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EVEN PERFECT NUMBERS AND THEIR EULER'S FUNCTION

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ABSTRACT. The purpose of this article is to prove some results on even perfect numbers and on their Euler's function. The results obtained are all straightforward deductions from well-known elementary number theory.

KEY WORDS AND PHRASES. Perfect number; triangular number; Euler's function; number of divisors function.
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1. INTRODUCTION.

A positive integer is called a perfect number if it is equal to the sum of its positive divisors excluding itself.

The n^{th} triangular number is the sum of the first n-positive integers

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) = T(n).$$

Euler's function $\phi(n)$ is the number of positive integers less than or equal to n and relatively prime to n.

The number of divisors function $d(n)$ is the number of positive divisors of n.

2. MAIN RESULTS.

The proof of the following Theorem 1 can be found in many elementary number theory books; see, for example, [1:p. 98].

THEOREM 1. If n is an even perfect number, there exists a prime 2^p-1 such that $n = 2^{p-1}(2^p-1)$.

THEOREM 2. If $T(p_1)$ is any even perfect number, where p_1 is prime, and if p_k is the first prime in the sequence $\{p_2, p_3, \ldots, p_j, \ldots\}$ where $p_j = 2p_{j-1}+1$, then $T(p_k)$ is the next even perfect number.

PROOF. It follows from Theorem 1 that an even perfect number is of the form $2^{n-1}(2^n-1)$, where 2^n-1 is prime. Now, $2^{n-1}(2^n-1)$ can be written as $T(p_1)$, where $p_1 = 2^n-1$. Let p_i be any composite term of the sequence $\{p_2, p_3, \ldots, p_j, \ldots\}$. It can be shown that $p_i = 2^{n+i-1}-1$, using the facts $p_i = 2^n-1$, and $p_j = 2p_{j-1}+1$. Now, it follows from Theorem 1 that $T(p_i) = 2^{n+i-2}(2^{n+i-1}-1)$ is not an even perfect number. Let p_k be the first prime in the sequence $\{p_2, p_3, \ldots, p_j, \ldots\}$. As before, $p_k = 2^{n+k-1} - 1$. Observe that $T(p_k) = 2^{n+k-2}(2^{n+k-1} - 1)$ is of the form $2^{m-1}(2^m - 1)$, where $2^m - 1$ is prime and thus $T(p_k)$ is an even perfect number by Theorem 1.

EXAMPLE. $T(3) = \frac{1}{2}(3)(4) = 6$, $T(7) = \frac{1}{2}(7)(8) = 28$.

$T(31) = \frac{1}{2}(31)(32) = 496$, $T(127) = \frac{1}{2}(127)(128) = 8128, \ldots$.

THEOREM 3. If $n = 2^{m-1}(2^m - 1)$, then, $n = 1^3 + 3^3 + \ldots + [2^{(m+1)/2} - 1]^3$.

PROOF. Observe that $2^{(m+1)/2} = 2k$, where $k = 2^{(m-1)/2}$. Now, consider $1^3 + 2^3 + 3^3 + \ldots + (2k-1)^3 + (2k)^3 = [1+2+3+\ldots+(2k-1) + (2k)]^2 = [\frac{1}{2}(2k)(2k+1)]^2$

which implies that $1^3 + 2^3 + 3^3 + \ldots + (2k-1)^3$

$$= k^2(2k+1)^2 - [2^3 + 4^3 + \ldots + (2k)^3]$$
$$= k^2(2k+1)^2 - 2^3(1^3 + 2^3 + \ldots + k^3)$$
$$= k^2(2k+1)^2 - 8(1 + 2 + \ldots + k)^2$$
$$= k^2(2k+1)^2 - 8[\frac{1}{2}k(k+1)]^2$$
$$= k^2(2k+1)^2 - 2k^2(k+1)^2 = k^2(2k^2 - 1).$$

Since $k = 2^{(m-1)/2}$, it follows that $1^3 + 3^3 + \ldots + [2^{(m+1)/2} - 1]^3 = 2^{m-1}(2^m - 1) = n$. The following Corollary 1, follows from Theorem 3.

COROLLARY 1. If n is an even perfect number $2^{p-1}(2^p - 1)$, then

$$n = 1^3 + 3^3 + \ldots + [2^{(p+1)/2} - 1]^3.$$

EXAMPLE. $496 = 1^3 + 3^3 + 5^3 + 7^3$; $p = 5$.

The proof of the following Theorem 4 can also be found in many elementary number theory books; see, for example [1: p. 63].

THEOREM 4. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$,

then $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k})$, where

$p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers.

As a consequence of Theorem 4, one can easily obtain Theorem 5, Corollary 2, and Corollary 3

THEOREM 5. $n = 2^{p-1}(2^p - 1)$ is an even perfect number if and only if

$\phi(n) = 2^{p-1}(2^{p-1} - 1)$, where $2^p - 1$ is prime.

COROLLARY 2. If n is an even perfect number, then $\phi(n) = n - 4^{p-1}$.

EXAMPLE. $\phi(8128) = \phi(2^6)\,\phi(127) = 4032 = 8128 - 4^6$.

COROLLARY 3. If n is an even perfect number, then $\phi(n) = \frac{n}{2} - 2^{p-2}$.

THEOREM 6. If $n_1, n_2, \ldots, n_k$ are k-distinct even perfect numbers,

then $\phi(n_1 n_2 \ldots n_k) = 2^{k-1}\,\phi(n_1)\,\phi(n_2) \ldots \phi(n_k)$.

PROOF. $\phi(n_1 n_2 \ldots n_k)$

$$= \phi[2^{p_1-1}(2^{p_1} - 1)\, 2^{p_2-1}(2^{p_2} - 1) \ldots 2^{p_k-1}(2^{p_k} - 1)]$$
$$= \phi[2^{p_1+p_2+\ldots+p_k-k}(2^{p_1} - 1)(2^{p_2} - 1) \ldots (2^{p_k} - 1)]$$
$$= \phi(2^{p_1+p_2+\ldots+p_k-k})\,\phi(2^{p_1} - 1)\,\phi(2^{p_2} - 1) \ldots \phi(2^{p_k} - 1)$$

$$= 2^{p_1+p_2+ \dots + p_k -k-1} (2^{p_1}-2)(2^{p_2}-2)\dots(2^{p_k}-2)$$

$$= 2^{k-1} \cdot 2^{p_1-1}(2^{p_1-1}-1) \cdot 2^{p_2-1}(2^{p_2-1}-1) \dots 2^{p_k-1}(2^{p_k-1}-1)$$

$$= 2^{k-1} \phi(n_1)\phi(n_2)\dots\phi(n_k).$$

The following Theorem 7 is proved in many books on elementary number theory; see, for example, [1: p. 96].

THEOREM 7. If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then $d(n) = \prod_{i=1}^{k} (1+\alpha_i)$, where p_i, $i=1, \dots, k$ are distinct primes and α_i, $i=1, \dots, k$ are positive integers, and $d(n)$ is the number of divisors function.

THEOREM 8. If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, and $d(n)$ is an even perfect number $2^{p-1}(2^p - 1)$, then

i) $p \geq k$.

ii) $\alpha_j = 2^{\mu_j}(2^p-1)-1$ for exactly one j such that $1 < j \leq k$ and $\mu_j \geq 0$.

iii) $\alpha_i = 2^{\mu_i} - 1$, where $\mu_i > 0$, $1 \leq i \leq k$, $i \neq j$.

iv) $\sum_{i=1}^{k} \mu_i = p - 1$.

PROOF. From Theorem 5, one obtains $d(n) = \prod_{i=1}^{k}(1+\alpha_i) = 2^{p-1}(2^p-1)$, which implies that (2^p-1) divides exactly one of the factors $(1+\alpha_i)$, $1 \leq i \leq k$, say $(1+\alpha_j)$. Thus $(1+\alpha_j) = (2^p-1)\cdot\lambda$ for some λ and exactly one j such that $1 \leq j \leq k$, and $(2^p-1)\cdot\lambda\cdot\prod_{\substack{i=1\\i\neq j}}^{k}(1+\alpha_i) = 2^{p-1}(2^p-1)$, that is, $\lambda\cdot\prod_{\substack{i=1\\i\neq j}}^{k}(1+\alpha_i) = 2^{p-1}$, which implies that $1+\alpha_i = 2^{\mu_i}$, $1 \leq i \leq k$, $1 \neq j$, $\mu_i > 0$; $\lambda = 2^{\mu_j}$, $\mu_j \geq 0$ and $\sum_{i=1}^{k}\mu_i = p-1$, which is *(iv)*.

Observe that $\sum_{i=1}^{k}\mu_i \geq k-1$ since $\mu_i > 0$ for $i \neq j$ and $\mu_j \geq 0$. Thus, $p-1 \geq k-1$ or $p \geq k$, which is *(i)*. Now, $(1+\alpha_j) = (2^p-1)\cdot\lambda = (2^p-1)\cdot 2^{\mu_j}$ for exactly one j, such that $1 \leq j \leq k$ and $\mu_j \geq 0$ implies that $\alpha_j = 2^{\mu_j}\cdot(2^p-1)-1$ for exactly one j such that $1 \leq j \leq k$ and $\mu_j \geq 0$, which proves *(ii)*.

Finally, $1 + \alpha_i = 2^{\mu_i}$, $1 \leq i \leq k$, $i \neq j$, $\mu_i > 0$ implies that $\alpha_i = 2^{\mu_i} - 1$, $1 \leq i \leq k$, $i \neq j$, $\mu_i > 0$, which proves (*iii*).

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