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TYPICALLY REAL FUNCTIONS AND TYPICALLY REAL DERIVATIVES

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ABSTRACT. Sufficient conditions, in terms of typically real derivatives, are given which force functions to be univalent.

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1. TYPICALLY REAL FUNCTIONS WITH A TYPICALLY REAL FIRST DERIVATIVE.

Let $D = \{z: |z| < 1\}$. Rogosinski [1] defined the class, T, of typically real functions as follows: If $f \varepsilon T$, then f is regular on D, $f(z) = z+a_2z^2+\cdots$, and $Im\{z\} = 0$ if and only if $Im\{f(z)\} = 0$. (See Goodman [2], p. 184.) The last part of this definition is equivalent to the statement that $Im\{z\} \neq 0$ if and only if $Im\{z\}Im\{f(z)\} > 0$. If $f \varepsilon T$, then f must be one-to-one on the real interval, $(-1,1)$. So, if $f \varepsilon T$, if $z,z' \varepsilon D$ with $z \neq z'$, and if $f(z) = f(z')$, then $Im\{z\}Im\{z'\} > 0$. These establish the following:

LEMMA 1. Let $f \varepsilon T$. Let $D^+ = D \cap \{z: Im\{z\} > 0\}$ and let $D^- = D \cap \{z: Im\{z\} < 0\}$. Then f is univalent on D if and only if f is univalent on each of D^+ and D^- separately.

The notion of a function which is typically real on D has nothing to do with its normalization. In what follows, it is convenient to say that a function, g, regular on D, is typically real on D if the following holds: $Im\{z\} = 0$ if and only if $Im\{g(z)\} = 0$. This is equivalent to saying that g is typically real on D provided that, for $x \varepsilon (-1,1)$ and for $z \varepsilon D$, then $Im\{z\} \neq 0$ if and only if $g'(x)Im\{z\}Im\{g(z)\} > 0$.

As is known, it is not necessarily the case that a function in T is univalent on D, e.g., $f(z) = z+z^3$. The following will show, however, that a simple additional requirement on functions in T will insure such univalence.

DEFINITION 1. Let $T' = \{f\varepsilon T$: f' is also typically real on D$\}$.

Barnard and Suffridge [3] have shown that if $f(z) = z+a_2z^2+\cdots \varepsilon T'$, then $|a_2| \leq (3\pi+2)/2\pi = 1.8183\cdots$ and that the result is sharp. We show the following:

THEOREM 1. If $f\varepsilon T'$, then f is univalent in D.

PROOF. It is enough to show that f is univalent in each of D^+ and D^- as defined in Lemma 1. Since f' is typically real in D it follows that $f''(0)\text{Im}\{f'(z)\} > 0$ for $z\varepsilon D^+$. Hence, f' maps the convex set, D^+, into a half-plane whose boundary passes through the origin. By a result of Noshiro [4] and of Warschawski [5], f is univalent on D^+. (See Goodman [2], p. 88.) Similarly, f is also univalent on D^-.

2. TYPICALLY REAL FUNCTIONS, ALL OF WHOSE DERIVATIVES ARE UNIVALENT.

In [6], Shah and Trimble introduced the class, E, of functions, normalized in D, such that $f\varepsilon E$ if and only if $f^{(n)}$ is univalent in D for $n = 0,1,2,\cdots$. ([7] provides a survey of results about E.) Among other things, they showed that if $f\varepsilon E$, then f is entire. Here, we wish to study results about functions in E which are typically real.

DEFINITION 2. Let ER be those functions in E such that if $f(z) = z+a_2z^2+\cdots$, then a_n is real for $n = 2,3,\cdots$. Let $\overline{ER}$ be those functions which are uniform limits on compact subsets of D of sequences in ER. Let ERP be those functions in ER such that $a_n > 0$ for $n = 2,3,\cdots$.

THEOREM 2. $f\varepsilon ER$ if and only if $f^{(n)}$ is typically real on D for $n = 0,1,2,\cdots$.

PROOF. If every $f^{(n)}$ is typically real on D, then Theorem 1 implies that each $f^{(n)}$ is univalent on D. Hence, $f\varepsilon ER$.

On the other hand, if a function, univalent on D, has real Maclaurin coefficients, it is well-known that the function is typically real on D. Hence, if $f\varepsilon ER$, then $f^{(n)}$ is typically real on D for $n = 0,1,2,\cdots$.

LEMMA 2. $\overline{ER} - ER$ is the set of polynomials with real Maclaurin coefficients such that each derivative of each polynomial including the polynomial itself, is either constant or univalent on D.

PROOF. Let $f \varepsilon (\overline{ER} - ER)$. Then there is a sequence, $\{f_k\}_{k=1}^{\infty}$, in ER which converges to f uniformly on compact subsets of D. Since the Maclaurin coefficients of each f_k are real, the Maclaurin coefficients of f must also be real. If $n\varepsilon\{0,1,2,\cdots\}$, then $\{f_k^{(n)}\}_{k=1}^{\infty}$ converges to $f^{(n)}$ uniformly on compact subsets of D. By Hurwitz's Theorem, $f^{(n)}$ is either univalent or constant on D. If $f^{(n)}$ is univalent on D for all n, then $f\varepsilon E$, which is impossible. Hence, there is some N such that $f^{(N)}$ is constant on D. So, if $n > N$, $f^{(n)}(z) \equiv 0$ on D. It follows that f is a polynomial of degree at most N.

Now let P be a polynomial with real Maclaurin coefficients such that each derivative of P, including P itself, is either constant or univalent on D. For $k\varepsilon\{1,2,\cdots\}$, let $r_k = 1-1/(k+1)$. Let $g(z) = (e^{\pi z}-1)/\pi$. (Note that $g\varepsilon ERP$.) Let N be the degree of F. Let $\{\delta_k\}_{k=1}^{\infty}$ be a sequence of positive numbers tending monotonically to 0. Define

$$F_k(z) = \frac{P(r_k z)+\delta_k g(z)}{r_k+\delta_k}.$$

Then $\{F_k\}_{k=1}^{\infty}$ converges to F uniformly on compact subsets of D. We now show that $F_k\varepsilon ER$ for all k.

The Maclaurin coefficients of each F_k are all real, so it is sufficient to show that, if $k\varepsilon\{1,2,\cdots\}$ and if $n\varepsilon\{0,1,2,\cdots\}$, then $F_k^{(n)}$ is univalent on D. If $n > N$, then $F_k^{(n)}(z) = \delta_k g^{(n)}(z)/(r_k+\delta_k)$, which is univalent on D. Since $r_k{}^N P^{(N)}(z)/(r_k+\delta_k)$ is constant, $F_k^{(N)}$ is also univalent on D. Suppose $n < N$. To show that $F_k^{(n)}$ is univalent on D, it is enough to show that, if $0 < \rho < 1$, then $F_k^{(n)}$ is one-to-one on $\{z: |z| = \rho\}$. Let $0 < \rho < 1$ and let $|z| = |\omega| = \rho$, $z \neq \omega$. Recall that, if h is univalent on D, then

$$\left|\frac{h(z)-h(\omega)}{z-\omega}\right| \geq \frac{1-\rho^2}{\rho^2}\frac{|(h(z)-h(0))(h(\omega)-h(0))|}{|h'(0)|}$$

$$\geq \frac{|h'(0)|(1-\rho)}{(1+\rho)^3}.$$

(See Duren [8], p. 127.) So,

$$\left|\frac{F_k^{(n)}(z)-F_k^{(n)}(\omega)}{z-\omega}\right| \geq \frac{r_k{}^n}{r_k+\delta_k}\left|\frac{P^{(n)}(r_k z)-P^{(n)}(r_k\omega)}{z-\omega}\right|$$

$$-\frac{\delta_k}{r_k+\delta_k}\left|\frac{g^{(n)}(z)-g^{(n)}(\omega)}{z-\omega}\right|$$

$$> \frac{(1/2)^N}{1+\delta_1}\frac{|P^{(n+1)}(0)|(1-\rho)}{(1+\rho)^3} - \frac{\delta_1}{(1/2)}\max_{|\zeta|=\rho}|g^{(n+1)}(\zeta)|$$

$$\geq \frac{(1/2)^N}{1+\delta_1}\frac{|P^{(n+1)}(0)|(1-\rho)}{(1+\rho)^3} - 2\delta_1\pi^N e^{\pi}.$$

Choose δ_1 so that this last expression is positive for $0 \leq n < N$. Then $F_k^{(n)}$ will be one-to-one on $\{z: |z| = \rho\}$. The proof of the lemma is done.

In what follows, it is convenient to write functions in $\overline{ER}$ as $z + \Sigma_{k=2}^{\infty} b_k z^k$, even though some of them may be polynomials.

THEOREM 3. Let $f \varepsilon ER$ and $g \varepsilon \overline{ER}$. Let $\lambda \varepsilon (0,1)$. Suppose $f(z) = z + \Sigma_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \Sigma_{k=2}^{\infty} b_k z^k$. Assume $a_k b_k \geq 0$ for all k. If $h(z) = \lambda f(z) + (1-\lambda) g(z)$, then $h \varepsilon ER$. Hence, ERP is a convex set.

PROOF. Since $a_k b_k \geq 0$, the signs of $h^{(n+1)}(0)$, $f^{(n+1)}(0)$, and $g^{(n+1)}(0)$ are all the same. So, if $z \varepsilon D$, $h^{(n+1)}(0) \mathrm{Im}\{z\} \mathrm{Im}\{h^{(n)}(z)\} = \lambda h^{(n+1)}(0) \mathrm{Im}\{z\} \mathrm{Im}\{f^{(n)}(z)\} + (1-\lambda) h^{(n+1)}(0) \mathrm{Im}\{z\} \mathrm{Im}\{g^{(n)}(z)\} > 0$ if and only if $\mathrm{Im}\{z\} \neq 0$. Hence, $h^{(n)}$ is typically real on D. By Theorem 2, $h \varepsilon ER$. If $f, g \varepsilon ERP$, then $a_k b_k > 0$ and so $[\lambda f + (1-\lambda) g] \varepsilon$ ERP, i.e., ERP is convex.

REMARK. Suffridge [9] has shown that, if $f \varepsilon ERP$ and if $f(z) = z + a_2 z^2 + \cdots$, then $a_{2k+1} \leq \pi^{2k}/(2k+1)!$ for $k = 1,2,\cdots$ and $a_{2k} \leq 2a_2 \pi^{2(k-1)}/(2k)!$. The inequalities are sharp. It is interesting that a_2 is necessarily involved in the bounds for the even coefficients but not for the odd.

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