph in Figure 2 read

$$b_3^2 r_3 y^3 r * b * r * r * b * r * b * b * r * b * r * b, \tag{2.1}$$

where $*$ stands for the noncommutative product in $K\langle\langle b, r\rangle\rangle$. As we can read in Figure 3 below, our trees have to satisfy the following consistency equations:

$$\begin{aligned} S_b &= b + b * (S_b + S_r) + b_3 b * R_b, \\ S_r &= r + r * (S_b + S_r) + r_3 r * R_r, \\ R_b &= b + b * (S_b + S_r) + yr * (1 - yr)^{-1} * b * (S_b + S_r) + yr * (1 - yr)^{-1} * b, \\ R_r &= r + r * (S_b + S_r) + yb * (1 - yb)^{-1} * r * (S_b + S_r) + yb * (1 - yb)^{-1} * r. \end{aligned} \tag{2.2}$$

Substituting the expressions of R_b and R_r into the equations for S_b and S_r, we get by elementary algebra the following system of equations:

$$\begin{aligned} S_b &= \underbrace{\left[b + b_3 b * (1 - yr)^{-1} * b\right]}_{=:\, A_b} * S_b + A_b * S_r + A_b, \\ S_r &= \underbrace{\left[r + r_3 r * (1 - yb)^{-1} * r\right]}_{=:A_r} * S_r + A_r * S_b + A_r, \end{aligned} \tag{2.3}$$

that is,

$$\begin{aligned} (1 - A_b) * S_b - A_b * S_r &= A_b, \\ -A_r * S_b + (1 - A_r) * S_r &= A_r. \end{aligned} \tag{2.4}$$

The system (2.4) can be solved easily with the result

$$\begin{aligned} S_b &= \left[1 - A_b * (1 - A_r)^{-1}\right]^{-1} * A_b * (1 - A_r)^{-1} \\ &= (1 - A_r) * (1 - A_r - A_b)^{-1} - 1. \end{aligned} \tag{2.5}$$

By symmetry, we also get

$$S_r = (1 - A_b) * (1 - A_r - A_b)^{-1} - 1. \tag{2.6}$$

Theorem 2.1. *Let $S_b = \sum_{\mathbf{x}} a_{\mathbf{x}} \mathbf{x}$ be the formal series given as part of the unique solution of (2.4) in $K\langle\langle b, r\rangle\rangle$. Then, all coefficients of words $\mathbf{x}$ starting with letter r are zero and the nontrivial coefficients $a_{\mathbf{x}}$ are given as finite sums*

$$a_{\mathbf{x}} = \sum_{i,j,k \in \mathbb{N}_0} m_{i,j,k}(\mathbf{x})\; b_3^i r_3^j y^k, \tag{2.7}$$

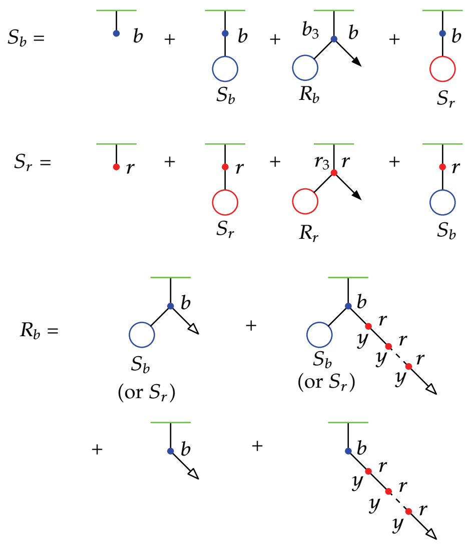

Figure 3: The possible building blocks from the perspective of an arbitrary vertex. The corresponding building blocks for R_r are obtained by exchanging colors and variables ($b \leftrightarrow r$).

where the multiplicities $m_{i,j,k}(\mathbf{x})$ *count the number of coloured hard-dimer configurations* D *on* $\mathbf{x}$ *with fixed* (i,j,k). *In particular, the evaluation* $a_{\mathbf{x}}(b_3 = 1, r_3 = 1, w = 1)$ *gives the number of CHDCs on the sequence* σ_n *corresponding to* $\mathbf{x}$. *The symmetric assertion holds for* S_r.

Remark 2.2. It is clear from (2.4) that $S = S_b + S_r$ solves the equation $(1-(A_b+A_r))*S = A_b+A_r$ with solution given by $S = (1-(A_b+A_r))^{-1}*(A_b+A_r)$. Moreover, for $S = \sum_{\mathbf{x}} c_{\mathbf{x}}\mathbf{x}$, the evaluation $c_{\mathbf{x}}(b_3 = 1, r_3 = 1, w = 1)$ gives the number of CHDCs on $\sigma_n \equiv \mathbf{x}$ irrespective of whether it starts with b or r.

Proof of Theorem 2.1. Let $\#$ be one of the subscripts b, r and decompose $S_\# = \sum_{n=0}^{\infty} S_\#^n$, where $S_\#^n$ comprises the sum of terms in $S_\#$ with length $|\mathbf{x}| = n$. In the same manner, we write $R_\# = \sum_{n=0}^{\infty} R_\#^n$. An elementary computation shows that $S_\# = \# + \{\text{higher order terms}\}$ and likewise for $R_\#$, which implies that a solution to (2.4) gives rise to sequences $(S_\#^n)_{n\geq -1}$ and $(R_\#^n)_{n\geq -1}$ obeying the following recursive equations ($n \geq 1$):

$$
\begin{aligned}
S_b^n &= b + b * \left(S_b^{n-1} + S_r^{n-1}\right) + b_3 b * R_b^{n-1},\\
S_r^n &= r + r * \left(S_b^{n-1} + S_r^{n-1}\right) + r_3 r * R_r^{n-1},\\
R_b^n &= b + b * \left(S_b^{n-1} + S_r^{n-1}\right) + \sum_{k=1}^{n-2} y^k r^k * b * \left(S_b^{n-1-k} + S_r^{n-1-k}\right) + \ y^{n-1} r^{n-1} * b,\\
R_r^n &= r + r * \left(S_b^{n-1} + S_r^{n-1}\right) + \sum_{k=1}^{n-2} y^k b^k * r * \left(S_b^{n-1-k} + S_r^{n-1-k}\right) + \ y^{n-1} b^{n-1} * r,
\end{aligned}
\tag{2.8}
$$

with initial conditions $S_\#^c = 0$, $R_\#^c = 0$ for $c \in \{-1, 0\}$ and $S_\#^1 = R_\#^1 = \#$. Vice versa, for sequences $(S_\#^n)_{n\geq -1}$, $(R_\#^n)_{n\geq -1}$ obeying (2.8) and related initial conditions, their sums $S_\# := \sum_{n=-1}^{\infty} S_\#^n$ and $R_\# := \sum_{n=-1}^{\infty} R_\#^n$ will satisfy (2.4). Therefore, we may equivalently look at the system (2.8). But

Figure 4: Two examples of uncompleted CHDCs.

the elements $S_{\#}^{n}$ contain precisely the sum of those terms from $K\langle\langle b, r\rangle\rangle$ which are algebraic counterparts of trees in $\mathbf{T}_{\mathrm{hcd}}$ that have n coloured vertices the first one of which is #. This is due to the fact that there is a one-to-one correspondence between trees from $\mathbf{T}_{\mathrm{hcd}}$ and trees of charge 0 that are constructed recursively according to the building blocks from Figure 3. For a fixed $\mathbf{x}$, the coefficient $a_{\mathbf{x}}$ is a sum of elements in K which stem from trees that are constructible for the particular sequence of blue and red vertices given by $\mathbf{x}$. Since the indeterminates b_3, r_3, y are commutative, trees will contribute the same term $b_3^i r_3^j y^k$, whenever the numbers of vertices corresponding to a b_3, r_3, y indeterminate, coincide. Therefore, the integer in front of $b_3^i r_3^j y^k$ gives the number of CHDCs with i blue dimers, j red dimers, and k inner vertices fixed. $\square$

3. The Solution S as a Recognizable Series

Here, we show in addition that $S = S_b + S_r$ is a recognizable series. Let A be a finite alphabet, A^* the corresponding free monoid of words, and R a semiring. A formal series $P = \sum_{\mathbf{x}\in A^*} a_{\mathbf{x}}\mathbf{x}$ with $a_{\mathbf{x}} \in R$ is called recognizable if there is some $n \geq 1$, a homomorphism of semirings (or simply a representation)

$$\mu : A^* \longrightarrow R^{n\times n}, \tag{3.1}$$

where $R^{n\times n}$ carries its multiplicative structure and tuples $\lambda, \gamma \in R^n$ such that for all words $\mathbf{x}$

$$a_{\mathbf{x}} = \lambda^{\top} \cdot \mu(\mathbf{x}) \cdot \gamma. \tag{3.2}$$

Further material on recognizable series is exposed in [3, 4].

In order to find a representation for S, we look at the prefixes of a given $\mathbf{x}$ and note that for a given prefix $\mathbf{w}$ of $\mathbf{x}$, we may decompose the set of CHDCs on $\mathbf{x}$ into three disjoint subsets by looking at the restriction of a CHDC to $\mathbf{w}$.

$X_1^{\mathbf{w}}$: the boundary points of dimers (if any) either lie entirely inside or outside of $\mathbf{w}$,

$X_2^{\mathbf{w}}$: a blue dimer starts in $\mathbf{w}$ and ends outside of $\mathbf{w}$,

$X_3^{\mathbf{w}}$: a red dimer starts in $\mathbf{w}$ and ends outside of $\mathbf{w}$.

Strictly speaking, we consider, in the above decomposition, the larger set of uncompleted CHDCs, meaning that the last vertex of a CHDC may be the starting or inner point of an uncompleted dimer; see Figure 4.

Now, the three sets are identified with the canonical basis elements of the K-module K^3; that is, $X_j^{\mathbf{w}} = (\delta_{ij})_{1\leq i\leq 3}$, $j \in \{1,2,3\}$. If the last vertex of $\mathbf{w}$ is blue or red, we assign matrices $(B)_{ij}$ or $(R)_{ij}$ to the latter. For this, let w_l be the last letter of $\mathbf{w}$. Then, the matrix entries are nonzero according to whether $X_i^{\mathbf{w}\setminus w_l} \subset X_j^{\mathbf{w}}$, $w_l \in \{b, r\}$, holds or does not hold. In accordance with the assignments of variables made in Section 2, we now choose entries $\#_3$ when a dimer of colour # is being completed and y when a dimer gets an additional inner vertex. The entries

1 indicate either that a single vertex is added or that a vertex becomes the starting point of a dimer. Explicitly, the matrices read

$$B = \begin{pmatrix} 1 & 1 & 0 \\ b_3 & 0 & 0 \\ 0 & 0 & y \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & y & 0 \\ r_3 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

We choose $\lambda = (1, 0, 0)$ which means that we start from the empty CHDC. In the end, the (complete) CHDCs can be read off by setting $\gamma = (1, 0, 0)$. We may summarize this discussion in the following.

Proposition 3.1. *The noncommutative formal series S is a recognizable series with representation*

$$\mu : \{b, r\}^* \longrightarrow K^{3\times 3}, \tag{3.4}$$

given by $\mu(b) = B$, $\mu(r) = R$ and tuples λ, γ.

The result above allows to study the asymptotic behaviour of the number of CHDCs on σ_n when n tends to infinity. According to Proposition 3.1, S is a recognizable series, and the evaluation at $b_3 = r_3 = y = 1$ again gives a recognizable series. Namely, for

$$M = S(b_3 = 1, r_3 = 1, y = 1), \tag{3.5}$$

the corresponding representation is given by $\mu' : \{b, r\}^* \to \mathbb{Z}^{3\times 3}$ through

$$\mu'(b) = B(b_3 = 1, y = 1), \qquad \mu'(r) = R(r_3 = 1, y = 1), \tag{3.6}$$

and the tuples are

$$\lambda' = \lambda, \qquad \gamma' = \gamma. \tag{3.7}$$

Note that according to Remark 2.2, $c_{\mathbf{x}} = (M, \mathbf{x})$ counts the number of CHDCs on the sequence of blue and red sites corresponding to $\mathbf{x}$.

It is convenient to discuss the asymptotics of CHDCs within the framework of ergodic dynamical systems. To be precise, let $B' := \mu'(b)$, $R' := \mu'(r)$ and define ν as the probability measure on $\{B', R'\}$ given by $\nu(B') = \nu(R') = 1/2$. Then, as ergodic dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, we choose the following.

(i) $\Omega = \{B', R'\}^{\mathbb{N}}$, that is, the set of sequences $\omega = (\omega_1, \omega_2, \ldots)$ with $\omega_i \in \{B', R'\}$.

If $g_n(\omega) = \omega_n$ denotes the coordinate maps then $\mathcal{F}$ should be the σ-algebra generated by the g_n's.

(ii) $\mathbb{P}$ is the product measure $\mathbb{P} = \prod_{n=1}^{\infty} \nu$.

(iii) θ is the shift operator $\theta(\omega) = (\omega_2, \omega_3, \ldots)$.

Upon turning $\{b, r\}^{\mathbb{N}}$ into a probability space that is isomorphic to $(\Omega, \mathcal{F}, \mathbb{P})$, we may state the following theorem.

Theorem 3.2. *For $\omega \in \{b,r\}^{\mathbb{N}}$, let $\boldsymbol{\omega}_n := (\omega_1,\ldots,\omega_n)$, and let $D_n(\omega)$ denote the number of CHDCs on $\boldsymbol{\omega}_n$. There exists a finite constant random variable α on $(\{b,r\}^{\mathbb{N}},\mathcal{F},\mathbb{P})$ such that for almost all $\omega \in \{b,r\}^{\mathbb{N}}$*

$$\frac{1}{n}\ln D_n(\omega) \longrightarrow \alpha, \quad \text{as } n \longrightarrow \infty. \tag{3.8}$$

This means that the number of CHDCs grows exponentially and that asymptotically this growth rate is the same for almost all ω.

Proof. If we define the matrix-valued random variables by

$$X_n(\omega) = g_1(\omega)g_2(\omega)\cdots g_n(\omega), \tag{3.9}$$

then

$$Y_n = -\ln\left(\lambda'^{\top}\cdot X_n\cdot\gamma'\right) \tag{3.10}$$

defines a subadditive sequence of random variables; that is,

$$Y_{m+n} \le Y_m \circ \theta^n + Y_n, \quad \forall m,n. \tag{3.11}$$

To see subadditivity, we first note that with the natural identifications

$$\boldsymbol{\omega}_n \longleftrightarrow \mathbf{v} : (\omega_1,\ldots,\omega_n) = \omega_1 * \cdots * \omega_n = \mathbf{v} \in \{b,r\}^*, \tag{3.12}$$

we may write $\lambda'^{\top}\cdot X_n(\omega)\cdot\gamma' = (M,\mathbf{v})$. Writing $\mathbf{x} = \mathbf{x}_n * \mathbf{x}_m$, with $|\mathbf{x}_n| = n$ and $|\mathbf{x}_m| = m$, subadditivity follows from the combinatorial fact that $(M,\mathbf{x}) \ge (M,\mathbf{x}_n)(M,\mathbf{x}_m)$. Now, Kingman's subadditive ergodic theorem, see [5] and [6, Theorem IV.1.2], applied to our ergodic dynamical system $(\Omega,\mathcal{F},\mathbb{P},\theta)$ implies that there exists a random variable β such that

$$\lim_{n\to\infty}\left(\frac{Y_n}{n}\right) = \beta = \mathbb{E}[\beta], \quad \mathbb{P}\text{-a.s.} \tag{3.13}$$

In addition, one has $\beta = \inf_n(n^{-1}\mathbb{E}[Y_n])$. If $\beta > -\infty$ convergence holds also in $L^1(\mathbb{P})$. A priori, it is not clear whether $\beta > -\infty$ which we clarify now. By Jensen's inequality, we find

$$\mathbb{E}\left[n^{-1}Y_n\right] \ge -n^{-1}\ln\left(\lambda'^{\top}\cdot\mathbb{E}[X_n]\cdot\gamma'\right) = -n^{-1}\ln\left(\lambda'^{\top}\cdot Z^n\cdot\gamma'\right), \tag{3.14}$$

where $Z = (B' + R')/2$. The eigenvalues of the matrix Z are $(3/2, 1/2, 0)$, and the diagonalization of Z shows that $\lim_{n\to\infty} - n^{-1}\ln(\lambda'^{\top}\cdot Z^n\cdot\gamma') = -\ln(3/2)$. This in turn implies

$\mathbb{E}[n^{-1}Y_n] \geq$ const., for all n, for some n-independent finite constant, and consequently $\beta = \inf_n(\mathbb{E}[n^{-1}Y_n]) \geq$ const. We have thus established almost sure convergence of

$$\frac{1}{n}\ln D_n = -\frac{Y_n}{n} \longrightarrow -\beta =: \alpha, \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

□

Remark 3.3. We conjecture that α from Theorem 3.2 is given by $\alpha = \ln(3/2)$. This conjecture is based on the following heuristic argument. From the explicit diagonalization of Z, we find that $\mathbb{E}[D_n] = \lambda'^{\top} \cdot Z^n \cdot \gamma' = (3/2)^{n-1}$ and from (3.14), we deduce $0 \leq \alpha \leq \ln(3/2)$. Moreover, by (3.8), we may write $D_n = b_n e^{\alpha n}$, with $(1/n)\ln b_n \rightarrow 0$, $\mathbb{P}$-a.s. Now, assuming $\alpha < \ln(3/2)$, it would follow that $\mathbb{E}[b_n] = 2/3\gamma^n$, with $\gamma > 1$, for which there is no evidence as far as numerical calculations are concerned.

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