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ON THE BOUNDEDNESS AND OSCILLATION OF SOLUTIONS TO $(m(t)x')' + a(t)b(x) = 0$

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ABSTRACT. In this paper, we discuss under what conditions the solutions to $(m(t)x')'$ $+ a(t)b(x)=0$ are bounded, oscillatory, and stable. Furthermore, some applications of these results are given during the discussion and proofs of the theorems.

KEY WORDS AND PHRASES. Bounded, oscillatory, stable, periodic, monotonic.
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1. INTRODUCTION.

In this article, we shall present some general theorems concerning the second order nonlinear differential equation,

$$(m(t)x')' + a(t)b(x) = 0 \tag{1.1}$$

For some related results concerning (1.1) see Bhatia [1] and Kroopnick [2]. A general boundedness theorem is now proven.

THEOREM 1. Suppose $a(t)$ and $m(t)$ are $\in C^1[0,\infty)$ and, furthermore, suppose $a(t) \geq a_o \geq 0$ and $m(t) \geq m_o \geq 0$ for some positive constants a_o and m_o. Also, let $m'(t) \leq 0$, $a'(t) \leq 0$, and let $b(x) \in C(-\infty, +\infty)$. Finally, if $\lim_{|x|\to\infty} B(x) = \int_0^x b(u)du = +\infty$, then $|x|$ and $|x'|$ are bounded as $t\to\infty$.

PROOF. By standard existence theory, (1.1) has at least one solution satisfying $x(0) = x_o$ and $x'(0) = \dot{x}_o$, and existing on some interval $[0,T)$, $T > 0$. Considering any such solution of (1.1) on $[0,T)$. Multiply (1) by $m(t)x'$ and integrate by parts from 0 to $t < T$ obtaining,

$$\tfrac{1}{2}(m(t)x'(t))^2 + a(t)m(t)B(x(t)) - \int_0^t B(x(s))(a'(s)m(s) + m'(s)a(s))ds = a(0)m(0)B(x(0)) + \tfrac{1}{2}(m(0)x'(0))^2 = a(0)m(0)B(x_o) + \tfrac{1}{2}(m(0)\dot{x}_o)^2 \tag{1.2}$$

Now all terms in (1.2) are positive for $|x|$ large except, perhaps, for $-\int_0^t B(x(s))(a'(s)m(s) + m'(s)a(s))ds = -\int_0^t B(x(s))d(a(s)m(s))$.

We shall show that this term is bounded from below. By hypothesis, $B(x) > -d^2$ so $-\int_0^t B(x(s))d(a(s)m(s)) > d^2\int_0^t d(am) > -a_o m_o d^2$. Consequently, should $|x|$ or $|x'|$ become large, the LHS of (1.2) becomes infinite which is impossible since the RHS of (1.2) is finite. That is, all terms in (1.2) must stay bounded. So the solution may be extended on $[0,\infty)$ and, therefore, both $|x|$ and $|x'|$ are bounded.

EXAMPLE 1. The above theorem shows that all solutions to $x'' + x^n - k = 0$ are bounded when n is an odd integer. Here we have $m(t) = a(t) = 1$ and $b(x) = x^n - k$. It is well known that when $k = 0$ all solutions are periodic.

REMARK 1. When a forcing term f(t) is introduced in (1.1) all solutions to (1.1) are still bounded provided $\int_0^x b(s)ds \geq M|x|^a (M > 0,\ a > 1)$ and $|x|$ sufficiently large. Specifically, consider

$$(m(t)x')' + a(t)b(x) = f(t) \tag{1.3}$$

Multiplying (1.3) by $m(t)x'$ and integrating from 0 to t as before we get,

$$\text{LHS of (2)} = \text{RHS of (2)} + \int_0^t f(s)m(s)x'(s)ds \tag{1.4}$$

Now by the mean value theorem, the term $\int_0^t f(s)m(s)x'(s)ds = f(p)m(p)(x(t) - x(0))$ $(0 < p < t)$. So long as f(t) is bounded and continuous the result follows since any unbounded solutions of (1.3) would approach infinity faster on the LHS of (1.4) than the RHS of (1.4) which is impossible.

The next theorem utilizes the following lemma of Utz [3].

LEMMA 1. Suppose w(t) is a real function for which w(t) is defined for $t \geq a$, a constant. (i) If for all $t \geq T$, $w'(t) < 0$ and $w''(t) \leq 0$, then $\lim w(t) = -\infty$. (ii) if for all $t \geq T$, $w'(t) > 0$ and $w''(t) \geq 0$, then $\lim_{t\to\infty} w(t) = +\infty$.

With the help of this lemma the following result may now be proven.

2. MAIN RESULTS

THEOREM 2. The hypotheses are the same as Theorem I. In addition, suppose $xb(x) > 0$ for $x \neq 0$, then all solutions to (1.1) are oscillatory.

PROOF. Suppose x(t) does not oscillate, then x(t) must be of fixed sign, say $x(t) > 0$ for $t \geq T \geq 0$ (a similar argument works for $x(t) < 0$). $x'(t)$ must also be of fixed sign. Otherwise, since $(m(t)x'(t))' = -a(t)b(x) < 0$, we have $x''(t) = -a(t)b(x) / m(t) < 0$ at points where $x'(t) = 0$. However, this implies x(t) has an infinite number of relative maxima which is impossible. Also, $x'(t) > 0$ for if $x'(t)$ were negative, then $x''(t) > 0$. Thus, by our lemma, we would have $\lim_{t\to\infty} x(t) = -\infty$ which is impossible since x(t) is bounded.

Upon integrating (1.1) from 0 to t we obtain,

$$m(t)x'(t) = m(0)\dot{x}_o - \int_0^t a(s)b(x(s))ds \tag{2.1}$$

Since $x'(t) > 0$, $\lim_{t\to\infty} x(t) = a_1 > 0$ and $\lim_{t\to\infty} x'(t) = 0$ since x(t)is bounded and ultimately monotonic. From (2.1), we see, however, that $\lim_{t\to\infty} x'(t) = -\infty$, a contradiction. Therefore, x(t) must oscillate. Furthermore, (1.2) implies that all solutions are stable since the LHS is positive and its size is determined by the RHS of (1.2).

Actually, all bounded solutions of (1.1) must be oscillatory under somewhat weaker conditions (cf. Bhatia [1], in his paper showed that oscillation of solutions occurs as long a $b'(x) \geq 0$ whether $x(t)$ is bounded or not). These conditions are discussed in the next theorem.

THEOREM 3. Suppose $a(t) \in C[0,\infty)$, $a(t) > 0$ and $\int_0^\infty a(t)du = \infty$. Furthermore, let $h_1 \geq m(t) \geq h_o > 0$ for some constants h_o, h_1, $m(t) \in C^1[0,\infty)$ and let $b(x) \in C(-\infty, +\infty)$, $xb(x) > 0$ for $x \neq 0$, then any bounded solution to (1.1) is necessarily oscillatory (note: $\int_0^{\pm\infty} b(u)du < \infty$ is possible).

PROOF. Suppose $x(t)$ does not oscillate, then using the same reasoning employed in Theorem II, we have $\lim_{t\to\infty} x(t) = k > 0$ and $\lim_{t\to\infty} x'(t) = 0$. However, from (2.1) we have $\lim_{t\to\infty} x(t) = -\infty$ which is impossible. Thus, $x(t)$ must oscillate.

EXAMPLE 2. Consider the differential equation

$$x'' + b(x) = 0 \tag{2.2}$$

where $b(x) \in C(-\infty, +\infty)$, $xb(x) > 0$ for $x \neq 0$. Let $V(x,x') = \frac{1}{2}x'^2 + \int_0^x b(u)du$. Since $\frac{dV}{dt} = x'x'' + b(x)x' = x'(-b(x)) + b(x)x' = 0$, we must have by Liapunov's theorem (Petrovski [4], p. 151) that any solution with initial conditions near the origin is stable and oscillatory by Theorem III. In fact, if $\int_0^{\pm\infty} b(u)du = \infty$, then all solutions are periodic.

EXAMPLE 3. The differential equation

$$x'' + x\exp(-x^2) = 0 \tag{2.3}$$

has as its general solution $x'^2 - \exp(-x^2) = k$. The solutions are unbounded when $k \geq 0$ and bounded for $k < 0$. The latter case corresponds to initial conditions near the origin of the (x,x') plane. So the trivial solution is stable in the sense of Liapunov.

With the help of Theorem III, the following general oscillation theorem may now be proven.

THEOREM 4. Suppose $a(t) \in C[0,\infty)$, $m(t) \in C^1[0,\infty)$, $b(x) \in C(-\infty, +\infty)$, $xb(x) > 0$ for $x \neq 0$, and $\lim_{|x|\to\infty} \int_0^x b(u)du = +\infty$. Furthermore, if $a(t) \geq a_o \geq 0$ and $m_1 \geq m(t) \geq m_o$ for positive constants a_o, m_o, m_1, then all solutions to (1.1) are oscillatory.

PROOF. Let $x(t)$ be a solution of (1.1). Suppose $x(t)$ does not oscillate. Then for $t \geq T \geq 0$, $x(t)$ must be of fixed sign. Reasoning as in Theorem II, assume $x(t) > 0$ (a similar argument works for $x(t) < 0$), then $x'(t)$ is of fixed sign so $x(t)$ is monotonic. Multiply (1.1) by $2x'(t)m(t)$ obtaining,

$$2m(t)x'(m(t)x')' + 2m(t)a(t)b(x)x' = 0 \tag{2.4}$$

Integrating from T to t we have,

$$(m(t)x'(t))^2 + \int_T^t 2a(s)m(s)b(x)x'ds = (m(T)x'(T))^2 \tag{2.5}$$

so (2.5) implies

$$2a_o m_o \int_{x(t)}^{x(t)} b(x)ds \leq (m(T)x'(T))^2 \tag{2.6}$$

That is, $x(t)$ must remain bounded. Therefore, by Theorem III, $x(t)$ oscillates.

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