RESEARCH NOTES

THE GENERALIZATION AND PROOF OF BERTRAND'S POSTULATE

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ABSTRACT. The purpose of this paper is to show that for $0 < r < 1$ one can determine explicitly an x_0 such that $\forall$ $x \geq x_0$, $\exists$ at least one prime between rx and x. This is a generalization of Bertrand's Postulate. Furthermore, the same procedures are used to show that if one can find upper and lower bounds for $\theta(x)$ whose difference is kx^{ρ}, then $\exists$ a prime between x and $x - Kx^{\rho}$, where $k, K > 0$ are constants, $0 < \rho < 1$ and $\theta(x) = \sum_{p \leq x} \ln p$, where p runs over the primes.

KEY WORDS AND PHRASES. Bertrand's Postulate, Primes, Intervals, Explicit bound for one prime in an interval
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1. INTRODUCTION.

Several authors (for example [1], [2]) have discussed estimates for differences between consecutive primes. The result of this note is a consequence of [2], for example; however, the methods used here are elementary and give explicit bounds for the range of validity.

The proof uses the work done by Lowell Schoenfeld [3]. In fact it is based on Theorem 7* from his paper which states that

$$|\theta(x)-x| < 0.0077629x/\ln x \qquad \text{for } x \geq e^{22} \tag{1.1}$$

where $\theta(x) = \sum_{p \leq x} \ln p$. (here p runs over the primes). Furthermore, it is based on a simple idea: if $\theta(x)-\theta(rx) > 0$ then there must be at least one prime between rx and x (here $0 < r < 1$). The importance of Theorem 1 is the following. by setting $r = 1/2$ we get Bertrand's Postulate. Hence, this theorem is a generalization of this postulate.

The importance of Theorem 2 is that it suggests that if $\rho = ½$ then $\exists$ a prime between x and $x - Kx^{½}$ where K is a positive constant. Moreover, Theorems 18 and 19 of a paper by J. Barkley Rosser and Lowell Schoenfeld [4] give numerical evidence for the hypothesis of Theorem 2 in the case of $\rho = ½$. Of course, if the Riemann hypothesis holds, then Theorem 10 of [3] states that

$$|\theta(x) - x| < \frac{1}{8\pi}\sqrt{x}\ln^2 x, \quad \text{if } 599 \leq x.$$

2. THEOREMS, LEMMA AND THEIR PROOFS.

THEOREM 1. Suppose $0 < r < 1$, let $a = 1 - r$, $b = (1-r)\ln r - .008(1+r)$ and $c = -.008\ln r$. If $x > e^{22}/r$ and $\ln x > (-b + \sqrt{b^2 - 4ac})/2a$, then $\exists$ prime p s.t. $rx < p \leq x$.

PROOF. We have by definition $\theta(x) = \sum_{p \leq x} \ln p$. Given a certain $0 < r < 1$ we want to find an x_0 s.t. $\forall x > x_0$ $\exists$ a prime p between rx and x. This means $\theta(x) - \theta(rx) > 0$. From (1.1) we have for $x \geq e^{22}/r$ the following:

$$\theta(x) - \theta(rx) > x(1-.008/\ln(x)) - rx\,(1 + .008/\ln(rx)).$$

What we need is

$$x(1 - .008/\ln(x)) - rx(1 + .008/\ln(rx)) > 0.$$

After several manipulations this becomes

$$(1-r)\ln^2 x + ((1-r)\ln r - .008(1+r))\ln x - .008\ln r > 0.$$

Let $y = \ln x$; then

$$(1-r)y^2 + ((1-r)\ln r - .008(1+r))\, y - .008\ln r > 0.$$

By letting $a = 1 - r$, $b = (1 - r)\ln r - .008(1+r)$ and $c = -.008\ln r$, we have

$$ay^2 + by + c > 0. \tag{2.1}$$

Therefore we must find y_0 such that $\forall y > y_0$, (2.1) will hold. Let $z = ay^2 + by + c$; since $a > 0$, the parabola opens upward.

By equating $z = 0$ we have

$$ay^2 + by + c = 0. \tag{2.2}$$

Now consider all the different types of roots in (2.2). They are the following:

If (2.2) has complex roots then $\forall y \in R$, (2.1) will hold.

If (2.2) has a double real root then $\forall y > -b/2a$, (2.1) will hold

Finally if (2.2) has distinct real roots then $\forall y > (-b + \sqrt{b^2 - 4ac})/2a$, (2.1) will hold.

However, regardless of the type of roots (2.2) has, if $y > (-b + \sqrt{b^2 - 4ac})/2a$ then (2.1) will hold. But $y = \ln x$. Therefore, if $\ln x > (-b + \sqrt{b^2 - 4ac})/2a$ then $\exists$ is a prime in that interval. Q.E.D.

THEOREM 2. Suppose $0 < \rho < 1$, let c, $c' \geq 0$ and $K > c' + c$. If x is sufficiently large and $x - cx^\rho < \theta(x) < x + c'x^\rho$, then $\exists$ a prime p s.t. $x - Kx^\rho < p < x$.

PROOF. We want to have a prime between $x - Kx^\rho$ and x. This means that $\theta(x) - \theta(x - Kx^\rho) > 0$. From the hypothesis we have

$$\theta(x) - \theta(x - Kx^\rho) > x - cx^\rho - (x - Kx^\rho + c'(x - Kx^\rho)^\rho).$$

What we need is $x - cx^\rho - (x - Kx^\rho + c'(x - Kx^\rho)^\rho) > 0$. Again after manipulation we have $K > c + c'(1 - K/x^{1-\rho})^\rho$, i.e. $K > c + c'$. Q.E.D.

LEMMA. Suppose $0 < r < 1$, let $a = 1 - r$, $b = (1 - r)\ln r - .008(1 + r)$ and $c = -.008\ln r$. If $z = ay^2 + by + c$ and $r \to 1$, then $(-b/2a,\, c - b^2/4a) \to (+\infty, -\infty)$ (this point is the vertex of the parabola).

PROOF

$$\begin{aligned} z &= ay^2 + by + c \\ &= a(y + b/2a)^2 + c - b^2/4a \end{aligned}$$

So $$\lim_{r\to 1} b/2a = \lim_{r\to 1}((1-r)\ln r - .008(1+r))/(1 - r) = -\infty.$$ Also $\lim_{r\to 1} c - b^2/4a = \lim_{r\to 1} -.008\ln r - ((1-r)\ln r - .008(1+r))^2/(4(1-r)) = -\infty$. Q.E.D. The significance of this Lemma is that the closer r is to 1 the larger x has to be in order to have at least one prime between rx and x. Of course, the lower bound for x is given explicitly by Theorem 1.

3. FINAL COMMENTS

It is obvious that with the aid of super computers we can find lower bounds for which Theorem 1 will still be valid. Although $x > e^{22}$ is a relatively "large" number without the aid of a computer, one could use Theorem 8 in a paper of J. Barkley Rosser and Lowell Schoenfeld [5] to obtain similar results. However, not only is the co-efficient not as sharp as the one used in (1.1), but also for r sufficiently close to 1, the Lemma guarantees that x becomes extremely large. In fact, by using Theorem 4 in [4] one can prove the following. Suppose $0 < r < 1$; let $a = 2(1-r)$, $b = 2(1-r)\ln r - r - 1$ and $c = -\ln r$. If $x > 563/r$ and $\ln x > (-b + \sqrt{b^2 - 4ac})/2a$, then $\exists$ prime p s.t. $rx < p \leq x$.

Now a simple proof of Bertrand's Postulate can be found in any elementary number theory book, for example, Niven and Zuckerman [6]. Finally, improvements to the bounds of $\theta(x)$ will greatly increase the importance of Theorem 2.

4. AKNOWLEDGEMENTS:

Originally this paper was based on the work done by J. Barkley Rosser and Lowell Schoenfeld [4]. However, the referee informed me that certain theorems had been improved, which I have now incorporated in this paper. I would like to thank him for his insight and also for other modifications.

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I would like to dedicate this paper to the memory of my teacher R.A. Smith.

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