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ON COMPLEX L_1-PREDUAL SPACES

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ABSTRACT - This paper contains some characterisations of complex L-balls, including interpolation theorems which are analogs of Edward's separation theorem for simplices.

KEY WORDS AND PHRASES: Boundary measures, Choquet ordering, Haar measure, Lindenstrauss space, Simplex.

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1. INTRODUCTION.

The purpose of this paper is to furnish some characterisations of complex L_1-predual spaces which are being known as Lindenstrauss spaces after [1]. The dual unit balls of such spaces now-a-days called L-balls have been characterised by many authors including Lazar & Lindenstrauss [2], Lazar [3], [4], Lau [5] and others when the spaces are real. But their complex versions far from being trivial follow-ups seem to be much complicated and in reality sometimes need ingenuity to be formulated even. This paper contains some complex versions of Lau's results [5] embodied in Theorem 3.

2. NOTATIONS AND PRELIMINARIES

For a compact convex subset K of a locally convex Hansdorff space E, $\partial_e K$ stands for the set of its extreme points; M(K) for the Banach space (with total variation as norm) of complex regular Borel measures on K; $M_1(K)$ for the set of members of M(K) with norm ≤ 1; C(K), A(K), P(K) for the space of all real-valued continuous functions, continuous affine functions, continuous convex functions on K respectively.

For bounded real-valued functions f on K, the upper envelope is denoted by $\hat{f}$ and the lower envelope by $\check{f}$. A measure μ is said to be a boundary measure if $|\mu|$ is maximal in the ordering of Choquet; in fact μ is a boundary measure iff $\mu(\hat{f}-f)=0$ for all $f \in C(K)$ [6;p.129]. We shall also write $\Gamma = \{Z \in ¢: |Z|=1\}$.

If V is a complex Banach space, the dual unit ball $K=(V^*)_1$ is convex and compact in the w*-topology. We define the map hom f as (hom f)(x) = $\int_\Gamma \bar{\alpha}\, f(\alpha x)\, d\alpha$ for semi-continuous function f on K, where $d\alpha$ is the unit Haar measure on Γ. Clearly hom f is Γ - homogeneous, i.e. (hom f) $(\beta x) = \beta$(hom f)(x) for $\beta \in \Gamma$. One can easily show that hom restricted to C(K) are norm-decreasing projections of C(K) onto the space of Γ - homogeneous continuous functions on K. The adjoint projection hom defined as hom $\mu=\mu \circ$ hom is also a norm decreasing w*-continuous projection of M(K) onto a linear subspace $M_{hom}(K)$ of M(K). We can write $(\text{hom } f)(x) = S_1 f(x) + i\, Tf(x)$

where $S_1 f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\theta \;f(xe^{i\theta})d\theta$, $Tf(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\theta f(xe^{i\theta})d\theta$

If we write $(Sf)(x) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\theta \; f(xe^{i\theta})d\theta$

(which multiplied by π gives what Roy [7] has defined as f_+), then

$2(S_1 f)(x) = \text{odd } Sf(x)$. (2.1)

Throughout the paper we shall write $A_o(K)$ for the set of all continuous Γ - homogeneous affine functions on $K=(V^*)_1$.

3. Main Results

For real Banach space V, the following results are recently proved.

Theorem 1. If K is the dual unit ball of a real Banach space V, then the following are equivalent:

(i) V is an L_1-predual space.

(ii) If μ_1, μ_2 are boundary measures on K having the same barycentre, then odd μ_1 = odd μ_2.

(iii) For $f \in P(K)$, odd$\hat{f}$ is affine.

(iv) For any $f \in P(K)$, $\hat{f}(0) = \frac{1}{2}\sup\{f(x) + f(-x): x\in K\} = \sup\{\text{even } f(x) : x \in K\}$.

(v) For any l.s.c. concave function f on K such that even $f \geq 0$, there exists a continuous affine symmetric function a on K such that $f \geq a$.

(vi) If f, -h are l.s.c. concave functions on K such that $h \leq f$ and $\sup\{\text{even } h(x) : x\in K\} \leq \inf\{\text{even } f(x) : x \in K\}$, then these exists a continuous affine symmetric function a on K such that $h \leq a \leq f$.

The equivalence of (i) - (iv) is due to Lazar [4] while that of (i), (v), (vi) is proved by Lau [5].

Many interesting developments are noted when efforts are made to obtain complex analogs of these results (many others not stated here) of real Lindenstrauss spaces. A brilliant step towards this have been made by Effros [8] who has shown that odd μ is to be replaced by hom μ in complex space. Olsen [9] has shown that the hypothesis even $f \geq 0$ in (v) is to be replaced by $\Sigma f(\zeta_k x) \geq 0$ for $\zeta_k \in \Gamma$ with $\Sigma\zeta_k = 0$, $x\in K$. Subsequently Roy [7] has tried to give complex analog of (iv), replacing odd $\hat{f}$ by odd$(Sf)^\wedge$. His formulation is rather partial. But [9] contains some interesting examples.

Below we give a characterisation of complex L_1-predual space V which is a kind of complex analog of Lau's result and is due to Olsen [9].

Theorem 2. If K is the dual unit ball of a complex Banach space V, then the following are equivalent:

(i) V is an L_1-predual space;

(ii) For every l.s.c. concave function f on K such that $\Sigma f(\zeta_k x) \geq 0$ for all $x\in K$ and $\zeta_k \in \Gamma$, $k = 1,2,\ldots,n$ with $\Sigma\, \zeta_k = 0$, there is an $a \in A_o(K)$ such that re $a(x) \leq f(x)$ for all $x\in K$.

We give in Theorem 3, some complex analogs of Lau's result. However to start with, we furnish a Lemma below:

Lemma 1. If μ be a non-zero positive measure on a compact convex subset K of a locally convex Hausdorf space E, then for all u.s.c. convex function f on K

$$f(x) \leq \mu(K)^{-1} \int f(y)d\mu \qquad \text{where } r(\mu) = x.$$

Proof: By a well-known result [10; I.2.2.], the stated inequality holds for $f \in P(K)$. Now applying Mokobodzki([10; I.5.1]) that for every u.s.c. convex function f, there is a dscending net $\{f_\alpha : f_\alpha \in P(K)\}$ which converges to f, we get the desired result.

Our main result is

Theorem 3. If K is the dual unit ball of a complex Banach space V, then the following are equivalent:

(i) V is an L_1-predual space;

(ii) If f is a l.s.c. concave function on K with even $(Sf)(x) \geq 0$ for all $x \in K$, then there exists an $a \in A_o(K)$ such that re $a \leq f$ on K;

(iii) If f, -h are l.s.c. concave functions on K such that $h \leq f$ and even $S\,h\,(x) \leq 0 \leq$ even $S\,f(y)$ for all $x,y \in K$; then there exists an $a \in A_o(K)$ such that $h \leq$ re $a \leq f$ on K.

(iv) If f, -h are l.s.c. concave functions on K such that $h \leq f$ and

$$\sup\{\sum_{k=1}^{n} \alpha_k h(\zeta_k x)\ ;\ x \in K,\ n \in N,\ 0 \leq \alpha_k,\ \Sigma\alpha_k = 1,\ \zeta_k \in \Gamma,\ \Sigma\alpha_k\zeta_k = 0\}$$

$$\leq 0 \leq \inf\{\sum_{k=1}^{n} \alpha_k f(\zeta_k x):\ x \in K, n \in N,\ 0 \leq \alpha_k,\ \Sigma\alpha_k = 1,\ \zeta_k \in \Gamma,\ \Sigma\alpha_k\zeta_k = 0\};$$

then there is an $a \in A_o(K)$ such that $h \leq$ re $a \leq f$ on K;

(v) If g is an u.s.c. convex function on K, then $\hat{g}(0) \leq \sup\{\Sigma\alpha_k g(\zeta_k x):$ $x \in K,\ n \in N,\ 0 \leq a_k;\ \Sigma a_k = 1,\ \zeta_k \in \Gamma,\ \Sigma a_k\,\zeta_k = 0\}$.

Proof . (i) $\to$ (ii).

We shall, in fact, show that (ii) is implied by Theorem 2 (ii). So let f be l.s.c. concave on K such that even $Sf(x) \geq 0$. We define

$$F(x) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \left[\cos\theta\ f(xe^{i\theta})\right] d\theta$$

Then $F(x) = 2\,Sf(x)$. Clearly F is l.s.c. concave. Let $\zeta_k \in \Gamma$ for $k = 1,2,\ldots,n$ be such that $\Sigma\,\zeta_k = 0$. Now note that $Sf(x) = S_1 f(x) +$ even $Sf(x)$ and that Σ Hom $f\,(\zeta_k x) = 0$. Thus $\Sigma F(\zeta_k x) = 2\ \Sigma Sf(\zeta_k x) = 2\Sigma S_1 f(\zeta_k x) + 2\Sigma$ even $Sf(\zeta_k x) =$ $2\ \Sigma$ even $Sf(\zeta_k x)$ which is ≥ 0 by hypothesis. Consequently by Theorem 2 (ii),, there is a function $b \in A_o(K)$ such that re $b \leq F$ on K.

We consider the measure $\mu = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos\theta\ \varepsilon\ (xe^{i\theta})d\theta$

where $\varepsilon(y)$ is the Dirac measure at y. By [6: p. 115], $r(\mu) = x$. Also $\mu(k) = \frac{4}{\pi}$. On applying Lemma 1, we have $2F(x) \leq \mu(K)f(x)$ i.e. $F(x) \leq \frac{2}{\pi} f(x)$. Putting $a = \frac{\pi b}{2}$, we have re $a \leq f$ on K.

(ii) → (iii). Let f, $-h$ be l.s.c. concave functions on K such that $h \leq f$ and even $S\,h(x) \leq 0 \leq$ even $Sf(y)$ for all $x,y, \in K$.

We first show that $h(x) + h(-x) \leq 0 \leq f(y) + f(-y)$ for all $x,y \in K$. Let us establish the last inequality, since the first one can be done similarly. To do so we take the measure

$$\mu = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos\theta \quad \varepsilon(xe^{i\theta})d\theta$$

and find as before by [6, p. 115] that $r(\mu) = x$, $\mu(K) = \frac{4}{\pi}$.
Now apply Lemma 1 to get $f(y) + f(-y) \geq 0$ from even $Sf(y) \geq 0$.
Now we define $F(x) = f(x) \wedge (-h)(-x)$.

Then F is l.s.c. concave. Moreover by the hypothesis and the inequalities just proved, we have $F(x) + F(-x) \geq 0$ for all $x \in K$ so that even $SF(x) \geq 0$.
By (ii) then we have an $a \in A_o(K)$ such that re $a \leq F$. This a is, in fact, the function with the desired property.

(iii) → (iv). Let f, $-h$ be two l.s.c. concave functions on K, which satisfy the conditions given in the hypothesis of (iv). Then clearly even $(Sh)(x)$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\theta\left[h(xe^{i\theta}) + h(-xe^{i\theta})\right]d\theta \leq 0$$

Similarly even $(Sf)(y) \geq 0$ for all $y \in K$.
So by (iii) there is an $a \in A_o(K)$ such that $h \leq$ re $a \leq f$ on K.

(iv) → (v). Suppose that g is an u.s.c. convex function and let
$g_o = \sup\{\sum_{k=1}^{n} \alpha_k g(\zeta_k x): \; x \in K;\; n \in N;\; 0 \leq \alpha_k;\; \Sigma\alpha_k = 1,\; \zeta_k \in \Gamma,\; \Sigma\alpha_k\zeta_k = 0\}$.
We assume that $g_o = 0$; there will be no loss of generality in the assumption since $(g + \alpha)\hat{}\,(0) = \hat{g}(0) + \alpha$
for positive real number α.

We define $F = -\sigma g$ where $(\sigma g)(x) = g(-x)$. Clearly F is l.s.c. concave. Since $g + \sigma g \leq 2g_o$ by the definition of g_o, it follows that $g \leq F$. Moreover for $n \in N$, $\alpha_k \geq 0$, $\Sigma\alpha_k = 1$, $\zeta_k \in \Gamma$, $\Sigma\alpha_k\zeta_k = 0$, $x \in K$, we have $\Sigma\alpha_k F(\zeta_k x) = -\Sigma\alpha_k g(-\zeta_k x)$
so that $\inf\{\sum_{k=1}^{n} \alpha_k F(\zeta_k x);\; x \in K,\; n\varepsilon,\; \alpha \geq 0\; \Sigma\alpha_k = 1,\; \zeta_k \in \Gamma,\; \Sigma\alpha_k\zeta_k = 0\} = -g_o = 0$.
Thus by (iv), there is an $h \in A_o(K)$ such that $g \leq$ re $h \leq F$. We put re $h = a \in A(K)$.
Now since $g \leq a$, we have $\hat{g}(0) \leq a(0)$. Again $-\sigma g \geq a - 2g_o \in A(K)$, so that $a(0) - 2g_o \leq (-g)\hat{}\,(0) = -\hat{g}(0)$.

Thus $\hat{g}(0) \leq a(0) \leq -\hat{g}(0) + 2g_o$ so that $\hat{g}(0) \leq g_o$ and the result follows.

(v) → (i) is the same as [7;p.103]

Note: Our result in (v) is sharper than Roy's result [7; Thm 3.3 (iii)] that for $g \in P(K)$, $\hat{g}(0) \leq \sup\{\Sigma\alpha_k g(\zeta_k x): x \in K,\; n \in N,\; 0 \leq \alpha_k,\; \Sigma\alpha_k = 1,\; \zeta_k \in \Gamma,\; \Sigma\alpha_k\zeta_k = 0\}$.
In fact this follows from (v) immediately, since the reverse inequality is evident from the concave character of $\hat{g}$.

REMARKS:

In analog with Lazar's selection theorem for Choquet simplex, Lazar & Lindenstrauss [2] formulated a selection theorem for real L-balls which was followed by a complex version by Olsen [9]. Our results which are chiefly complex analogue of Lau's result [3] seem to resemble Edward's interpolation theorem [10; [II.3-10]]for simplices.

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