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TWO NEW FINITE DIFFERENCE METHODS FOR COMPUTING EIGENVALUES OF A FOURTH ORDER LINEAR BOUNDARY VALUE PROBLEM

RIAZ A. USMANI

Department of Applied Mathematics
University of Manitoba
Winnipeg, Manitoba, Canada
R3T 2N2

MANABU SAKAI

Department of Mathematics
University of Kagoshima
Kagoshima, Japan 890

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ABSTRACT. This paper describes some new finite difference methods of order 2 and 4 for computing eigenvalues of a two-point boundary value problem associated with a fourth order differential equation of the form $(py'')'' + (q - \lambda r)y = 0$. Numerical results for two typical eigenvalue problems are tabulated to demonstrate practical usefulness of our methods.

KEY WORDS AND PHRASES. Band-matrices, finite-difference methods, generalized eigenvalue problem, positive definite matrices, two-point boundary value problems.
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1. INTRODUCTION.

We shall consider the fourth order linear differential equation

$$\frac{d^2}{dx^2}\left[p(x)\,\frac{d^2y}{dx^2}\right] + [q(x) - \lambda r(x)]y = 0, \quad -\infty \le a \le x < b < \infty, \tag{1.1}$$

associated with the following pairs of homogeneous boundary conditions

$$y(a) = y(b) = y''(a) = y''(b) = 0. \tag{1.2}$$

Such boundary value problems occur in applied mathematics, engineering and modern physics, (see ref. [1-4]. In the differential equation (1.1) the functions $p(x)$, $q(x)$, $r(x) \in C[a,b]$ and satisfy the conditions

$$p(x) > 0,\ q(x) \ge 0 \ \text{ and } r(x) > 0,\ x \in [a,b]. \tag{1.3}$$

We cannot compute the exact values of the eigenvalues λ for which the boundary value problem (1.1) - (1.2) has a nontrivial eigensolution $y(x)$ for arbitrary chocies of the functions $p(x)$, $q(x)$ and $r(x)$. We resort to numerical methods for computing approximate values of λ. The most commonly used technique for approximating λ for which the system (1.1) - (1.2) has a nontrivial eigenfunction $y(x)$ is by finite difference methods.

Recently, the author [2] has analysed some new finite different methods of order 2 and 4 for computing eigenvalues of a two point boundary value problem involving the differential equation (1.1) with $p(x) \equiv 1$ associated with one of the following pairs of homogeneous boundary conditions:

$$\begin{aligned} &(a)\quad y(a) = y(b) = y'(a) = y'(b) = 0 \\ &(b)\quad \text{the same boundary conditions as (1.2)} \\ &(c)\quad y(a) = y'(a) = y''(b) = y'''(b) = 0. \end{aligned} \tag{1.4}$$

Chawla and Katti [3] have developed a numerical finite difference method of order 2 for approximating the lowest eigenvalue λ of the system (1.1) - (1.4(a)) with $p(x) \equiv 1$. A fourth order method was later developed by Chawla [4] for the numerical treatment of the same problem. This latter method leads to a generalized seven-band symmetric matrix eigenvalue problem.

Let λ be any eigenvalue of the system (1.1) - (1.2) and let $y(x) \not\equiv 0$ be the corresponding eigenfunction. Then on multiplying (1.1) by $y(x)$ and integrating the resulting equation from a to b, we find after integration by parts and on using (1.2), that

$$\lambda = \frac{\int_a^b p(y'')^2\, dx + \int_a^b qy^2 dx}{\int_a^b ry^2 dx} > 0 \tag{1.5}$$

in view of (1.3).

The purpose of this brief report is to present two new finite difference methods for computing approximate values of λ for the system (1.1) - (1.2). These methods lead to generalized five-band and nine-band symmetrix matrix eigenvalue problems and provide $0(h^2)$ and $0(h^4)$ -convergent approximations for the eigenvalues.

2. A SECOND ORDER METHOD

For a positive integer $N \geq 5$, let $h = (b-a)/(N+1)$ and $x_i = a + ih$, $i = 0(1)N+1$. We shall designate $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$. Note that the differential system (1.1) - (1.2) is equivalent to

$$\begin{aligned} &(a)\quad y''(x) = v(x)/p(x)\ ,\ y(a) = y(b) = 0\ , \\ &(b)\quad v''(x) + [q(x) - \lambda r(x)]\, y(x) = 0\ , \\ &\qquad v(a) = v(b) = 0. \end{aligned} \tag{2.1}$$

Now the central difference approximation to 2.1(a) is

$$-y_{i-1} + 2y_i - y_{i+1} + h^2(v_i/p_i) + \frac{h^4}{12} y^{(4)}(\Theta_i) = 0\ , \tag{2.2}$$

$$\Theta_i \in (x_{i-1}, x_{i+1})\ ,\ i = 1(1)N\ .$$

The preceding system can be conveniently written in matrix form

$$JY + h^2 P^{-1} V + \frac{h^4}{12} T_1 = 0 \tag{2.3}$$

where $Y = (y_i)$, $V = (v_i)$, $T_1 = (\rho_i)$ are N-dimensional column vectors with $\rho_i = y^{(4)}(\Theta_i)$, $P = \text{diag}\,(p_i)$, and $J = (j_{mn})$ is a tridiagonal matrix so that

$$j_{mn} = \begin{cases} 2, & m = n \\ -1, & |m - n| = 1 \\ 0, & |m - n| > 1 \; . \end{cases} \tag{2.4}$$

In an analogous manner, on discretizing 2.1(b), we get

$$JV - h^2QY + \lambda h^2RY + \frac{h^4}{12} T_2 = 0 \tag{2.5}$$

where $Q = \text{diag}\,(q_i)$, $R = \text{diag}\,(r_i)$ and $T_2 = (\sigma_i)$ with $\sigma_i = v^{(4)}(\phi_i)$, $\phi_i \in (x_{i-1}, x_{i+1})$. Next, we eliminate v between (2.3) and (2.5) to obtain

$$AY \equiv (JPJ + h^4Q)Y = \lambda h^4RY + \Gamma \; , \tag{2.6}$$

where

$$\Gamma = \frac{1}{12}[h^6T_2 - h^4JPT_1] \; . \tag{2.7}$$

It can be verified that the matrix $A = JPJ + h^4Q$ is a five-band symmetric matrix. Now, in (2.6), neglect truncation error Γ, replace Y by $\tilde{Y}$, then our method for computing approximations Λ for λ of the system (1.1) - (1.2) can be expressed as a generalized seven-band symmetric matrix eigenvalue problem

$$A\tilde{Y} = \Lambda h^4R\tilde{Y} \; . \tag{2.8}$$

In fact the matrix JPJ is a positive definite matrix and hence for any step-size $h > 0$, the approximations Λ for λ by (2.8) are real and positive for all $p(x) > 0$ and $r(x) > 0$. That our method provides $O(h^2)$ convergent approximations Λ for λ can be established following Grigorieff [5]. We omit the proof of convergence for brevity.

3. A FOURTH ORDER METHOD

Following Shoosmith [6] the boundary value problems 2.1(a) and 2.1(b) are discretized by the finite difference scheme

$$\begin{aligned} &\text{(a)}\quad 14y_0 - 29y_1 + 16y_2 - y_3 = h^2[y_0'' + 12y_1''] \; , \\ &\text{(b)}\quad (1 - \tfrac{\delta^2}{12})\delta^2 y_i = h^2 y_i'' \; , \quad i = 2(1)N-1, \\ &\text{(c)}\quad -y_{n-2} + 16y_{N-1} - 29y_N + 14y_{N+1} = h^2[12y_N'' + y_{N+1}''] \; . \end{aligned} \tag{3.1}$$

It turns out the boundary value problem 3.1(a) gives rise to the linear equations

$$M\tilde{Y} + 12h^2p^{-1}\tilde{V} = 0 \; . \tag{3.2}$$

Similarly, for the system 2.1(b), we obtain the linear equations

$$M\tilde{V} = 12h^2Q\tilde{Y} - 12\Lambda h^2R\tilde{Y} \; , \tag{3.3}$$

where the five-band N x N matrix M is given by

$$M = \begin{bmatrix} 29 & -16 & 1 & & & & \\ -16 & 30 & -16 & 1 & & & \\ 1 & -16 & 30 & -16 & 1 & & \\ - & - & - & - & - & - & - \\ & & 1 & -16 & 30 & -16 & 1 \\ & & & 1 & -16 & 30 & -16 \\ & & & & 1 & -16 & 29 \end{bmatrix} . \tag{3.4}$$

The elimination of $\tilde{V}$ from (3.2) and (3.3) gives our method for computing Λ for λ of (1.1) - (1.2) in the form

$$(MPM + 144h^4Q)\tilde{Y} = 144\Lambda h^4 R\tilde{Y} , \tag{3.5}$$

where the matrix MPM is a nine-band positive definite matrix and hence for any step-size $h > 0$, the approximations Λ for λ by (3.5) are real and positive for all $p(x), r(x) > 0$. As before, it can be proved from the results of Grigorieff [5] that our present method provided $0(h^4)$ convergent approximations Λ for λ.

4. NUMERICAL RESULTS

In order to illustrate our methods of order 2 and 4 for the approximation of λ satisfying (1.1) - (1.2), we consider the eigenvalue problems:

$$[(1 + x^2)y'']'' + [\frac{1}{(1 + x^2)} - \lambda(1 + x)^4]y = 0 , \tag{4.1}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0 .$$

The smallest eigenvalue λ_1 = 22.754, 058, 480, . . .

$$[e^x y'']'' + [\sin x - \lambda \cos x]y = 0 , \tag{4.2}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0 .$$

The smallest eigenvalue of the system (4.2) is λ_1 = 181.345, 488, 233, . . . We list the approximations Λ_1 for λ_1 and the relative errors $| 1 - \frac{\lambda_1}{\Lambda_1} |$ for various values of the step-size h. It is readily verified that the relative errors (Table I) based on generalized eigenvalue problem (2.8) provide $0(h^2)$ - convergent approximations for the smallest eigenvalue of the system (4.1) and (4.2). Similarly, the relative errors (Table II) based on the generalized eigenvalue problem (3.5) do indeed provide $0(h^4)$ -convergent approximations for the smallest eigenvalue of the systems (4.1) and (4.2).

TABLE I

Results based on (2.8), second order approximations

Problem	N	Λ_1	$\left\lvert 1 - \frac{\lambda_1}{\Lambda_1} \right\rvert$
(4.1)	7	22.187	2.557-2*
	15	22.610	6.352-3
	31	22.718	1.586-3
	63	22.745	3.962-4
	127	22.752	9.907-5
	255	22.753	2.480-5
(4.2)	7	176.641	2.664-2
	15	180.159	6.588-3
	31	181.048	1.642-3
	63	181.271	4.103-4
	127	181.327	1.025-4
	255	181.341	2.560-5

*We write 2.557-2 for 2.557×10^{-2}.

TABLE II

Results based on (3.5), 4th order approximations

Problem	N	Λ_1	$\left\lvert 1 - \frac{\lambda^1}{\Lambda_1} \right\rvert$
(4.1)	7	22.746, 419	3.358-4
	15	22.753, 574	2.129-5
	31	22.754, 027	1.358-6
	63	22.754, 056	1.078-7
(4.2)	7	181.244, 637	5.564-4
	15	181.339, 089	3.529-5
	31	181.345, 093	2.175-6
	63	181.345, 470	9.728-8

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