AN ORDERED SET OF NÖRLUND MEANS

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ABSTRACT. Nörlund methods of summability are studied as mappings from $\ell_1$ into $\ell_1$.
Those Nörlund methods that map $\ell_1$ into $\ell_1$ are characterized. Inclusion results
are given and a class of Nörlund methods is shown to form an ordered abelian semi-
group.

KEY WORDS AND PHRASES. Inclusion Theorem, $\ell_1$ method, ordered abelian semigroup,
Nörlund method.

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1. INTRODUCTION.

Let $p$ be a complex sequence $p_0 \dagger 0$, and let $P_n = \sum_{k=0}^{n} p_k$, $n=0,1,2,\ldots$, denote
the $n+1$-st partial sum of $p$. Suppose the sequence $P_n$ is eventually non-zero. Let
$K$ be the least positive integer so that $P_n \dagger 0$ for all $n \geq K$. Define the Nörlund
method of summability $N_p$ by
\[ N_p[n,k] = \begin{cases} 
\frac{P_{n-k}}{P_n}, & \text{if } 0 \leq n < K, \ k \leq n, \\
\frac{P_{n-k}}{P_n}, & \text{if } n \geq K, \ k \leq n \text{ and} \\
0, & \text{otherwise.} 
\end{cases} \]

Let \( N \) denote the collection of all such Norlund methods \( N_p \). If \( P_n \to 0 \) for all \( n \geq 0 \), that is \( K = 0 \), then

\[ N_p[n,k] = \begin{cases} 
\frac{P_{n-k}}{P_n}, & \text{if } k \leq n \text{ and} \\
0, & \text{otherwise.} 
\end{cases} \]

Throughout we let \( \hat{P}_n = P_0 \) if \( 0 \leq n < K \) and \( \hat{P}_n = P_n \) if \( n \geq K \). The \( N_p \) transform of a sequence \( x \) is then given by \( N_p x \), where

\[ (N_p x)_n = (1/\hat{P}_n) \sum_{k=0}^{n} P_{n-k} x_k \]

for all \( n \geq 0 \).

A matrix summability method is called an \( \ell \)-\( \ell \) method if and only if it maps the space \( \ell \) into itself. In [4], Knopp and Lorentz proved that the matrix method \( A \) is \( \ell \)-\( \ell \) if and only if there exists some \( M > 0 \) such that

\[ \sup_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} |a_{nk}| \right\} < M. \]

In the special case of a Norlund method \( N_p \) we have:

**THEOREM 1.** The Norlund method \( N_p \) is \( \ell \)-\( \ell \) if and only if

(i) \( p \in \ell \), and

(ii) \( \hat{P}_n \to 0 \) as \( n \to \infty \).

**PROOF.** First suppose that (i) and (ii) hold. Since \( p \in \ell \) it implies \( \lim_{n} \hat{P}_n \) exists and by (ii) is non-zero. So there exist strictly positive numbers \( n \)
H and \( \delta \) such that \( \sum |p_n| < H \) and \( |\hat{p}_n| > \delta \) for all \( n \geq 0 \). Thus for each fixed \( k \),

\[
\sum_{n=k}^{\infty} |N[n,k]| = \sum_{n=k}^{\infty} |p_{n-k}/\hat{p}_n| < (1/\delta) \sum_{n=k}^{\infty} |p_{n-k}|
\]

\[
< H/\delta.
\]

Thus by the Knopp-Lorentz Theorem \( N \) is \( \ell - \ell \).

Now suppose that \( N \) is \( \ell - \ell \). Then

\[
\sum_{n=k}^{\infty} |p_{n-k}/\hat{p}_n| = O(1).
\]

In particular if \( n = k \) it implies \( |1/\hat{p}_n| = O(1) \) and hence \( \hat{p}_n \to 0 \) as \( n \to \infty \).

Now suppose that \( p \notin \ell \). We assert that the series \( \sum (|p_n/\hat{p}_n|) \) diverges.

For sufficiently large \( n \),

\[
|\hat{p}_n| = |P_n| = |\sum_{k=0}^{n} p_k| \leq |p_0| = S_n.
\]

Then for sufficiently large \( N \)

\[
\sum_{n=N+1}^{N+m} (|p_n/\hat{p}_n|) \geq \frac{1}{S_N} \sum_{n=N+1}^{N+m} |p_n| = 1 - S_{N+1}/S_{N+m}.
\]

But \( S_{N+m} \to \infty \) as \( m \to \infty \). So choose \( m \) large enough such that \( S_{N+m} > 2S_N \).

Then \( \sum_{n=N+1}^{N+m} (|p_n/\hat{p}_n|) > \frac{1}{2} \). The theorem now follows.

**COROLLARY 1** [1, Theorem 4]. Let \( N_p \) be a Norlund method with \( p_n \geq 0 \), \( p_0 > 0 \). Then \( N_p \) is \( \ell - \ell \) if and only if \( p \in \ell \).

2. We make the following definitions.
DEFINITION. Let \( N \) denote the collection of all \( N \in \mathcal{N} \) that are \( \xi \)-methods.

DEFINITION. Let \( \xi(N) \) consist of all sequences \( x \) such that \( N \) is in \( \xi \).

DEFINITION. Given two Norlund methods \( N_p \) and \( N_q \), \( N_p \) is \( \xi \)-stronger than \( N_p \) if and only if \( \xi(N_p) \subseteq \xi(N_q) \). The method \( N_q \) is strictly \( \xi \)-stronger than \( N_p \) provided \( \xi(N_p) \subset \xi(N_q) \), and \( N_p \) and \( N_q \) are \( \xi \)-equivalent provided \( \xi(N_p) = \xi(N_q) \).

DEFINITION. Given \( N_p \) and \( N_q \) define formally:

\[
(1) \quad p(z) = \sum_{n=0}^{\infty} p_n z^n, \quad P(z) = \sum_{n=0}^{\infty} \hat{P}_n z^n,
\]

\[
(2) \quad q(z) = \sum_{n=0}^{\infty} q_n z^n, \quad Q(z) = \sum_{n=0}^{\infty} \hat{Q}_n z^n, \quad \text{and}
\]

\[
(3) \quad a(z) = p(z)/q(z) = \sum_{n=0}^{\infty} a_n z^n, \quad b(z) = q(z)/p(z) = \sum_{n=0}^{\infty} b_n z^n.
\]

The next propositions follow by an argument similar to the one used in Theorem 18 of [3].

**Proposition 1.** If \( N_p \in \mathcal{N}_\xi \), then the series \( \sum_{n=0}^{\infty} p_n z^n \) and \( \sum_{n=0}^{\infty} \hat{P}_n z^n \) converge for \(|z| < 1\).

**Proposition 2.** If \( N_p, N_q \in \mathcal{N}_\xi \), then the series \( a(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( b(z) = \sum_{n=0}^{\infty} b_n z^n \) have positive radii of convergence and, moreover

\[
(1) \quad p_n = a_n q_0 + \ldots + a_0 q_n, \\
(2) \quad \hat{P}_n = a_n \hat{Q}_0 + \ldots + a_0 \hat{Q}_n, \\
(3) \quad q_n = b_n p_0 + \ldots + b_0 p_n, \quad \text{and}
\]

\[
(4) \quad \hat{Q}_n = b_n \hat{P}_0 + \ldots + b_0 \hat{P}_n.
\]

**Proposition 3.** Suppose \( N_p \in \mathcal{N}_\xi \) and the sequence \( S_n = \{S_n\} \) is in \( \xi(N_p) \). Then the series \( S(z) = \sum_{n=0}^{\infty} S_n z^n \) has positive radius of convergence.
PROOF. Let \( h(z) = 1/p(z) \). By Proposition 1, \( p(z) \) defines an analytic function for \(|z|<1\). Since \( p_0 \neq 0 \), by the continuity of \( p(z) \) there exists some \( a \in (0,1) \) such that \( p(z) \neq 0 \) for all \( z \in (-a,a) \). Therefore \( h(z) = \sum_{n=0}^{\infty} h_n z^n \) has positive radius of convergence. Now for \( n \geq 0 \),

\[
\sum_{n=0}^{\infty} [h_n \hat{P}_k (N,S)_k] = S_n.
\]

Since \( S \in \ell(N_p) \), it follows that the series \( \sum_{k=0}^{\infty} \hat{P}_k (N,S)_k z^k \) converges for \(|z|<1\). Therefore \( \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{P}_k (N,S)_k z^k \) has positive radius of convergence since

\[
\sum_{n=0}^{\infty} S_n z^n = \{h(z)\} \{\hat{P}_k (N,S)_k z^k\},\quad k=0
\]

3. The symmetric product \( g = p*q \) of the sequences \( p \) and \( q \) is defined by

\[
g_n = p_0 q_n + \ldots + p_n q_0 \quad \text{for all } n \geq 0.\]

Given Norlund methods \( N_p \) and \( N_q \) in \( N \) we say \( g = N_p \ast N_q \) is the symmetric product of \( N_p \) and \( N_q \) provided \( N_s \in N \).

In order to prove an inclusion result for two Norlund methods in \( N_k \) we need the following lemma.

Lemma 1. Let the complex sequences \( p \) and \( q \) be given and define \( r = p* q \).

Suppose \( N_p, N_r \in N \). Then in order that \( \ell(N_p) \subseteq \ell(N_r) \) it is necessary and sufficient that there is some \( M > 0 \), independent of \( k \), such that

\[
|\hat{P}_k| \sum_{n=k}^{\infty} \frac{|q_{n-k} / \hat{R}_n|}{M} < M.
\]

Proof. Let \( x = \{x_n\} \) be any sequence. Then

\[
(N_r x)_n = (1/\hat{R}_n) \sum_{k=0}^{\infty} [q_{n-k} \hat{P}_k (N,r)_k].
\]
Let $e_{nk} = q_{n-k}/R_n$ if $k \leq n$, and 0 if $k > n$. Then by the Knopp-Lorentz Theorem

$(\ell(N_p) \subseteq \ell(N_r))$ if and only if there exists some $M > 0$ such that

$$\sup_{k} \{ \sum_{n=k}^{\infty} |e_{nk}| \} < M.$$ 

That is,

$$\sup_{k} \{ \sum_{n=k}^{\infty} |q_{n-k}/R_n| \} < M.$$ 

The lemma can now be used to get the desired inclusion result.

**Theorem 2.** Suppose $N_p, N_q \in \mathbb{N}_{\ell}$. The $(\ell(N_p) \subseteq \ell(N_q))$ if and only if

$$b \{ b_n \} \in \ell.$$ 

**Proof.** In the previous lemma replace the sequence $q$ with the sequence $b$ which implies $r = p*b = q$. Then $(\ell(N_p) \subseteq \ell(N_r) = \ell(N_q))$ if and only if there exists some $M > 0$, independent of $k$, such that

$$\sup_{n=k}^{\infty} \{ |\hat{P}_k| \sum_{n=k}^{\infty} |b_{n-k}/\hat{Q}_n| \} < M.$$ 

But since $N_p, N_q \in \mathbb{N}_{\ell}$, $|\hat{P}_k|$ and $|\hat{Q}_k|$ respectively are bounded by two strictly positive constants. Thus

$$\sup_{n=k}^{\infty} \{ |\hat{P}_k| \sum_{n=k}^{\infty} |b_{n-k}/\hat{Q}_n| \} < M,$$ 

is equivalent to

$$b \in \ell.$$ 

**Corollary 2.** Suppose $N_p, N_q \in \mathbb{N}_{\ell}$. Then

(i) $(\ell(N_p) = \ell(N_q))$ if and only if both $a = \{a_n\} \in \ell$ and $b = \{b_n\} \in \ell$ and

(ii) $(\ell(N_p) \subseteq \ell(N_q))$ if and only if $a = \{a_n\} \notin \ell$ and $b = \{b_n\} \in \ell$.

**Corollary 3.** Suppose $N_p \in \mathbb{N}_{\ell}$ and $h(z) = 1/p(z)$. Then $(\ell(N_p) = \ell$ if and only if $h \in \ell$. 

Proof. Let $I$ be the identity matrix so that $t(I) = t$. Then

\[ I(z) = \sum_{n=0}^{\infty} i_n z^n = 1; \]
that is $i_0 = 1$ and $i_n = 0$ for all $n \geq 1$. Therefore

\[ a(z) = p(z)/I(z) = p(z) \quad \text{and} \quad b(z) = I(z)/p(z) = h(z). \]
The corollary now follows.

Example. The binary matrix $B = (b_{nk})$ is given by $b_{nk} = 1$, if $n=k=0$, $1/2$, if $k=n-1$ or $k=n$, $n > 1$, and 0 otherwise. Thus $B$ is the $l-\ell$ Norlund method defined by $p_0 = 1 = p_1$, $p_n = 0$ for $n > 2$. It now follows by Corollary 3 that $l \subseteq l(B)$.

The next result addresses the question of when two Norlund methods $N_P$, $N_Q \in N_\ell$ are comparable. We prove that changing only the first term in the generating sequence $N_P$ can result in a method $N_Q \in N_\ell$ satisfying

\[ l(N_P) \cap l(N_Q) = \ell. \]
That is, the methods are not comparable.

Theorem 3. Suppose $N_P \in N_\ell$. Let $q = (p_0', p_1', \ldots)$ with $p_0' \neq p_0$.

If the sequence $q$ satisfies $\lim_{n \to \infty} |Q_n| > 0$ then $N_Q$ is $\ell-\ell$ and moreover

\[ l(N_P) \cap l(N_Q) = \ell. \]

Proof. Since $q \in \ell$, $N_Q \in N_\ell$, $|\hat{P}_n|$ and $|\hat{Q}_n|$ respectively lie between two strictly positive constants. Now for any sequence $x$

\[ (N_Q x)_n = (p_0' - p_0)x_n/\hat{Q}_n + [(N x)_n][\hat{P}_n/\hat{Q}_n], \]
and hence

\[ |(N_Q x)_n| + |\hat{P}_n/\hat{Q}_n||N x)_n| \geq |p_0' - p_0||x_n||\hat{Q}_n|. \]
Therefore, if $x \in l(N_P)$, $x \notin \ell$, then $x \notin l(N_Q)$.

Similarly if $x \in l(N_Q)$, $x \notin \ell$, then $x \notin l(N_P)$.

Corollary 4. Suppose $p$ is a positive number sequence in $\ell$. Let

\[ q = (p_0', p_1', \ldots), \]
where $p_0' > 0$ and $p_0' \neq p_0$. Then $N_P$ and $N_Q$ are $\ell-\ell$ with

\[ l(N_P) \cap l(N_Q) = \ell. \]
Theorem 4. Suppose \( p \) is a sequence in \( \ell \), with \( p_n \) eventually non-zero. Let 
\[ q = (p_1, p_2, \ldots) \]. If \( q_n \) is eventually non-zero, then \( N_p, N_q \in \ell \) with 
\[ \ell(N_p) \cap \ell(N_q) = \ell. \]

The next theorem is a special case of Theorem 2.

Theorem 5. If \( N \) and \( N \) are Norlund methods with

(i) \( p_{n+1}/p_n \geq p_n/p_{n-1}, n > 0, p_0 = 1, p_n > 0 \),
(ii) \( q_n > 0, q_0 = 1 \), and
(iii) \( p, q \in \ell \),

then \( \ell(N_p) \subseteq \ell(N_q) \).

Proof. By Theorem 22 of [3]

\[ \{p(z)\} = 1 - \frac{c_1 z}{c_2 + \ldots}, \text{ where } c_n > 0 \text{ for all } n \geq 0 \text{ and } \sum c_n \leq 1. \]

Then for small \( |z| \), \( q(z)/p(z) = \sum \frac{q_n \gamma_0 + \ldots + q_0 \gamma_n}{n=0} z^n \), where \( \gamma_n = -c_n \) for \( n > 0 \) and \( \gamma_0 = 1 \). So that if \( q(z)/p(z) = \sum \frac{b_n z^n}{n=0}, \) we have

\[ \sum_{n=0}^{\infty} |b_n| \leq \sum_{n=0}^{\infty} (|q_0| |\gamma_n| + \ldots + |q_n| |\gamma_0|) \]

\[ = \left\{ \sum_{n=0}^{\infty} |q_n| \right\} \left\{ \sum_{n=0}^{\infty} |c_n| \right\} \]

\[ < \infty. \]

Therefore by Theorem 2, \( \ell(N_p) \subseteq \ell(N_q) \).

Corollary 6. If \( N \) and \( N \) are \( \ell-\ell \) Norlund methods such that

(i) \( p_{n+1}/p_n \geq p_n/p_{n-1}, n > 0, p_0 = 1, p_n > 0 \),
and

(ii) \( q_{n+1}/q_n \geq q_n/q_{n-1}, n > 0, q_0 = 1, q_n > 0 \),

then \( \ell(N_p) = \ell(N_q) \).
For example if \( p_n = \rho^n \) and \( q_n = \tau^n \) where \( 0 < \rho, \tau < 1 \), then \( \ell(N_p) = \ell(N_q) \).

Moreover by Corollary 3, \( \ell(N_p) = \ell = \ell(N_q) \). If \( \gamma_n = 1/n^k \), \( k > 1 \) and \( n > 0 \), then \( \ell(N_p) = \ell(N_q) \).

4. We now show that \( N_q \) forms an ordered abelian semigroup. The order relation is set inclusion between the absolute summability fields, and the binary operation is the symmetric product of the generating sequences. We need the following lemmas.

Lemma 2. Suppose \( N_p \in N_q \) and \( p \) is a sequence for which \( \lim_{n \to \infty} \hat{p} = \hat{P} \neq 0 \). Let \( r = p*q \). Then \( \lim_{n \to \infty} (\hat{R}_n/\hat{P}_n) = 1 \).

Proof. We assert that \( N_q \) is a regular method. By the Silverman-Toeplitz Theorem, see for example [6], it suffices to show that

(i) \( \lim_{n \to \infty} \hat{R}_n/\hat{Q}_n = 1 \) for each fixed \( \nu \geq 0 \), and

(ii) \( \sum_{\nu=0}^{\infty} \hat{Q}_n = 0 \).

But since \( N_q \in N_q \), (i) and (ii) follow. Now for all \( n \) sufficiently large

\[
\hat{R}_n/\hat{Q}_n = (1/\hat{Q}_n) \sum_{k=0}^{n} r_k = (1/\hat{Q}_n) \sum_{k=0}^{n} q_{n-k} \hat{P}_k,
\]

which is the \( N_q \) transform of the convergent sequence \( \{P_n\} \). Thus

\[
\lim_{n \to \infty} (\hat{R}_n/\hat{Q}_n) = 1.
\]

Lemma 3. If \( N_p, N_q \in N_q \) and \( r = p*q \), then \( N_r \in N_q \).

Proof. By Lemma 1, we have \( \lim_{n \to \infty} \hat{R}_n = \hat{P}_n \). In order to have \( N_r \in N_q \), we need to show \( N_r \) is an \( \ell \)-method. By Theorem 1, it suffices to show that \( r \in \ell \).

But

\[
\sum_{n=0}^{\infty} |r_n| \leq \left( \sum_{n=0}^{\infty} |p_n| \right) \left( \sum_{n=0}^{\infty} |q_n| \right).< \infty.
\]
Lemma 4. If $N_p, N_q \in \mathcal{N}_\mathcal{L}, r = p*q$, then $N_r$ is $\mathcal{L}-\mathcal{L}$ with

$$\mathcal{L}(N_p) \cup \mathcal{L}(N_q) \subseteq \mathcal{L}(N_r).$$

Proof. By Lemma 2, $N_r \in \mathcal{N}_\mathcal{L}$. By Theorem 2 it follows that

$$\mathcal{L}(N_p) \cup \mathcal{L}(N_q) \subseteq \mathcal{L}(N_r).$$

Lemma 5. Suppose $N_p, N_q, N_s \in \mathcal{N}_\mathcal{L}, \mu = p*s$, and $\upsilon = q*s$.

(i) If $\mathcal{L}(N_p) \subseteq \mathcal{L}(N_q)$, then $\mathcal{L}(N_\mu) \subseteq \mathcal{L}(N_\upsilon)$.

(ii) If $\mathcal{L}(N_p) \subseteq \mathcal{L}(N_q)$, then $\mathcal{L}(N_\mu) \subseteq \mathcal{L}(N_\upsilon)$.

Proof of (i). Let $b(z) = q(z)/p(z)$ and $c(z) = \upsilon(z)/\mu(z)$. By Theorem 2, $b \in \mathcal{L}$. We need to show that $c \in \mathcal{L}$. Now for $|z| < 1$,

$$\upsilon(z) = \sum_{n=0}^{\infty} c(z) = \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} q(z)s(z).$$

Similarly $\mu(z) = p(z)s(z)$. Therefore

$$\sum_{n=0}^{\infty} = c(z) = \upsilon(z)/\mu(z) = b(z) = \sum_{n=0}^{\infty} b(z) n.$$
Proposition 5. There exists infinite chains of Nörlund methods from $N_k$.

Proof. Let $p^{(1)}(z) = \sum_{n=0}^{\infty} p_n^{(1)} z^n$, where \(\{p_n^{(1)}\} \in \ell_k\), \(p_n^{(1)} \geq 0\) for \(n \geq 0\), and

\[ p_0^{(1)} > 0. \]

Then $N(p^{(1)}) \in N_k$. Define

\[ p^{(n)}(z) = (1+z)^{n-1} \sum_{k=0}^{\infty} p_k^{(n-1)} z^k, \text{ for } n \geq 2. \]

Then $N(p^{(n)}) \in N_k$ for $n \geq 1$. Moreover by Proposition 4 we have

\[ \ell(N) \subseteq \ell(N^{(1)}) \subseteq \ell(N^{(2)}) \subseteq \cdots \subseteq \ell(N^{(n)}) \subseteq \cdots. \]

The next theorem is a different version of Lemma 3.

Theorem 7. If $N \in N_k$, \(\lim_{n \to \infty} P_n = P + 0\), and $r = p*q$, then $\ell(N_p) \subseteq \ell(N_r)$.

Proof. By Lemma 1 it suffices to show there exists some $M > 0$, independent of $k$, such that

\[ \left| \frac{p_k}{q_{n-k}} \right| \leq M. \]

Since $N \in N_k$, there exists some $H > 0$ such that

\[ \left| \frac{\hat{q}_{n-k}}{\hat{q}_n} \right| < H \]

for all $n \geq k$ and for all $k \geq 0$. Then by Lemma 1, we have

\[ \left| \frac{\hat{p}_{n-k}}{\hat{r}_n} \right| = \left| \frac{\hat{p}_{n-k} / \hat{R}_n}{\hat{q}_{n-k} / \hat{Q}_n} \right| < \infty \]

for all $n \geq k$. Thus the result follows.

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REFERENCES


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