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CO-CONVEXIAL REFLECTOR CURVES WITH APPLICATIONS

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ABSTRACT. The concept of reflector curves for convex compact sets of reflecting type in the complex plane was introduced by the authors in a recent paper (to appear in J. Math. Anal. and Appln.) in their attempt to solve a problem related to Stieltjes and Van Vleck polynomials. Though, in the said paper, certain convex compact sets (e.g. closed discs, closed line segments and the ones with polygonal boundaries) were shown to be of reflecting type, it was only conjectured that all convex compact sets are likewise. The present study not only proves this conjecture and establishes the corresponding results on Stieltjes and Van Vleck polynomials in its full generality as proposed earlier by the authors, but it also furnishes a more general family of curves sharing the properties of confocal ellipses.

KEY WORDS AND PHRASES. Generalized Lame' differential equations, Stieltjes polynomials, Van Vleck polynomials, and co-convexial reflector curves.
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1. INTRODUCTION.

The present study has been motivated by a recent conjecture (cf. authors [1, concluding Remarks (i)]) that every convex compact subset of the complex plane is of reflecting type. This arose while solving a problem related to stieltjes and Van Vleck polynomials. In this paper we are able to prove this conjecture by the introduction of a nice function v: $\underline{C} \to \underline{R}_+$ ($\underline{C}$,$\underline{R}$ and $\underline{R}_+$ denote the set of all complex real and non-negative real numbers, respectively). In fact, Section 2 is primarily intended to establish some relevant properties of the function v that is solely responsible for materializing, in Section 3, the family of the so-called co-convexial reflector curves needed to prove the said conjecture. Besides, this family of reflector curves does present an interesting geometrical feature in as much as it provides an analogous theory of confocal ellipses under very general conditions. Finally, section 4 highlights certain applications of the theory of co-convexial reflector curves by obtaining some new results on the zeros of stieltjes and Van Vleck polynomials, some of which were only predicted in [1] and [2].

Before proceeding further, it is desirable to explain certain notations and terminology to be used later. Unless mentioned otherwise, K denotes a convex compact body (i.e. a convex compact set with an interior point) in the complex plane. Given any nonempty subset S of the complex plane, we denote by $K(S)$, $\mathring{S}$, ∂S and $|\partial S|$ the convex hull, the interior, the boundary and the length of the boundary of S, respectively. For $z \notin K$, we shall write $\alpha(z,K)$ to denote the angle subtended by K at z (cf. 1). Also, for every $z \in \underline{C}$, we write $K_z = K(K \cup \{z\})$. It may be noted that

$$K \begin{cases} = K_z & \forall \ z \in K, \\ \subsetneqq K_z & \forall \ z \notin K. \end{cases} \tag{1.1}$$

It is known (cf.[3, Thm. 12.20]) that K has a rectifiable boundary (with length denoted by $|\partial K|$). The following special case of a theorem in Valentine 3, Theorem 12.6 is interesting to record for future references.

THEOREM 1.1. If K,K' are convex compact bodies in $\underline{C}$ such that $K \subsetneqq K'$, then $|\partial K| < |\partial K'|$.

In view of this and (1.1) we have

$$|\partial K_z| \begin{cases} = |\partial K| & \forall \ z \in K, \\ > |\partial K| & \forall \ z \notin K. \end{cases} \tag{1.2}$$

2. THE REFLECTOR FUNCTION.

We begin with the following definition.

DEFINITION 2.1. Given K, we define the function $v: \underline{C} \to (0, +\infty)$ by $v(z) = |\partial K_z|$ for every $z \in \underline{C}$ and call v the <u>reflector function</u> for K.

Here we remark (cf [3, Theorem 12.7]) that v is continuous with $v \geq |\partial K|$ and $v(z) \to \infty$ as $z \to \infty$ along any continuous path in $\underline{C}$. Since a continuous image of a connected set is connected, we observe that $v(\underline{C}) = [|\partial K|, +\infty)$.

LEMMA 2.2. <u>If</u> G <u>is a ray with base at an interior point</u> a <u>of</u> K <u>and cutting</u> ∂K <u>at</u> b, <u>then</u> v(z) <u>increases strictly and continuously from</u> $|\partial K|$ <u>to</u> $+\infty$ <u>as</u> z <u>moves away from</u> b <u>along</u> G.

PROOF. If $z, z' \in G - \mathring{K}$ such that $|z' - a| > |z - a|$, then $K_z \subsetneqq K_{z'}$, and Theorem 1.1 implies that $|\partial K_z| < |\partial K_{z'}|$. Consequently, $v(z) < v(z')$ and the lemma is established.

We shall often use the following notations. Given $z \notin K$ (a convex compact set), we let a_z, a'_z denote the unique extreme points of K closest to z on the respective supporting lines (possibly coincident) of K through z (cf.[1] or [4]), labelled in such a manner that the movement along ∂K_z from z to a_z via a'_z gives a clockwise orientation to ∂K_z. We then let A_z(resp. A'_z) denote the supporting ray with base at z which passes through a_z(resp. a'_z). Also B_z shall denote the ray with base at z that bisects the angle between A_z and A'_z , i.e. B_z bisects the angle $\alpha(z,K)$. A line through z perpendicular to B_z will be denoted by T_z and the closed half-plane (not containing K) determined by T_z will be denoted by H_z.

DEFINITION 2.3. Given a convex compact subset K of $\underline{C}$ and a line L (not cutting K), we say that a point $z \in L$ is a reflector point of K in L if $T_z = L$. We write such a point as $z = z(K,L)$.

Observe that $z = z(K,T_z)$ for all $z \notin K$. However, the fact that every line L (not cutting K) has a unique reflector point of K in L follows from the following lemma.

LEMMA 2.4. *Given a convex compact subset K of C, let L be a directed line (not cutting K with a preassigned positive direction. For each $z \in L$, if $\alpha(A_z)$, $\alpha(A'_z)$ and $\alpha(B_z)$ denote the angles which A_z, A'_z and B_z, respectively, make with the positive direction of L, then*

(a) *each of $\alpha(A_z)$, $\alpha(A'_z)$ and $\alpha(B_z)$ increases strictly and continuously with range $(0,\pi)$ as z traverses the whole line L in the positive direction,*

(b) *there exists a unique $\zeta \in L$ such that $\zeta = \zeta(K,L)$.*

PROOF. (a) If $z,w \in L$ such that $w \neq z$ and w is in the positive direction from z, then $\alpha(A_w) > \alpha(A_z)$. For, otherwise, $A_z \cap A_w = \emptyset$ and A_z would not cut K. Hence $\alpha(A_z)$ increases strictly as z moves along L in the positive direction. Next, if $w \to z$ along L from either direction, the monotonicity of (A_w) forces the expression $|\alpha(A_w) - \alpha(A_z)|$ to approach zero. For, otherwise, A_w would cease to be a supporting ray of K for w sufficiently close to z. Hence $\alpha(A_z)$ is continuous on L. This Proves the assertion about $\alpha(A_z)$ if one observes that $\alpha(A_z)$ approaches 0 and π as z approaches the negative and positive ends of L, respectively. The proof for $\alpha(A'_z)$ is similar and the one for $\alpha(B_z)$ is immediate, since the sum of two increasing functions is increasing.

(b) Since the range of $\alpha(B_z)$ is $(0,\pi)$, the property of $\alpha(B_z)$ in part (a) establishes the statement in part (b)

This completes the proof of Lemma 2.4.

For proving our next lemma, we introduce the following notations: Given $a,b \in \partial K$, we write $\partial(a,b)$ for the portion of ∂K from a to b described in the clock-wise direction of ∂K. The length of $\partial(a,b)$ will be denoted by $|\partial(a,b)|$.

LEMMA 2.5. If $z_o \notin K$, *then*

(a) *$v(z)$ increases strictly and continuously from $v(z_o)$ to $+\infty$ as z moves away from z_o along Tz_o,*

(b) *$v(z) > v(z_o)$ for every $z \in H_{z_o} - \{z_o\}$,*

(c) $v(z_o) = \min_{z \in H_{z_o}} v(z) = \min_{z \in T_{z_o}} v(z)$.

PROOF. (a) let $z, w \in T_{z_o}$ such that $|w - z_o| > |z - z|$ and z,w do not lie on opposite sides of z_o. Here z may coincide with z_o. Let the directed line segment from z to w be taken as the positive direction of T_{z_o}. Suppose $A'_z \cap A'_w = \{b_z\}$.
Then

$$v(w) - v(z) = (|w - a'_w| + |w - a_w| + |\partial(a_z, a_w)| + |\partial(a'_w, a_z)|)$$
$$- (|z-a'_z| + |z - a_z| + |\partial(a'_z, a'_w)| + |\partial(a'_w, a_z)|)$$
$$= \{|w-a'_w| - (|z - a'_z| + |\partial(a'_z, a'_w)|)\} + \{(|w - a_w| + |\partial(a_z, a_w)|) - |z - a_z|\}$$
$$= r + s . \tag{2.1}$$

In case $a'_w \notin A'_z$ (resp. $a_z \notin A_w$) the convex body $K(\partial(a'_z, a'_w) \cup \{z\})$ (resp. $K\{w, a_w, a_z\}$) is contained in the convex body $K\{z, b_z, a'_w\}$ (resp. $K(\partial(a_z, a_w) \cup \{w\})$). Application of Theorem 1.1 yields

$$|z-a'_z| + |\partial(a'_z, a'_w)| \leq |z - b_z| + |b_z - a'_w|$$

and

$$|w - a_w| + |\partial(a_z, a_w)| \geq |w - a_w| + |a_w - a_z| \geq |w - a_z|.$$

Therefore (cf.(2.1))

$$r \geq |w - b_z| - |z - b_z|,$$

$$s \geq |w - a_z| - |z - a_z|.$$

Hence

$$v(w) - v(z) \geq (|w - a_z| + |w - b_z|) - (|z - a_z| + |z - b_z|) \equiv m'-m. \tag{2.2}$$

Observe that m', m are the lengths of the major axes of the confocal ellipses E, E' passing through z, w, respectively, and belonging to the family $\sum$ of all confocal ellipses with focii at a_z and b_z. If $\sigma = K\{a_z, b_z\}$, then $\alpha(z,\sigma) = \alpha(z,K)$ and so B_z is also the bisector of $\alpha(z,\sigma)$, where $\alpha(B_z) \geq \alpha(B_{z_o}) = \pi/2$ by Lemma 2.4 applied to K and T_{z_o}. The same lemma, when applied to σ and T_{z_o}, provides a unique point $w_o \in T_{z_o}$ such that $w_o = w_o(\sigma, T_{z_o})$ and such that either $w_o = z_o$ or w_o and w lie on opposite sides of z. If E_o is the member of $\sum$ through w_o, then T_{z_o} is tangent to E_o at w_o and any other member (cutting T_{z_o}) of $\sum$ must interest T_{z_o} in exactly two distinct points on opposite sides of w_o. Since z and w do not lie on opposite sides of w_o, the ellipse E is enclosed by E'. Consequently (cf.(2.2))

$$v(w) - v(z) \geq m' - m > 0$$

which establishes part (a) of the lemma.

(b) Let $a \in \overset{o}{K} \cap B_{z_o}$. Given $z \in H_{z_o} - \{z_o\}$, consider the directed ray G through z with base at a. Let G cut T_{z_o} at w. If $w = z_o$, we are done by Lemmas 2.2. In case $w \neq z_o$, Lemma 2.2 and part (a) above gives $v(z) \geq v(w) > v(z_o)$. Part (b) is thus established.

(c) The same technique as in the proof of part (b) above confirms that, for each $w \in T_{z_o}$, there exists a $z \in H_{z_o} - T_{z_o}$ such that $v(w) < v(z)$. Now part (c) follows from part (b).

The proof of Lemma 2.5 is thus complete.

In view of Lemma 2.4(b) and Lemma 2.5(c), we remark that the function v attains a minimum on every line L, not cutting K, at the reflection point of K in L. However, for any other line L', v attains a minimum at every point in $K \cap L'$.

3. REFLECTOR CURVES.

Given K, we define a relation '~' betweem elements of $\underline{C}$ as follows:

$$z \sim z' \text{ if and only if } v(z) = v(z').$$

Then '~' defines an equivalence relation on $\underline{C}$ and the equivalence class C_z, containing z, is given by

$$C_z = \{w \in \underline{C} \mid v(w) = v(z)\}.$$

Thus, v partitions $\underline{C}$, via the equivalence relation '~', into mutually disjoint equivalence classes C_z ($z\varepsilon\underline{C}$), one of them being K itself (Note that $C_z=K$ if and only if $z\varepsilon K$.) For each $z\varepsilon\underline{C}$, we see that $C_z= v^{-1}\{k\}$, where $k = v(z) \varepsilon[|\partial K|,+\infty)$. So each C_z is a closed and bounded set, because v is continuous and $v(z_n) \to +\infty$ as the sequence $z_n \to \infty$. Also, since $v(\underline{C}) = [|\partial K|,+\infty)$, for each $k\varepsilon[|\partial K|,+\infty)$ there exists a point $z\varepsilon\underline{C}$ such that the class $C_z= v^{-1}\{k\}$. All this can be summed up in the following.

PROPOSITION 3.1. The family $\{C_z\}_{z\varepsilon\underline{C}}$ has the following properties:

(a) C_z is compact and $z\varepsilon C_z$ for every $z\varepsilon\underline{C}$;

(b) Either $C_z \cap C_{z'} = \emptyset$ (which happens if and only if $v(z) \neq v(z')$) or $C_z = C_{z'}$ (which holds if and only if $v(z) = v(z')$);

(c) The family of all disjoint equivalence classes, $\{C_z\}_{z\varepsilon\underline{C}}$, is in 1-1 correspondence with the interval $[|\partial K|, +\infty)$.

Next, we prove the following results. By a curve we mean a continuous arc whose initial point coincides with its terminal point.

PROPOSITION 3.2. (a) Every $C_z (z\notin K)$ is a Jordan curve enclosing K (For $z\varepsilon K$, $C_z= K$). (b) C_z is enclosed by $C_{z'}$ if and only if $v(z) < v(z')$.

PROOF. (a) Choose $a\varepsilon \overset{\circ}{K}$ and a ray G_o, with base at a, as the initial line for measuring angles. For each $\theta\varepsilon[0,2\pi]$, let G_θ denote the ray, with base at a , making an angle θ with G_o. For a fixed $z\notin K$, so that $v(z)=k > |\partial K|$, consider

$$C_z = \{w|v(w) = k\}.$$

Lemma 2.2 allows us to choose a unique point w on each ray G_θ such that $v(w) = k$. This enables us to define a mapping $\Gamma:[0,2\pi] \to \underline{C}$ such that $\Gamma(\theta) \varepsilon G_\theta$ and $v(\Gamma(\theta)) = k$ for all $\theta\varepsilon[0,2\pi]$. Continuity of v then implies that Γ is continuous. Observe that $\Gamma(\theta_1)\neq\Gamma(\theta_2)$ if $\theta_1\neq\theta_2$ and that $\Gamma(\theta)\notin K$ for all θ. Furthermore, application of Lemma 2.2 to the ray $G_o = G_{2\pi}$ yields $\Gamma(0) = \Gamma(2\pi)$ and completes the proof.

(b) The proof follows from Propositions 3.2(a) and 3.1(b), together with Lemma 2.2.

PROPOSITION 3.3. Each $C_z(z\notin K)$ is a convex curve.

PROOF. For each $z\notin K$, let H'_z (resp. H''_z) denote the closed (resp. open) half plane determined by T_z which contains K. Now consider C_z for a fixed $z\notin K$. By Lemma 2.5,

$$C_z \cap H'_w = C_z \;\forall\; w\varepsilon C_z.$$

That is,

$$C_z \subset \cap \{H'_w|w\varepsilon C_z\} .$$

But (since $w\notin H''_w$ for all $w\varepsilon C_z$)

$$C_z \cap \{H''_w|w\varepsilon C_z\} = \emptyset \;\forall\; w\varepsilon C_z.$$

Therefore, every point of C_z is a boundary point of $K(C_z)$. That is, C_z is a convex curve, and the lemma is established.

A regular Jordan curve C, lying outside a nonempty convex compact set K, is called a reflector curve for K (cf.[1, Definition 2.1]) if the normal at every point $c\varepsilon C$ is along B_c, the bisector of the angle $\alpha(c,K)$. A nonempty convex compact subset K of $\underline{C}$ is said to be of reflecting type (cf.[1, Definition 2.3]) if it has a unique convex reflector curve, enclosing K, through every point $z\notin K$ (it may be noted that any two such reflector curves for K must necessarily be either identical or disjoint). The family of all nonempty convex compact subsets of $\underline{C}$ of reflecting type will be denoted by F. It is known [1, Remark 2.4] that F contains closed discs and closed line segments (the only convex sets without interior points) and also convex bodies with polygonal boundary. In the remainder of this section we establish that F contains all nonempty convex compact subsets of $\underline{C}$.

PROPOSITION 3.4. Every C_z, $z \notin K$, is a reflector curve for K.

PROOF. Given $z \notin K$, let $w \in C_z$. By Lemma 2.5(b) we know that $C_z \cap T_w = \{w\}$ and $C_z \subset H'_w$, where H'_w is as in the proof of Proposition 3.3. To prove regularity of C_z, it is sufficient to prove that any line L through w, not cutting K and different from T_w, must cut C_z at precisely one more point other than w. Let us assign a positive direction to such a line L so that the resulting directed line L makes a positive acute angle with the ray B_w. By Lemma 2.4, there exists a unique reflector point ζ of K in L, with $\zeta \neq w$, such that ζ lies on the positive side of L from w. Since $\zeta \notin K$ and $T_\zeta = L$, we apply lemma 2.5(a) to ζ and obtain a unique point $w' \in L$ such that w and w' lie on opposite sides of ζ and such that $v(w) = v(w')$. Since $w, z \in C_z$, we conclude that $w' \in C_z$. Moreover, we further conclude that $v(c) \neq v(w)$ for any $c \in L$, $c \neq w, w'$. That is, L cuts C_z at only one point $w' (\neq w)$. Consequently, T_w is tangent to C_z at w. Now Proposition 3.2(a) completes the proof,

Propositions 3.3 and 3.4 assert that, for each $z \notin K$, C_z is a convex reflector curve for K passing through z and enclosing K. In fact, we claim that if C' is any reflector curve for K passing through z then $C' = C_z$. For, otherwise, we obtain a point $z' \in C' - C_z$. Now consider $C_{z'}$. Since $\{C_z\}_{z \notin K}$ is a nonintersecting family of convex regular curves which is everywhere dense (cf. Propositions 3.1-3.3) in the region between C_z and $C_{z'}$ and, since C' is a convex regular curve passing through z and z', we conclude that there exists a point $\zeta \in C'$ such that C' cuts C_ζ at a positive angle. This contradicts the fact that C' and C_ζ must touch each other at ζ (both being reflector curves for K). Thus, we have established the following theorem which answers affirmatively the conjecture made earlier in [1, Concluding Remarks(1)].

THEOREM 3.5. *If K is a nonempty convex compact subset of $\underline{C}$, then K belongs to F.*

REMARK. Though we have proved Theorem 3.5 for a convex body K, but it remains valid also for a convex set K without interior points (see the paragraph immediately preceding Proposition 3.4).

It is interesting to note that the family of co-convexial reflector curves (for a given K) generalizes the notion of confocal ellipses, which we obtain by taking K to be a closed line segment. In this direction we refer the interested reader to Hartman and Valentine [5].

4. APPLICATIONS.

The theory of reflector curves discussed in Section 3 finds its application in predicting the location of the zeros of stieltjes and Van Vleck polynomials which arise as polynomial solutions of the *generalized Lame's differential equation*

$$\frac{d^2w}{dz^2} + \left[\sum_{j=1}^{p} \alpha_j/(z-a_j) \right] \frac{dw}{dz} + \frac{\Phi(z)}{\Pi_{j=1}^{p}(z-a_j)} w = 0, \qquad (4.1)$$

where $\Phi(z)$ is a polynomial of degree at most (p-2) and where α_j, a_j are complex constants. It is known (cf.[6],[7,p.36]) that there exist at most C(n + p - 2, p - 2) polynomial solutions V(z) (called Van Vleck polynomials) such that, for $\Phi(z) = V(z)$, the equation (4.1) has a polynomial solution S(z) of degree n (called stieltjes polynomials.).

The differential equation (cf.[2],[8],[9])

$$\frac{d^2w}{dz^2} + \left[\sum_{j=1}^{p} \alpha_j \left\{ \frac{\prod_{t=1}^{n_j-1} (z-b_{jt})}{\prod_{s=1}^{n_j} (z-a_{js})} \right\}\right] \frac{dw}{dz} \tag{4.2}$$

$$+ \frac{\Phi(z)}{\prod_{j=1}^{p} \prod_{s=1}^{n_j} (z-a_{js})} w=0 ,$$

where $\Phi(z)$ is a polynomial of degree at most $(n_1+n_2+\ldots+n_p-2)$ and where α_j, a_{js}, b_{jt} are complex constants, can always be written in the form (4.1) by expressing each fraction (in the coefficient of dw/dz) into its partial fractions. In fact, (4.2) is surely of the form (4.1) if $n_j=1$ for all j. It may also be observed (as in case of (4.1)) that there exists at most

$$C(n + n_1 + n_2 + \ldots + n_p - 2, n_1 + n_2 + \ldots + n_p - 2)$$

polynomials V(z) such that, for $\Phi(z)=V(z)$, the differential equation (4.2) has a polynomial solution S(z) of degree n. That is, there do exist Stieltjes polynomials S(z) and Van Vleck polynomials V(z) associated with the differential equation (4.2). For convenience, we shall write

$$q = \max\{n_1,n_2,\ldots,n_p\}. \tag{4.3}$$

Throughout this section, unless mentioned otherwise, K will denote a nonempty convex compact subset of $\underline{C}$. We write $R_z = K(C_z)$ for each $z \notin K$ and call it the <u>reflector region</u> for K determined by z (cf.[1],[2]). Given K and $\phi (0 < \phi \leq \pi)$, we recall (cf.[7,p.31][1],[2]) that the <u>star-shaped</u> region $S(K,\phi)$ is given by

$$S(K,\phi) = \{z\epsilon\underline{C} \mid \alpha(z,K) \geq \phi\}.$$

Given K, $\gamma \in [0, \pi/2)$ and an integer $q \geq 1$, we write (cf.[2])

$$S_{\gamma,q} = S(K, (\pi-2\gamma)/(2q-1))$$

and denote by $K_{\gamma,q}$ the intersection of all the reflector regions R_z containing $S_{\gamma,q}$. Then (cf.[2]) $K_{\gamma,q}$ is a convex compact subset such that

$$K \subset S_{\gamma,q} \subset K_{\gamma,q} \quad .$$

In particular, $K \subsetneq S_{,q}$ for $\gamma > 0$, $K = S_{0,1}$, and $K = S_{0,1} = K_{0,1}$. Sometimes, for q = 1, we shall write

$$S_{\gamma,1} \equiv S_\gamma \quad\text{and}\quad K_{\gamma,1} \equiv K_\gamma \ .$$

Now the main theorem of this paper (e.g. Theorem 3.5) answers an earlier conjecture (cf. 1 , concluding Remark(i)) in the affirmative and we obtain the following result.

THEOREM 4.1. *In the differential equation* (4.1), if

$$|\arg \alpha_j| \le \gamma < \pi/2 \qquad \forall\ j=1,2,\ldots,p$$

and if the points a_j (j=1,2,...p) *lie in a convex compact subset* K, *then the zeros of each nth degree Stieltjes (resp. Van Vleck) polynomial lie in the region* K_γ.

Similarly, we get the following general version of Theorems (2.3) and (2.4) in [2] concerning the differential equation (4.2).

THEOREM 4.2. *In the differential equation* (4.2), *if*

$$|\arg \alpha_j| \le \gamma < \pi/2 \qquad \forall\ j=1,2,\ldots,p$$

and if all the points a_{js} *and* b_{jt} *(occuring in* (4.2) *lie in a convex compact subset* K, *then the zeros of each stieltjes* (resp. *Van Vleck*) *polynomial lie in* $K_{\gamma,q}$, *where* q *is as in* (4.3).

REMARK 4.3. (I) For q=1, Theorem 4.2 reduces to Theorem 4.1.

(II) Theorem 4.1 (resp. Theorem 4.2) is a generalization of Theorem 3.1 (resp. Theorems (2.3) and (2.4) in [1] (resp. [2]).

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